The log crystalline specialization of $\mathrm{A}_{\mathrm{inf}}$-cohomology in the semistable case

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Abstract

We apply the theory of saturated logarithmic de Rham–Witt complexes to prove the log crystalline comparison of $\mathrm{A}_{\mathrm{inf}}$-cohomology in the case of semistable reduction.

1. Introduction

Let $\mathfrak{X}$ be a formal scheme over $\mathcal{O}_C$ with semistable reduction[1], where C is a completed algebraic closure of a finite extension of $\mathbf{Q}_p$. Let $k = \mathcal{O}_C/\mathfrak{m}_{\mathcal{O}_C} = \mathcal{O}_C^\flat/\mathfrak{m}_{\mathcal{O}_C^\flat}$ be the residue field of $\mathcal{O}_C$. Write $\underline{\mathfrak{X}}$ for the formal log scheme with the divisorial log-structure from its mod p fiber $\mathfrak{X}_{\mathcal{O}_C/p}$. In [BMS18], Bhatt–Morrow–Scholze constructs a cohomology theory $R\Gamma_{\mathrm{A}_{\mathrm{inf}}}(\mathfrak{X})$ valued in the derived category of Fontaine's period ring $\mathrm{A}_{\mathrm{inf}} = W(\mathcal{O}_C^\flat)$, which interpolates various p-adic cohomology theories. We shall briefly review their construction below in Section 3 and refer the reader to [BMS18] and [CK17] for more detail.

This paper studies the specialization of the $\mathrm{A}_{\mathrm{inf}}$-cohomology $R\Gamma_{\mathrm{A}_{\mathrm{inf}}}(\mathfrak{X})$ along the base change $\widetilde{\vartheta} : W(\mathcal{O}_C^\flat) \to W(k)$, obtained by the Witt vector functoriality from the map $\mathcal{O}_C^\flat \to k \xrightarrow{\varphi^{-1}} k$.

Theorem 1 *There is a Frobenius compatible quasi-isomorphism*

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})\widehat{\otimes}^{\mathbf{L}}_{A_{\mathrm{inf}}} W(k) \cong R\Gamma_{\text{log-cris}}(\underline{\mathfrak{X}_k}/W(k)),$$

relating the specialization of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ to $W(k)$ with the log crystalline cohomology of the special fiber of $\mathfrak{X}$ over k. Here the (derived) completion on the left is p-adic.

This result has recently been obtained by Cesnavicius and Koshiwara [CK17] from an "absolute" crystalline comparison over $\mathrm{A}_{\mathrm{cris}}$, generalizing [BMS18]. In our approach, we avoid the comparison over the larger ring $\mathrm{A}_{\mathrm{cris}}$, and directly analyze the base change to $W(k)$.

The key ingredient we use to prove Theorem 1 is the theory of saturated logarithmic de Rham–Witt complexes developed in [Yao19b]. When $\mathfrak{X}$ has good reduction over $\mathcal{O}_C$, Theorem 1 is proved in [BLM18] as an application of the saturated de Rham–Witt complexes without log structures. However, in the semistable case, the nontrivial log structures impose additional difficulties, as it is not clear how to read off the logarithmic data from the $\mathrm{A}_{\mathrm{inf}}$-cohomology. In the rest of the introduction we outline the proof of Theorem 1.

We first recall the theory of saturated log de Rham–Witt complexes. Let N be a monoid and equip k with a log structure $N \to k$ where $N\backslash\{0\} \mapsto 0$. Let $\underline{R} = (R, M)$ be a log algebra over $\underline{k} = (k, N)$. Let $\mathrm{DA}^{\log,p}$ be the category of log Dieudonné algebras, which consists of commutative differential graded algebras (A^*, d), equipped with a Frobenius map F, a log structure L and a log

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[1]More generally, we may assume that $\mathfrak{X}$ has generalized semistable reduction over $\mathcal{O}_C$ as in Subsection 3.1.

derivation δ, satisfying $dF = pFd, \delta F = pF\delta$ and other compatibility relations (see Section 2). Also recall the décalage operator η_p on a p-torsion free cochain complex A^*, giving a sub-complex $\eta_p(A^*) \subset A^*$ defined by

$$\eta_p A^i := \{x \in p^i A^i | dx \in p^{i+1} A^{i+1}\}.$$

For each p-torsion free object A^* in $\mathrm{DA}^{\log,p}$, there is a morphism $\phi_F : A^* \to \eta_p(A^*)$ of log Dieudonné algebras. If ϕ_F turns out to be an isomorphism, then A^* is naturally equipped with the Verschiebung maps V. The full subcategory $\mathrm{DA}^{\log,p}_{\mathfrak{str}}$ of *strict* log Dieudonné algebras consists of log Dieudonné algebras $A^* \in \mathrm{DA}^{\log,p}$ such that ϕ_F is an isomorphism and A^* is complete with respect to the V-filtration (see Section 2). The saturated log de Rham–Witt complex $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ is a certain "initial object" in $\mathrm{DA}^{\log,p}_{\mathfrak{str}}$. The main result of [Yao19b] is the following.

Theorem 2 ([Yao19b]) *Let $\underline{X}$ be a coherent log scheme over $\underline{k}$. There is an* étale *sheaf $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ valued in log Dieudonné algebras, such that on a local chart $\underline{R} = (R, M)$ of $\underline{X}$, there is a canonical F and V compatible isomorphism of cdga's $\mathcal{W}\omega^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} \Gamma(\mathrm{Spec}\, R, \mathcal{W}\omega^*_{\underline{X}/\underline{k}})$. Suppose that $\underline{X}$ is log-smooth of log-Cartier type, then $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ agrees with the Hyodo–Kato complex and computes the log crystalline cohomology $R\Gamma_{\text{log-cris}}(\underline{X}/W(\underline{k}))$.*

We need another ingredient. Consider the category DC of Dieudonné complexes, which are cochain complexes equipped with Frobenius maps (satisfying the usual relation $dF = pFd$). As before, we have a full subcategory $\mathrm{DC}_{\mathfrak{str}}$ consisting of strict Dieudonné complexes. Denote by $\widehat{D}(\mathbf{Z}_p)^{L\eta_p}$ the $L\eta_p$-fixed points in derived p-complete $\mathbf{Z}_p$-modules.

Theorem 3 ([BLM18]) *The canonical functor $\mathrm{DC}_{\mathfrak{str}} \longrightarrow \widehat{D}(\mathbf{Z}_p)^{L\eta_p}$ is an equivalence of categories. In particular, objects in $\widehat{D}(\mathbf{Z}_p)^{L\eta_p}$ admit canonical cochain representatives.*

Now we return to the setup of Theorem 1 and let $\underline{k} = (k, \mathcal{O}_C \backslash \{0\})$. Denote by $\mathfrak{X}_C$ the adic generic fiber of $\mathfrak{X}$ over C. Recall from [BMS18] the definition of $A\Omega_{\mathfrak{X}} := L\eta_\mu(R\nu_* \mathbf{A}_{\mathrm{inf},\mathfrak{X}_C})$, where $L\eta_\mu$ is the derived décalage operator with $\mu := [\epsilon] - 1 \in \mathrm{A_{inf}}$, $\nu : (\mathfrak{X}_C)_{\text{proét}} \to \mathfrak{X}_{\text{ét}}$ is the nearby cycle map (see Section 3), and $\mathbf{A}_{\mathrm{inf},\mathfrak{X}_C} = W(\widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}_C})$ is the basic period sheaf constructed in [Sch13]. The sheaf $\mathbf{A}_{\mathrm{inf},\mathfrak{X}_C}$ is equipped with a natural Frobenius, which induces a Frobenius φ on $A\Omega_{\mathfrak{X}}$ and the $\mathrm{A_{inf}}$-cohomology $R\Gamma_{\mathrm{A_{inf}}}(\mathfrak{X}) := R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$.

The first issue we need to overcome (in attempting to invoke Theorem 2) is that $\underline{\mathfrak{X}_k}$ is not a coherent log scheme (hence not log-smooth over $\underline{k}$). This turns out not to be a serious issue once we work with small enough local charts for $\underline{\mathfrak{X}_k}$. More precisely, we shall prove Theorem 1 by restricting to small enough affine opens $\mathrm{Spf}\, S \in \mathfrak{X}_{\text{ét}}$ which admit semistable coordinates (see Subsection 3.1). For such S, let $A\Omega_S := R\Gamma(\mathrm{Spf}\, S, A\Omega_{\mathfrak{X}}) \in D(\mathrm{A_{inf}})$. Write $\underline{S_k} = \underline{S} \otimes_{\mathcal{O}_C} k$ where S is equipped with the divisorial log structure from its mod p fiber $\mathrm{Spec}\, S/p$. Theorem 1 will follow from the following key local statement.

Theorem 4 *There is a canonical Frobenius-compatible quasi-isomorphism*

$$\mathcal{W}\omega^*_{\underline{S_k}/\underline{k}} \xrightarrow{\sim} A\Omega_S \widehat{\otimes}^{\mathbf{L}}_{\mathrm{A_{inf}}} W(k).$$

For this we first construct a quasi-isomorphism

$$A\Omega_S \widehat{\otimes}^{\mathbf{L}}_{\mathrm{A_{inf}}} W(k) \cong L\eta_p(A\Omega_S \widehat{\otimes}^{\mathbf{L}}_{\mathrm{A_{inf}}} W(k)).$$

This puts $A\Omega_S \widehat{\otimes}^{\mathbf{L}}_{\mathrm{A_{inf}}} W(k)$ in the category $\widehat{D}(\mathbf{Z}_p)^{L\eta_p}$, so we get a strict Dieudonné algebra A^*_S by the discussion above. To upgrade A^*_S to a log Dieudonné algebra is more subtle compared to the non-logarithmic setting, since it is difficult to read off log structures from the derived category (for example we do not know a version of Theorem 2.8 that takes log structures into account). To remedy this, we first fix a choice of semistable coordinates $\square : S^\square \to S$, from which one obtains a chart $(S, M^\square)$ for the log structure where $M^\square$ is particularly simple. Using explicit computations for this chart, we are able to define log structures and log derivations on A_S^* :

$$\alpha^\square : M^\square \to A_S^0, \qquad \delta^\square : M^\square \to A_S^1.$$

This way, A_S^* becomes a strict log Dieudonné algebra. From the universal property of saturated log de Rham–Witt complexes, we get a map of strict log Dieudonné algebras

$$\tau^\square : \mathcal{W}\omega^*_{\underline{S_k}/\underline{k}} \longrightarrow A_S^*.$$

The upshot is that, even though the logarithmic data depend on coordinates, *on the underlying complex the morphism $\tau^\square$ does not depend on this choice.* In other words, we have

Proposition 5 *As a morphism of Dieudonné algebras, the map $\tau^\square$ is independent of the choice of semistable coordinates. In particular, ignoring log structures on both sides, we get a canonical isomorphism $\tau : \mathcal{W}\omega^*_{\underline{S_k}/\underline{k}} \longrightarrow A_S^*$ of Dieudonné algebras.*

Remark 6 *The final step replies on the Hodge-Tate (and de Rham) comparison (Theorem 3.7) of A_{inf}-cohomology. In fact, we only need to analyze $A\Omega_S$ along the Hodge–Tate specialization $A_{\text{inf}} \to \mathcal{O}_C$, instead of objects such as $A\Omega_S \otimes^{\mathbf{L}} A_{\text{cris}}^{(m)}$ (see [CK17]). Moreover, our approach sees no difference in treating formal schemes of semistable type or generalized semistable type.*

Remark 7 *In a forthcoming paper we give another proof of Theorem 1, by constructing a functorial map from $R\Gamma_{(\text{log-})\text{cris}}(\underline{\mathfrak{X}}_{\mathcal{O}_C/p})$ to $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})\widehat{\otimes}^{\mathbf{L}} A_{\text{cris}}$. The map we construct there goes in the opposite direction compared to the one used in [BMS18] and [CK17]. Moreover, our construction is enough to recover most of the results in loc.cit., including the B_{st} comparison theorem (in fact in a slightly more general setting). See [Yao19a] and the introduction thereof.*

Outline of the paper

In Section 2 we briefly recall the theory of saturated log de Rham–Witt complexes, and state the precise versions of Theorem 2 and 3. In Section 3 we recall the construction of A_{inf}-cohomology via perfectoid spaces following [BMS18] and [CK17], and carry out some local computations in an example of the standard semistable reduction. Section 4 is the technical core of the paper, where we prove the main following the strategy outlined above.

Conventions

We use the language of derived ∞-categories in the article, where we mostly follow the conventions of Section 10 of [BLM18]. In particular, for a commutative ring A, we use $\mathcal{D}(A)$ to denote the derived ∞-category of A-modules and identify the usual (triangulated) derived category $D(A)$ with the homotopy category of $\mathcal{D}(A)$. For log geometry we follow the conventions of summarized in the appendix of [Yao19b] and freely use results there.

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2. Saturated log de Rham–Witt complexes

In this section we recall the necessary background on log de Rham–Witt complexes needed for the proof of the main theorem.

2.1 Log Dieudonné algebras

First recall the décalage operator

Definition 2.1 *Let R be a ring and $\mu \in R$ a nonzero divisor. Let (M^*, d) be a cochain complex of μ-torsion free R-modules, then $(\eta_\mu M)^* \subset M^*[\frac{1}{\mu}]$ is defined to be the sub-complex given by*

$$(\eta_\mu M)^i = \{x \in \mu^i M^i : dx \in \mu^{i+1} M^{i+1}\}.$$

The operator η_μ kills μ-torsion in cohomology. More precisely, $H^i(\eta_\mu M^*) \cong H^i(M^*)/H^i(M^*)[\mu]$. In particular, η_μ descends to the derived category, inducing a functor $L\eta_\mu : D(R) \to D(R)$. For other properties of η_μ and $L\eta_\mu$ we refer the reader to §6 of [BMS18].

Let $\underline{W} = (W(k), N)$ be a log algebra.

Definition 2.2 *A p-compatible log Dieudonné $\underline{W}$-algebra* [2] *is a tuple (A^*, L, d, δ, F), where*

- *$(A^* = \oplus_{i \geq 0} A^i, d)$ is a cdga over W,*
- *$\underline{A}^0 = (A^0, \alpha : L \to A^0)$ is a log algebra over $\underline{W}$,*
- *$\delta : L \to A^1$ is a map of monoids, and*
- *$F : A^* \to A^*$ is a graded algebra homomorphism.*

The data are required to satisfy the following conditions:

(1) $F(x) \equiv x^p \mod p$ for all $x \in A^0$; while $F(\alpha(l)) = \alpha(l)^p$ for all $l \in L$.

(2) $dF(x) = pF(dx)$ for all $x \in A^*$.

(3) $(d : A^0 \to A^1, \delta : L \to A^1)$ is a log derivation of $\underline{A}^0/\underline{W}$, where we further require that the composition $L \xrightarrow{\delta} A^1 \xrightarrow{d} A^2$ is 0.

Let A^* be a p-torsion free log Dieudonné algebra, then the Frobenius F determines a map of cochain complexes $\phi_F : A^* \to \eta_p A^*$ by sending $x \mapsto p^n F(x)$ for $x \in A^n$. A log Dieudonné algebra A^* is saturated if it is p-torsion free and ϕ_F is an isomorphism. Suppose that A^* is saturated, then for each degree$i$, the composed map

$$\phi_F : A^i \xrightarrow{F} \{x \in A^i : dx \in pA^{i+1}\} \xrightarrow{\times p^i} (\eta_p A)^i$$

is an isomorphism, hence F is injective and $F(A^*)$ contains pA^*. Therefore, for each $x \in A^n$, there is a unique element Vx such that $F(Vx) = px$. In other words saturated log Dieudonné complexes are equipped with a natural Verschiebung map.[3] In particular, suppose that A^* is saturated, for each $r \geq 1$, we may form the quotient

$$W_r(A^*) := A^*/(V^r A^* + dV^r A^*),$$

which is a cdga as $V^r A^* + dV^r A^* \subset A^*$ is a differential graded ideal. Define the V-completion of A^* to be the limit $W(A^*) := \varprojlim W_r A^*$ along the natural projection maps.

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[2] The term p-compatibility refers to the condition that $F(\alpha(l)) = \alpha(l)^p$ (compare with [Yao19b]), and will be omitted in the rest of the paper since all log Dieudonné algebras are assumed to be p-compatible here. Moreover, when no confusion should arise, we simply denote a log Dieudonné algebra by A^*.

[3] It is straightforward to check that $FV = VF = p, FdV = d, Vd = p\, dV$, and $xVy = V(Fx \cdot y)$, etc.

Definition 2.3 *A saturated (p-compatible) log Dieudonné algebra A^* is strict if the canonical map $A^* \longrightarrow W(A^*)$ is an isomorphism.*

Notation. Denote the p-compatible log Dieudonné algebras by $\mathrm{DA}^{\log,p}_{/\underline{W}}$ where morphisms are the obvious ones. We often suppress notations and write $\mathrm{DA}^{\log,p} = \mathrm{DA}^{\log,p}_{/\underline{W}}$ when $\underline{W}$ is understood. The full subcategory of $\mathrm{DA}^{\log,p}$ spanned by strict algebras is denoted by $\mathrm{DA}^{\log,p}_{\mathfrak{str}}$.

Remark 2.4 *Let A^* be a saturated log Dieudonné algebra then A^0/VA^0 is reduced. If A^* is in addition strict, then each A^i is p-complete, and A^0 is identified with the Witt-vectors of $W_1(A)^0 = A^0/VA^0$, where Frobenius F on A^0 corresponds to the Witt vector Frobenius.*

Finally we record the following useful lemma.

Lemma 2.5 ([BLM18] Corollary 2.7.4) *A map $f : A^* \to B^*$ between two strict log Dieudonné algebras is an isomorphism iff its reduction $f : W_1(A^*) \to W_1(B^*)$ is an isomorphism.*

2.2 Saturated log de Rham–Witt complexes

Let $\mathrm{Alg}^{\log}_{\underline{k}}$ be the category of log algebras over $\underline{k}$. There is a functor $\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}} \longrightarrow \mathrm{Alg}^{\log}_{\underline{k}}$ given by $A^* \mapsto \underline{A}^0/V(A^0)$, where $\underline{A}^0/V(A^0)$ consists of the data $(A^0/V(A^0), \alpha : L \to A^0 \twoheadrightarrow A^0/V(A^0))$.

Definition 2.6 *Let $\underline{R} = (R, M) \in \mathrm{Alg}^{\log}_{\underline{k}}$ be a log algebra over $\underline{k}$. The saturated log de Rham–Witt complex $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ of $\underline{R}$ is a strict log Dieudonné $W(\underline{k})$-algebra in $\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}$ equipped with an isomorphism $e : \underline{R} \xrightarrow{\sim} \mathcal{W}_1\omega^0_{\underline{R}/\underline{k}}$, such that for any $A^* \in \mathrm{DA}^{\log,p}_{\mathfrak{str}}$, the natural map*

$$\mathrm{Hom}_{\mathrm{DA}^{\log}_{\mathfrak{str}}}(\mathcal{W}\omega^*_{\underline{R}/\underline{k}}, A^*) \longrightarrow \mathrm{Hom}_{\mathrm{Alg}^{\log}_{\underline{k}}}(\underline{R}, \underline{A}^0/V(A^0))$$

is a bijection.

The following theorem is proved in Section 4, 5 and 6 of [Yao19b]:

Theorem 2.7 *$\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ exists and glues to a sheaf $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ on the* étale *site $X_{\text{ét}}$ when $\underline{X}$ is a coherent log scheme over $\underline{k}$. If moreover $\underline{X}$ is log-smooth of log-Cartier type over $\underline{k}$, then there is a natural Frobenius compatible quasi-isomorphism $R\Gamma_{\text{ét}}(X, \mathcal{W}\omega^*_{\underline{X}/\underline{k}}) \cong R\Gamma_{\text{log-cris}}(\underline{X}/W(\underline{k}))$.*

2.3 $L\eta_p$-fixed points of the p-complete derived category

In this subsection we consider the category DC of Dieudonné complexes. A Dieudonné complex (M^*, d, F) is a cochain complex (M^*, d) of abelian groups equipped with a Frobenius operation $F : M^* \to M^*$ satisfying $dF = pFd$.[4] The full subcategory $\mathrm{DC}_{\mathfrak{str}}$ of strict Dieudonné complexes is similarly defined as above. Consider the subcategory $\widehat{D}(\mathbf{Z})_p$ of $D(\mathbf{Z})$ (viewed as a triangulated category) generated by derived p-complete objects. For applications later, we also need the derived ∞-category enhancement $\mathcal{D}(\mathbf{Z})$ of its homotopy category $D(\mathbf{Z})$, and similarly define $\widehat{\mathcal{D}}(\mathbf{Z})_p \subset \mathcal{D}(\mathbf{Z})$. Denote by $\widehat{\mathcal{D}}(\mathbf{Z})_p^{L\eta_p}$ (resp. $\widehat{D}(\mathbf{Z})_p^{L\eta_p}$) the $L\eta_p$ fixed points of $\widehat{\mathcal{D}}(\mathbf{Z})_p$ (resp. $\widehat{D}(\mathbf{Z})_p$), which consists of an object X equipped with an isomorphism $X \xrightarrow{\sim} L\eta_p X$.

[4] For future references: a Dieudonné algebra is a triple (A^*, d, F) where (A^*, d) is a cdga concentrated in nonnegative degrees and F is a graded ring homomorphism satisfying $dF = pFd$ and $F(x) \equiv x^p \mod p$ on A^0.

Theorem 2.8 ([BLM18] Theorem 7.3.4 & 7.4.8) [5]

The natural functor $\mathrm{DC}_{\mathfrak{str}} \longrightarrow \widehat{D}(\mathbf{Z})_p^{L\eta_p}$ *factors through the following sequences of equivalences of categories:*

$$\mathrm{DC}_{\mathfrak{str}} \xrightarrow{\sim} \widehat{\mathcal{D}}(\mathbf{Z})_p^{L\eta_p} \xrightarrow{\sim} \widehat{D}(\mathbf{Z})_p^{L\eta_p}.$$

We refer the interested readers to Section 7 of [BLM18] for more detail.

Remark 2.9 *We describe an explicit quasi-inverse functor (which is not presented in [BLM18] but not difficult to obtain). Let* $(X, \psi : X \xrightarrow{\sim} L\eta_p X)$ *be an* $L\eta_p$ *fixed point in the category* $\widehat{D(\mathbf{Z})}_p$*. For each* $r \geq 1$*, we define the cochain complexes* (X_n^*, β_n) *by*

$$X_r^i := H^i(X \otimes^{\mathbf{L}} \mathbf{Z}/p^r),$$

where the differentials β_r *are given by the Bockstein differentials (associated to* $X \otimes^{\mathbf{L}} Z/p^r \to X \otimes^{\mathbf{L}} Z/p^{2r} \to X \otimes^{\mathbf{L}} Z/p^r$*). Next define maps*

$$\mu_r : H^k(X \otimes^{\mathbf{L}} \mathbf{Z}/p^r) \longrightarrow H^k(L\eta_p X \otimes^{\mathbf{L}} \mathbf{Z}/p^{r-1})$$

by sending $y \in H^k(X \otimes^{\mathbf{L}} \mathbf{Z}/p^r)$ *to* $p^k y \in H^k(L\eta_p X \otimes^{\mathbf{L}} \mathbf{Z}/p^{r-1})$*. This leads to the "restriction maps"* $R_r := X_r^* \longrightarrow X_{r-1}^*$ *by setting* $R_r := \psi^{-1} \circ \mu_r$*, namely the composition*

$$\begin{array}{ccc} H^*(X \otimes^{\mathbf{L}} \mathbf{Z}/p^r) & \xrightarrow{\mu_r} & H^*(L\eta_p X \otimes^{\mathbf{L}} \mathbf{Z}/p^{r-1}) \\ & \searrow^{R_r} & \downarrow{\scriptstyle \psi^{-1}} \\ & & H^*(X \otimes^{\mathbf{L}} \mathbf{Z}/p^{r-1}) \end{array}$$

We build a cochain complex $X^* := \varprojlim_{R_r} X_r^*$ *by taking the inverse limit along the restriction maps* R_r*. Equipped with Bockstein differentials,* X^* *becomes a Dieudonné complex as follows: the canonical projection* $\mathbf{Z}/p^r \to \mathbf{Z}/p^{r-1}$ *and map* $\mathbf{Z}/p^{r-1} \xrightarrow{\times p} \mathbf{Z}/p^r$ *respectively induce the maps*

$$F : X_r^* \to X_{r-1}^*, \qquad V : X_{r-1}^* \to X_r^*.$$

They are compatible with R_r *and therefore induce operators* F, V *on* X^**.*

3. $\mathrm{A_{inf}}$-cohomology theory

In this section we ecall the theory of $\mathrm{A_{inf}}$-cohomology developed in [BMS18] and [CK17] via perfectoid spaces. In particular, we need the de Rham and Hodge–Tate comparisons (Theorem 3.7) and the fact that $L\eta$ commutes with a certain form of global sections on small enough objects (Theorem 3.4). We also carry out some local computations.

Notation 3.1 – *For simplicity let* C *be a completed algebraic closure of* $W(k)[\frac{1}{p}]$*. Let* $\mathfrak{m}$ *be the maximal ideal of* $\mathcal{O}_C$*. Fix a choice of* $p^{\mathbf{Q}} \subset C$ *(in particular a choice of compatible* p^n*th roots of unity* $\{\zeta_{p^n}\} \in \mu_{p^n}(C)$*) and consider the element* $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$*. This determines*

$$\mu := [\epsilon] - 1, \quad \xi := \frac{\mu}{\varphi^{-1}(\mu)} = \sum_{i=0}^{p-1} [\epsilon]^{\frac{i}{p}}, \quad \widetilde{\xi} := \varphi(\xi)$$

[5] This equivalence is already hinted in [IR83] (and in [HK94] for the log version). For example, compare the inverse functor described below and the Hyodo–Kato construction in Subsection 6.1 of [Yao19b]

in Fontaine's period ring $A_{\inf} = W(\mathcal{O}_C^\flat)$, *where* $\mathcal{O}_C^\flat$ *is the tilt of* $\mathcal{O}_C$. *The ring* $A_{\inf}$ *is equipped with the* (p,μ)*-adic topology.*

- *Let* θ *be the surjection* $A_{\inf} \twoheadrightarrow \mathcal{O}_C$ *lifting the canonical morphism* $\mathcal{O}_C^\flat \to \mathcal{O}_C/p$, *and define* $\widetilde{\theta} := \theta \circ \varphi^{-1}$. *Therefore we have* $\ker(\theta) = (\xi)$ *(resp.* $\ker(\widetilde{\theta}) = (\widetilde{\xi})$*). We will call the map* θ *(resp.* $\widetilde{\theta}$*) the de Rham (resp. Hodge-Tate) specialization map.*
- *Let* $\widetilde{\vartheta} : A_{\inf} \to W(k)$ *be the unique lift of the quotient map* $A_{\inf} \xrightarrow{\widetilde{\theta}} \mathcal{O}_C \to k$, *which will be referred to as the (log) crystalline specialization map. All maps* $A_{\inf} \to W(k)$ *in the rest of the article are assumed to be* $\widetilde{\vartheta}$ *unless otherwise stated.*

3.1 Semistabe coordinates and log structures

Let $\mathfrak{X}$ be a p-adic formal scheme over $\mathcal{O}_C$ of generalized semistable reduction, which means that there exists a covering by affine opens $\mathfrak{U} = \operatorname{Spf} R$ in the étale topology of $\mathfrak{X}$, such that each $\mathfrak{U}$ admits an étale morphism (called a *semistable coordinate*)

$$\square : \mathfrak{U} \longrightarrow \operatorname{Spf} R^\square = \operatorname{Spf}(R_1^\square \widehat{\otimes} \cdots \widehat{\otimes} R_s^\square)$$

over $\mathcal{O}_C$, where each $R_i^\square$ is of the form

$$\mathcal{O}_C\langle T_0, ..., T_r, T_{r+1}^{\pm 1}, ..., T_d^{\pm 1}\rangle / (T_0 \cdots T_r - \varpi)$$

with $0 \le r \le d$, where $\varpi \in \mathfrak{m}\backslash\{0\}$ a non-unit of rational valuation. The adic generic fiber $\mathfrak{X}_C := \mathfrak{X}_\eta^{\mathrm{ad}}$ is smooth over C by the requirement above. We equip $\mathcal{O}_C$ (resp. $\mathcal{O}_C/p$) with the log structure $\mathcal{O}_C\backslash\{0\} \hookrightarrow \mathcal{O}_C$ (resp. its pullback log structure), and $\mathfrak{X}$ its "divisorial log structure" $M_{\mathfrak{X}}$ given by the sheafification of $(\mathcal{O}_{\mathfrak{X}}[\frac{1}{p}])^\times \cap \mathcal{O}_{\mathfrak{X}}$. The log scheme $\underline{\mathfrak{X}}_{\mathcal{O}_C/p}$ over $\operatorname{Spec} \underline{\mathcal{O}_C}/p$ and its special fiber $\underline{\mathfrak{X}}_k$ are equipped with the pullback log structures. Similarly define $\underline{\mathfrak{U}}$ and $\underline{\mathfrak{U}_k}$.

Note that the semistable coordinate morphism $\mathfrak{U} = \operatorname{Spf} R \to \operatorname{Spf}(R_1^\square \widehat{\otimes} \cdots \widehat{\otimes} R_s^\square)$ is the p-adic completion of (the $\mathcal{O}_C$-base change of) some étale $\mathcal{O}$-morphism

$$\begin{aligned} U \quad \longrightarrow \quad & \operatorname{Spec}\Big(\mathcal{O}[T_{h,i}, T_{h,j}^{\pm 1}]_{1\le h\le s} / \big(\prod T_{h,i} - \varpi_h\big)_{1\le h \le s}\Big) \\ = & \operatorname{Spec}\Big(\mathcal{O}[T_{1,i}, T_{1,j}^{\pm 1}]/(T_{1,0}\cdots T_{1,r_1} - \varpi_1)\Big) \times \cdots \\ & \times \operatorname{Spec}\Big(\mathcal{O}[T_{s,i}, T_{s,j}^{\pm 1}]/(T_{s,0}\cdots T_{s,r_s} - \varpi_s)\Big) \end{aligned}$$

where $\mathcal{O} \subset \mathcal{O}_C$ is a discretely valued subring, and $\varpi_1, ..., \varpi_s \in \mathcal{O}$. We still call this a semistable coordinate and continue to denote it by $\square$. It determines a choice of charts for the divisorial log structure on U as follows. Let $\widetilde{M}^\square$ be the following push-out of monoids.

$$\begin{array}{ccc} \mathbf{N}^s & \xrightarrow{\text{diag}} & \mathbf{N}^{r_1+1} \oplus \cdots \oplus \mathbf{N}^{r_s+1} \\ \downarrow & & \downarrow \\ \mathcal{O}\backslash\{0\} & \longrightarrow & \widetilde{M}^\square = \Big(\bigoplus_{1\le h\le s} \mathbf{N}^{r_h+1}\Big) \sqcup_{\mathbf{N}^s} (\mathcal{O}\backslash\{0\}) \end{array}$$

Here the top horizontal map is the "diagonal" map, given by $1 \mapsto (1, ..., 1)$ on each $\mathbf{N} \to \mathbf{N}^{r_h+1}$; and the left vertical map $\mathbf{N}^s \to \mathcal{O}\backslash\{0\}$ is $(m_1, ..., m_s) \longmapsto \varpi_1^{m_1} \cdots \varpi_s^{m_s}$. Then the log structure $(\mathcal{O}_U[\frac{1}{p}])^\times \cap \mathcal{O}_U$ has a chart given by $\widetilde{M}^\square \to \Gamma(U, \mathcal{O}_U)$, where $(\{m_{h,i}\}_{\substack{1\le h\le s \\ 0\le i\le r_h}}, a)$ gets mapped to $\big(\prod (T_{h,i})^{m_{h,i}}\big) \cdot a \in \Gamma(U, \mathcal{O}_U)$. Moreover, the pre-log structure described by the chart above is the base change of

$$\begin{array}{ccc} M^{\square} := \bigoplus_{1\le h\le s} \mathbf{N}^{r_h+1} & \longrightarrow & R_{\circ} := \Gamma(U, \mathcal{O}_U) \\ {\scriptstyle \mathrm{diag}}\uparrow & & \uparrow \\ N^{\square} := \mathbf{N}^s & \longrightarrow & \mathcal{O} \end{array}$$

along $(\mathcal{O}, N^{\square} = \mathbf{N}^s) \to (\mathcal{O}, \mathcal{O}\backslash\{0\})$, here the underlying map of rings is identity and the monoid map is $(m_1, ..., m_s) \longmapsto \varpi_1^{m_1} \cdots \varpi_s^{m_s}$.

In the following remarks we remedy the issue that the log structures on $\underline{\mathfrak{X}}$ are not coherent.

Remark 3.2 *Although $\underline{\mathfrak{U}} = \operatorname{Spf} \underline{R}$ is not log-smooth over $\underline{\mathcal{O}_C}$ (as the log structures are not fine), we may treat it as if it were so, since the map $\underline{\mathcal{O}_C} \to \underline{R}$ is the base change of the log-smooth morphism $(\mathcal{O}_C, N^{\square}) \longrightarrow (R, M^{\square})$ of fine log rings along $(\mathcal{O}_C, N^{\square}) \longrightarrow (\mathcal{O}_C, \mathcal{O}\backslash\{0\})$. Consequently, the (continuous) log differentials $\omega^{*,\mathrm{ct}}_{\underline{R}/\underline{\mathcal{O}_C}}$ and $\omega^{*,\mathrm{ct}}_{(R,M^{\square})/(\mathcal{O}_C,N^{\square})}$ are isomorphic, and the $\mathcal{O}_{\mathfrak{X}}$-module $\omega^{*,\mathrm{ct}}_{\underline{\mathfrak{X}}/\underline{\mathcal{O}_C}}$ is locally free of finite rank. For the special fiber $R_k = R \otimes_{\mathcal{O}_C} k$ of $\mathfrak{U}$, the local chart $(R_k, \widetilde{M^{\square}})$ of $\underline{\mathfrak{X}_k}$ over $(k, \mathcal{O}_C\backslash\{0\})$ is the base change along $(k, N^{\square}) \to (k, \mathcal{O}_C\backslash\{0\})$ of $\underline{k}^{\square} := (k, N^{\square}) \longrightarrow (R_k, M^{\square})$, which is log-smooth of log-Cartier type. Hence*

$$\omega^*_{(R_k,M^{\square})/\underline{k}^{\square}} \cong \omega^*_{(R_k,\widetilde{M^{\square}})/(k,\mathcal{O}_C\backslash\{0\})} \cong \omega^{*,\mathrm{ct}}_{\underline{R}/\underline{\mathcal{O}_C}} \otimes_{\mathcal{O}_C} k$$

is locally free over R_k and satisfies the Cartier isomorphism. More importantly, by Remark 4.16 in [Yao19b], the comparison with de Rham complexes (in particular the conclusion of Proposition 4.22 thereof) still applies to $\mathcal{W}\omega^*_{(R_k,M^{\square})/\underline{k}^{\square}}$ and the étale *sheaf $\mathcal{W}\omega^*_{\underline{\mathfrak{X}_k}/\underline{k}}$, where $\underline{k} = (k, \mathcal{O}\backslash\{0\})$. Therefore, the map $(R_k, M^{\square}) \to \underline{R_k} = (R_k, \Gamma_{\text{ét}}(\mathfrak{U}, M_{\mathfrak{X}}))$ induces an isomorphism*

$$\mathcal{W}\omega^*_{(R_k,M^{\square})/\underline{k}^{\square}} \xrightarrow{\sim} \mathcal{W}\omega^*_{\underline{R_k}/\underline{k}}$$

on the underlying Dieudonné algebras, by Corollary 2.10 of loc.cit.

Remark 3.3 *In our setup, we still have $\mathcal{W}\omega^*_{\underline{\mathfrak{X}_k}/\underline{k}} \xrightarrow{\sim} W^{\mathrm{HK}}\omega^*_{\underline{\mathfrak{X}_k}/\underline{k}}$,*[6] *and both compute the log crystalline cohomology of $\underline{\mathfrak{X}_k}$. The first claim essentially follows from the same line of arguments as the proof of Theorem 6.1 in [Yao19b] the key being the Cartier isomorphism (which still holds by the previous remark). The rest follows from the fact that there is a canonical quasi-isomorphism $W_n^{\mathrm{HK}}\omega^*_{\underline{\mathfrak{X}_k}/\underline{k}} \xrightarrow{\sim} Ru^{\log}_{\underline{X}/W_n,*}(\mathcal{O}_{\underline{X}/\underline{W_n}})$ for each $n \geq 1$. Note that the proof of Theorem 4.19 in [HK94] goes through (this again amounts to the Cartier isomorphism).*

3.2 $\mathbf{A}_{\mathrm{inf}}$-cohomology

Following [BMS18], we define $A\Omega_{\mathfrak{X}} := L\eta_{\mu} R\nu_*(\mathbf{A}_{\mathrm{inf},\mathfrak{X}_C})$, where $\nu : \mathfrak{X}_{C,\text{proét}} \to \mathfrak{X}_{\text{ét}}$ is the composition of $\mathfrak{X}_{C,\text{proét}} \to \mathfrak{X}_{C,\text{ét}}$ and $\mathfrak{X}_{C,\text{ét}} \to \mathfrak{X}_{\text{ét}}$, and $\mathbf{A}_{\mathrm{inf},\mathfrak{X}_C} := W(\widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}_C})$ is the period sheaf on the pro-étale site of $\mathfrak{X}_C$ which takes an affinoid perfectoid object (S, S^+) to $W(S^{+,\flat})$ (cf. [Sch13]). The Frobenius automorphism on $\widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}_C}$ lifts to a Frobenius on $\mathbf{A}_{\mathrm{inf},\mathfrak{X}_C}$. By functoriality this induces an $\mathrm{A}_{\mathrm{inf}}$-linear Frobenius map $\varphi_{\mathfrak{X}} : A\Omega_{\mathfrak{X}} \to \varphi_* A\Omega_{\mathfrak{X}}$, which is the composition [7]

$$A\Omega_{\mathfrak{X}} \xrightarrow{\widetilde{\varphi}_{\mathfrak{X}}} \varphi_* L\eta_{\varphi(\mu)} R\nu_* \, \mathbf{A}_{\mathrm{inf},\mathfrak{X}_C} = \varphi_* L\eta_{\widetilde{\xi}} A\Omega_{\mathfrak{X}} \to \varphi_* A\Omega_{\mathfrak{X}}.$$

[6] Even though in the original work of [HK94], all log schemes are assumed to be fine, we may still define the Hyodo–Kato complex by taking the inverse limit of $R^* u^{\log}_{\underline{X}/W_n,*}(\mathcal{O}_{\underline{X}/\underline{W_n}})$ as discussed in Subsection 6.1 of [Yao19b].

[7] The isomorphism $\widetilde{\varphi}_{\mathfrak{X}}$ is called the divided Frobenius in [BLM18].

We consider the site $\mathfrak{X}_{\text{ét,aff}}$ consisting of open affine formal schemes $\mathfrak{U} \in \mathfrak{X}_{\text{ét}}$ which admit étale semistable coordinates, where the topology is given by étale coverings. As in [BMS18], there are two presheaves on $\mathfrak{X}_{\text{ét,aff}}$ that play important roles for us, which we define now. Let $\mathcal{D}(\mathfrak{X}_{\text{ét,aff}}, \mathrm{A}_{\text{inf}})$ be the derived ∞-category of sheaves of A_{inf}-modules on $\mathfrak{X}_{\text{ét,aff}}$, and $\mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \mathcal{D}(\mathrm{A}_{\text{inf}}))$ the ∞-category of presheaves valued in $\mathcal{D}(\mathrm{A}_{\text{inf}})$. [8] Let

$$\iota : \mathcal{D}(\mathfrak{X}_{\text{ét,aff}}, \mathrm{A}_{\text{inf}}) \longrightarrow \mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \mathcal{D}(\mathrm{A}_{\text{inf}}))$$

denote canonical fully faithful functor which takes $\mathcal{F}$ to $\iota(\mathcal{F}) : \mathfrak{U} \mapsto R\Gamma(\mathfrak{U}, \mathcal{F})$. ι admits a left adjoint j^{-1} which is the sheafification functor. We will interchangeably denote $\iota(\mathcal{F})$ by $\mathcal{F}^{\text{pre}}$ to emphasize that it is the presheaf associated to $\mathcal{F}$ (viewed as a sheaf taking values in the derived ∞-category $\mathcal{D}(\mathrm{A}_{\text{inf}})$). Now we define the following two objects in $\mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \mathcal{D}(\mathrm{A}_{\text{inf}}))$: [9]

- $A\Omega_{\mathfrak{X}}^{\text{pre}} := \iota(A\Omega_{\mathfrak{X}}) = (L\eta_\mu R\nu_* \mathbf{A}_{\text{inf}})^{\text{pre}}$;
- $A\Omega_{\mathfrak{X}}^{\text{psh}} := L\eta_\mu(\iota(R\nu_* \mathbf{A}_{\text{inf}})) = L\eta_\mu(R\nu_* \mathbf{A}_{\text{inf}})^{\text{pre}}$.

Unwinding definitions, we know that on each affine open $\mathfrak{U} = \operatorname{Spf} R \in \mathfrak{X}_{\text{ét,aff}}$, the global sections of the two presheaves are

$$A\Omega_{\mathfrak{X}}^{\text{pre}}(\mathfrak{U}) = R\Gamma_{\text{ét}}(\operatorname{Spf} R, A\Omega_{\mathfrak{X}}); \qquad A\Omega_{\mathfrak{X}}^{\text{psh}}(\mathfrak{U}) = L\eta_\mu R\Gamma_{\text{proét}}(U, \mathbf{A}_{\text{inf}})$$

where $U = \mathfrak{U}_\eta^{\text{ad}}$ is the adic generic fiber of $\mathfrak{U}$. Both presheaves sheafify to $A\Omega_{\mathfrak{X}}$, so we have a natural map $A\Omega_{\mathfrak{X}}^{\text{psh}} \longrightarrow A\Omega_{\mathfrak{X}}^{\text{pre}}$ by adjunction. To simplify notation we will often write $A\Omega_R := A\Omega_{\mathfrak{X}}^{\text{pre}}(\mathfrak{U})$ and $A\Omega_R^{\text{psh}} := A\Omega_{\mathfrak{X}}^{\text{psh}}(\mathfrak{U})$.

3.3 Local perfectoid covers and group cohomology

We now recall the local theory of A_{inf}-cohomology for small enough $\operatorname{Spf} R$ in $\mathfrak{X}_{\text{ét}}$, mostly developed in [BMS18] and [CK17]. The strategy is to compute both $A\Omega_R^{\text{psh}}$ and $A\Omega_R$ (thus to relate them) using continuous group cohomology. Let us assume that $\operatorname{Spf} R \in \mathfrak{X}_{\text{ét}}$ admits an étale coordinate map $\square : \operatorname{Spf} R \to \operatorname{Spf}(R_1^\square \widehat{\otimes} \cdots \widehat{\otimes} R_s^\square)$, with each

$$R_h^\square = \mathcal{O}_C\langle T_{h,i}, T_{h,j}^{\pm 1}\rangle / (\prod T_{h,i} - \varpi_h)$$

where the variables range over $0 \le i \le r_h$, $r_h + 1 \le j \le d_h$ and $\varpi_h = p^{q_h}$ for some $q_h \in \mathbf{Q}_{>0}$ as in Subsection 3.1. In particular, our choice of $p^{\mathbf{Q}} \in \mathcal{O}_C$ determines a choice of $\varpi_h^{1/p^m} \in \mathcal{O}_C$ for each $m \ge 1$.

3.3.1 *Construction of R_∞* For each $m \ge 1$, we define

$$R_{h,m}^\square = \mathcal{O}_C\langle T_{h,i}^{1/p^m}, T_{h,j}^{\pm 1/p^m}\rangle / (\prod T_{h,i}^{1/p^m} - \varpi_h^{1/p^m}),$$

and then let $R_m^\square := R_{1,m}^\square \widehat{\otimes} \cdots \widehat{\otimes} R_{s,m}^\square$. In the direct limit we get "perfectoid covers"

$$R_{h,\infty}^\square := (\varinjlim R_{h,m}^\square)^\wedge, \qquad R_\infty^\square := R_{1,\infty}^\square \widehat{\otimes} \cdots \widehat{\otimes} R_{s,\infty}^\square.$$

[8] The homotopy category of $\mathcal{D}(\mathfrak{X}_{\text{ét,aff}}, \mathrm{A}_{\text{inf}})$ is identified with the usual triangulated derived category $\mathrm{D}(\mathfrak{X}_{\text{ét,aff}}, \mathrm{A}_{\text{inf}})$. Similarly, the homotopy category of $\mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \mathcal{D}(\mathrm{A}_{\text{inf}}))$ is $\mathrm{D}(\mathfrak{X}_{\text{ét,aff}}^{\text{pre}}, \mathcal{D}(\mathrm{A}_{\text{inf}}))$ where $\mathfrak{X}_{\text{ét,aff}}^{\text{pre}}$ is the site consisting of the same objects as $\mathfrak{X}_{\text{ét,aff}}$ but equipped with the indiscrete (= chaotic) Grothendieck topology.

[9] The presheaf $A\Omega_{\mathfrak{X}}^{\text{pre}}$ is denoted by $A\Omega_{\mathfrak{X}}^{\text{sm}}$ in [BLM18] in the case of good reductions; the presheaf $A\Omega_{\mathfrak{X}}^{\text{psh}}$ agrees with the notation used in [CK17], where it is considered as an object in the homotopy category.

These perfectoid $\mathcal{O}_C$-algebras admit explicit decompositions as $\mathcal{O}_C$-modules by

$$R_\infty^\square = \widehat{\bigoplus}_{a_{h,i}} \mathcal{O}_C T_{1,0}^{a_{1,o}} \cdots T_{1,d_1}^{a_{1,d_1}} \cdots T_{s,0}^{a_{s,0}} \cdots T_{s,d_s}^{a_{s,d_s}} = R^\square \oplus M_\infty^\square,$$

where the completed direct sum ranges over $a_{h,0}, ..., a_{h,d_h} \in \mathbf{Z}[\frac{1}{p}]$ for $1 \le h \le s$, such that each $a_{h,i} \ge 0$ and $\prod_i a_{h,i} = 0$ for each h. The $R^\square$-module decomposition $R_\infty^\square = R^\square \oplus M_\infty^\square$ follows the same notation used in [CK17], where $M_\infty^\square$ denotes the completed direct sum of all $\mathcal{O}_C \cdot T_{1,0}^{a_{1,o}} \cdot \cdots \cdot T_{s,d_s}^{a_{s,d_s}}$ such that some supscripts $a_{h,i}$ or $a_{h,j}$ are non-integral (that belongs to $\mathbf{Z}[\frac{1}{p}] \backslash \mathbf{Z}$). This allows us to obtain

$$R_\infty := (R \otimes_{R^\square} R_\infty^\square)^\wedge = R \oplus M_\infty.$$

Write $\mathbf{Z}_p(1) = (\varprojlim_{m \ge 0} \mu_{p^m}(\mathcal{O}_C))$, and define for each $1 \le h \le s$ the abelian group

$$\Delta_h := \Big\{ (\epsilon_{h,0}, ..., \epsilon_{h,d_h}) \in \mathbf{Z}_p(1)^{\oplus (d_h+1)} \Big| \epsilon_{h,0} + \cdots + \epsilon_{h,d_h} = 0 \Big\}.$$

Finally we set $\Delta := \Delta_1 \oplus \cdots \oplus \Delta_s$, with each Δ_h isomorphic to $\mathbf{Z}_p^{\oplus d_h}$, generated by

- $\gamma_{h,i} = (-1, 0, ..., 0, 1, 0, ..., 0)$ for $i = 1, ..., r_h$, with 1 at the i^{th} entry;
- $\gamma_{h,j} = (0, ..., 0, 1, 0, ..., 0)$ for $j = r_h + 1, ..., d_h$, with 1 at the j^{th} entry.

Under this identification, the element $\gamma_{h,i}$ acts on $R_{h,m}^\square$ by sending

$$T_{h,0}^{1/p^m} \mapsto \zeta_{p^m}^{-1} T_{h,0}^{1/p^m}, \quad T_{h,i}^{1/p^m} \mapsto \zeta_{p^m} T_{h,i}^{1/p^m}, \quad T_{h,k}^{1/p^m} \mapsto T_{h,k}^{1/p^m} \text{ for } k \neq 0, i$$

(resp. $\gamma_{h,j}$ acts by $T_{h,j}^{1/p^m} \mapsto \zeta_{p^m} T_{h,j}^{1/p^m}$, $T_{h,k}^{1/p^m} \mapsto T_{h,k}^{1/p^m}$ for $k \neq j$). The product Δ therefore acts on $R_m^\square$, and continuously on $R_\infty^\square$ (resp. R_∞).

3.3.2 *Construction of $A(R^\square)$* We next observe that

$$(R_{h,\infty}^\square)^\flat = \Big(\varinjlim_m \mathcal{O}_C^\flat [U_{h,i}^{1/p^m}, U_{h,j}^{\pm 1/p^m}] / (\prod U_{h,i}^{1/p^m} - (\varpi_h^\flat)^{1/p^m}) \Big)^\wedge$$

where $U_{h,i}^{1/p^m}$ denotes the elements

$$U_{h,i}^{1/p^m} = (T_{h,i}^{1/p^m}, T_{h,i}^{1/p^{m+1}}, T_{h,i}^{1/p^{m+2}}, ...) \in (R_{h,\infty}^\square)^\flat = \varprojlim_{y \mapsto y^p} R_{h,\infty}^\square$$

for each $i = 0, ..., r_h$. We likewise define elements $U_{h,j}$ for $j = r_h + 1, ..., d_h$.[10][11] From the description of $(R_\infty^\square)^\flat$ and the decomposition $R_\infty^\square = R^\square \oplus M_\infty^\square$, we have

$$\begin{aligned} \mathrm{A_{inf}}(R_\infty^\square) &\cong \widehat{\bigotimes}_h \Big(\varinjlim_m \mathrm{A_{inf}} \big[X_{h,i}^{1/p^m}, X_{h,j}^{\pm 1/p^m} \big] / (\prod X_{h,i}^{1/p^m} - [\varpi_h^\flat]^{1/p^m}) \Big)^\wedge \\ &\cong \widehat{\bigoplus}_{\substack{a_{h,0}, ..., a_{h,d_h} \in \mathbf{Z}[\frac{1}{p}]_{\ge 0} \\ \prod_i a_{h,i} = 0, \, 1 \le h \le s}} \mathrm{A_{inf}} \cdot X_{1,0}^{a_{1,o}} \cdots X_{1,d_1}^{a_{1,d_1}} \cdots X_{s,d_s}^{a_{s,d_s}} \\ &\cong A(R^\square) \oplus N_\infty^\square \end{aligned}$$

[10] We reserve the symbols $X_{h,i}$ for the Teichumuller representatives of $U_{h,i}$ in the Witt vectors.

[11] The tilt $(R_\infty)^\flat$ of R_∞ can be identified with the $p^\flat$-adic completion of R', where $(R_\infty^\square)^\flat \to R'$ is any lift of the étale algebra $R_\infty^\square/p \to R_\infty/p$ to an étale $(R_\infty^\square)^\flat$-algebra.

with $X_{h,i}^{1/p^m} = [U_{h,i}^{1/p^m}]$ (likewise for $X_{h,j}$), and all completions being (p,μ)-adic. We pose to remark that in particular, we have $\widetilde{\theta}(X_{h,i}^{1/p^m}) = T_{h,i}^{1/p^{m+1}}$ under the map $\widetilde{\theta} : \mathrm{A}_{\mathrm{inf}}(R_\infty^\square) \to R_\infty^\square$. Here the subring $A(R^\square)$ is given by

$$\mathrm{A}_{\mathrm{inf}}(R^\square) \cong \widehat{\bigotimes}_h \Big(\mathrm{A}_{\mathrm{inf}}\left[X_{h,i}, X_{h,j}\right] / (\prod X_{h,i} - [\varpi_h^\flat]) \Big)^{\widehat{}}_{(p,\mu)}$$

We have similar decompositions for $\mathrm{A}_{\mathrm{inf}}(R_\infty) = W(R_\infty^\flat) = A(R) \oplus N_\infty$. Moreover, $A(R)$ is (by construction) formally étale over $A(R^\square)$, and also (p,μ)-adically complete. The group $\Delta = \oplus \Delta_h$ acts on $\mathrm{A}_{\mathrm{inf}}(R_\infty^\square)$ (resp. $\mathrm{A}_{\mathrm{inf}}(R_\infty)$) by functoriality, and the action respects the decomposition $A(R^\square) \oplus N_\infty^\square$ (resp. $A(R) \oplus N_\infty$) described above. Moreover, by the same analysis in [BMS18] (or [CK17]), for each i the (continuous) group cohomology $H^i(\Delta, N_\infty^\square)$ (resp. $H^i(\Delta, N_\infty)$) is entirely μ-torsion, hence $L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, N_\infty) = 0$ and consequently $L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, \mathrm{A}_{\mathrm{inf}}(R_\infty))$ only involves the action of Δ on $A(R)$.

3.3.3 $\widetilde{\Omega}_R$ *and* $A\Omega_R$ Let us write $\widehat{\mathcal{O}}^+$ for the pro-étale sheaf $\widehat{\mathcal{O}}^+_{\mathfrak{X}_C}$. By almost purity ([Sch13] Theorem 4.10), the perfectoid pro-étale Δ-cover $\mathfrak{U}_{\eta,\infty}^{\mathrm{ad}} = \mathrm{Spa}(R_\infty[\frac{1}{p}], R_\infty)$ over $U = \mathfrak{U}_\eta^{\mathrm{ad}} := (\mathrm{Spf}\, R)_\eta^{\mathrm{ad}}$ gives rise to an almost quasi-isomorphism $e : R\Gamma_{\mathrm{ct}}(\Delta, R_\infty) \xrightarrow{\sim}_a R\Gamma_{\mathrm{pro\acute{e}t}}(U, \widehat{\mathcal{O}}^+)$.[12] Let $\widetilde{\Omega}_{\mathfrak{X}} := L\eta_{\zeta_p - 1} R\nu_* \widehat{\mathcal{O}}^+$ and define $\widetilde{\Omega}_{\mathfrak{X}}^{\mathrm{psh}}$ and $\widetilde{\Omega}_{\mathfrak{X}}^{\mathrm{pre}}$ as in Subsection 3.2.

Theorem 3.4 ([CK17] 3.9, 3.20, 4.6) *There are quasi-isomorphisms*

$$L\eta_{\zeta_p-1}(R\Gamma_{\mathrm{ct}}(\Delta, R_\infty)) \xrightarrow{\sim} L\eta_{\zeta_p-1}(R\Gamma_{\mathrm{pro\acute{e}t}}(U, \widehat{\mathcal{O}}^+)) \xrightarrow{\sim} R\Gamma(\mathfrak{U}, \widetilde{\Omega}_{\mathfrak{X}})$$

where the the quasi-isomorphism is given by $L\eta_\mu(e)$ and the second comes from the canonical map $\widetilde{\Omega}_{\mathfrak{X}}^{\mathrm{psh}} \to \widetilde{\Omega}_{\mathfrak{X}}^{\mathrm{pre}}$. Analogously, there are quasi-isomorphisms

$$L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A_{\mathrm{inf}}(R_\infty)) \xrightarrow{\sim} L\eta_\mu R\Gamma_{\mathrm{pro\acute{e}t}}(U, \mathbf{A}_{\mathrm{inf}}) \xrightarrow{\sim} R\Gamma(\mathfrak{U}, A\Omega_{\mathfrak{X}}).$$

Thus, the canonical map $A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \xrightarrow{\sim} A\Omega_{\mathfrak{X}}^{\mathrm{pre}}$ is an isomorphism of presheaves in $\mathrm{Psh}(\mathfrak{X}_{\mathrm{\acute{e}t,aff}}, \mathcal{D}(A_{\mathrm{inf}}))$.

Remark 3.5 *For notational simplicity, we denote the asserted quasi-isomorphisms by*

$$\widetilde{\Omega}_R^{\mathrm{gp}} \xrightarrow{\sim} \widetilde{\Omega}_R^{\mathrm{psh}} \xrightarrow{\sim} \widetilde{\Omega}_R, \quad \text{and} \quad A\Omega_R^{\mathrm{gp}} \xrightarrow{\sim} A\Omega_R^{\mathrm{psh}} \xrightarrow{\sim} A\Omega_R.$$

The proof of the claim $\widetilde{\Omega}_R^{\mathrm{gp}} \xrightarrow{\sim} \widetilde{\Omega}_R^{\mathrm{psh}} \xrightarrow{\sim} \widetilde{\Omega}_R$ is similar to [BMS18]. The proof of the isomorphism $A\Omega_R^{\mathrm{gp}} \xrightarrow{\sim} A\Omega_R^{\mathrm{psh}}$ uses Lemma 3.18 of [CK17],[13] which reduces to the computation of $W(\mathfrak{m}^\flat)$-torsion of $H^i_{\mathrm{ct}}(\Delta, A_{\mathrm{inf}}(R_\infty)/\mu)$. In the case when $s = 1$ in the product $R^\square = R_1^\square \widehat{\otimes} \cdots \widehat{\otimes} R_s^\square$, this computation is carried out in 3.15 – 3.19 of loc.cit. but the same argument goes through in our setting. Finally, the proof of the isomorphism $A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \xrightarrow{\sim} A\Omega_{\mathfrak{X}}^{\mathrm{pre}}$ uses the following claim: the natural map $A\Omega_{\mathfrak{X}} \otimes^{\mathbf{L}}_{\widetilde{\theta}} \mathcal{O}_C \xrightarrow{\sim} \widetilde{\Omega}_{\mathfrak{X}}$ is a quasi-isomorphism. The proof of this claim involves commuting $L\eta_\mu$ with certain tensor products, which ultimately relies on

Lemma 3.6 ([Bha17] Proposition 5.16) *Suppose f, g are nonzerodivisors in a ring A, and $K \in D(A)$ satisfies that each $H^i(K \otimes^{\mathbf{L}} A/f)$ is g-torsion free, then*

$$(L\eta_f K) \otimes^{\mathbf{L}} A/g \xrightarrow{\sim} L\eta_{\overline{f}}(K \otimes^{\mathbf{L}} A/g)$$

is a quasi-isomorphism, where $\overline{f} \in A/g$ is the image of f.

[12] this, of course, depends on the choice of coordinates.

[13] This is a variant of Lemma 8.11 and Proposition 9.12 of [BMS18], which states that if a map $f : B \to B'$ in $D(\mathrm{A}_{\mathrm{inf}})$ is an almost quasi-isomorphism in the sense that $W(\mathfrak{m}^\flat)$ kills $\mathrm{Cone}(f)$, and if $H^i(B \otimes^{\mathbf{L}} \mathrm{A}_{\mathrm{inf}}/\mu)$ is $W(\mathfrak{m}^\flat)$-torsion free, then $L\eta_\mu$ turns f into an actual quasi-isomorphism.

3.3.4 *The Hodge-Tate (and de Rham) specialization*

Theorem 3.7 ([CK17], 4.11) *There is a (non-canonical) isomorphism of cdga's*

$$H^*(\widetilde{\Omega}_{\mathfrak{X}}) \xrightarrow{\sim} \omega^*_{\mathfrak{X}/\mathcal{O}_C},$$

where the differential on $H^*(\widetilde{\Omega}_{\mathfrak{X}}) = H^*(A\Omega_{\mathfrak{X}} \otimes^{\mathbf{L}} A_{\inf}/\widetilde{\xi})$ is the Bockstein differential.[14] *Consequently, we have a natural quasi-isomorphism*

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbf{L}}_{A_{\inf},\theta} \mathcal{O}_C \cong \omega^*_{\underline{\mathfrak{X}}/\underline{\mathcal{O}_C}}.$$

Remark 3.8 *The key is to show that $H^1(\widetilde{\Omega}_{\mathfrak{X}}) \cong \omega^1_{\mathfrak{X}/\mathcal{O}_C}\{-1\}$. When $\mathfrak{X}$ has semistable reduction, [CK17] uses formal GAGA and a form of the Grothendieck Existence Theorem in the non-Noetherian setup to reduce to the case of good reduction (Theorem 8.3 of [BMS18]). The requirements in their argument are the following: write $\mathfrak{U} = \operatorname{Spf} R \in \mathfrak{X}_{\text{ét,aff}}$ as the formal p-adic completion of the base change of an affine scheme $\mathfrak{U}_0$ over a discretely valued subring $\mathcal{O} \subset \mathcal{O}_C$, then we need that (i) the p-adic completion morphism $\mathfrak{U} \to \mathfrak{U}_0$ of log ringed* étale *sites is strict; and (ii) the complement $\mathfrak{U}_0 \backslash \mathfrak{U}_0^{\text{sm}}$ of the smooth locus $\mathfrak{U}_0^{\text{sm}} \subset \mathfrak{U}_0$ over $\operatorname{Spec} \mathcal{O}$ has codimension at most 2. Both are satisfied for $\mathfrak{X}$ in our setup.*

3.4 A basic example

We record a simple example of the Koszul complex that computes $L\eta_\mu R\Gamma_{\text{ct}}(\Delta, \mathrm{A}_{\inf}(R_\infty))$, which provides a summary for the previous discussion and will be helpful later.

Example 3.9 *In this example let $\varpi = p^q$ for some $q \in \mathbf{Q}_{>0}$ and*

$$R = R^\square = \mathcal{O}_C\langle T_0, T_1\rangle/(T_0T_1 - \varpi).$$

Therefore $A(R) = A(R^\square) = A_{\inf}\langle X_0, X_1\rangle/(X_0X_1 - [\varpi^\flat])$. In this setup, $\Delta \cong \mathbf{Z}_p(1)$ has a generator γ which acts on X_0 by $[\epsilon]^{-1}$ and on X_1 by $[\epsilon]$. After applying $L\eta_\mu$, the group cohomology $L\eta_\mu R\Gamma_{\text{ct}}(\Delta, A_{\inf}(R))$ is computed by the complex

$$\eta_\mu\Big(A(R) \xrightarrow{\gamma-1} A(R)\cdot e_1\Big) = A(R) \xrightarrow{\gamma-1} ([\epsilon]-1)\cdot A(R)\cdot e_1.$$

This is in turn quasi-isomorphic to the free complex

$$A(R) = A_{\inf}\langle X_0, X_1\rangle/(X_0X_1 - [\varpi^\flat]) \xrightarrow{\frac{\gamma-1}{[\epsilon]-1}} A_{\inf}\langle X_0, X_1\rangle/(X_0X_1 - [\varpi^\flat])\cdot e_1,$$

where e_1 is our symbol for a dummy basis in degree 1 (this is denoted by dlogX_1 *in [BMS18]). The differential in the last complex is explicitly given by*

(1) $X_0^m \mapsto -([\epsilon]^{-m} + \cdots + [\epsilon]^{-1})X_0^m \cdot e_1$,

(2) $X_1^m \mapsto (1 + [\epsilon] + \cdots + [\epsilon]^{m-1})X_1^m \cdot e_1$.

• ***base change to $\mathcal{O}_C$.*** *Note that the map $\widetilde{\theta} : A_{\inf}(R_\infty) \to R_\infty$ sends $X_i \mapsto T_i^{1/p}$, hence the base change $\big(L\eta_\mu R\Gamma_{\text{ct}}(\Delta, A_{\inf}(R))\big) \otimes^{\mathbf{L}}_{\widetilde{\theta}} \mathcal{O}_C$ is given by*

$$\mathcal{O}_C\langle T_0^{1/p}, T_1^{1/p}\rangle/(T_0^{1/p}T_1^{1/p} - \varpi^{1/p}) \longrightarrow \mathcal{O}_C\langle T_0^{1/p}, T_1^{1/p}\rangle/(T_0^{1/p}T_1^{1/p} - \varpi^{1/p})\cdot e_1$$

with differentials

[14] To get a canonical isomorphism, one needs $\omega^i_{\mathfrak{X}/\mathcal{O}}\{-i\}$ instead on the right hand side, where $\{-1\}$ denotes the Breuil-Kisin twist.

(1) $T_0^{m/p} \mapsto -(\zeta_p^{-m} + \cdots + \zeta_p^{-1}) T_0 \cdot e_1$,

(2) $T_1^{m/p} \mapsto (1 + \cdots + \zeta_p^{m-1}) T_1 \cdot e_1$.

This complex is quasi-isomorphic to

$$\mathcal{O}_C\langle T_0, T_1\rangle/(T_0T_1 - \varpi) \xrightarrow{0} \mathcal{O}_C\langle T_0, T_1\rangle/(T_0T_1 - \varpi) \cdot e_1.$$

• ***base change to*** $W(k)$***.*** *Similarly, since* $\widetilde{\vartheta}([\varpi^\flat]) = 0$ *the twisted crystalline base change* $\big(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A_{\mathrm{inf}}(R))\big) \otimes^{\mathbf{L}} W(k)$ *is computed by*

$$W(k)\langle T_0^{1/p}, T_1^{1/p}\rangle/(T_0^{1/p}T_1^{1/p}) \longrightarrow W(k)\langle T_0^{1/p}, T_1^{1/p}\rangle/(T_0^{1/p}T_1^{1/p}) \cdot e_1$$

with differentials

(1) $T_0^{m/p} \mapsto -mT_0^{m/p} \cdot e_1$,

(2) $T_1^{m/p} \mapsto mT_1^{m/p} \cdot e_1$.

Note that, since each $T_i^{p^{r-1}}$ *maps to* $\pm p^r T_i^{p^{r-1}} \cdot e_1$*, we have*

$$T_i^{p^{r-1}} \in H^0\Big(L\eta_\mu R\Gamma_{\mathrm{ct}}\big(\Delta, A_{\mathrm{inf}}(R)\big) \otimes^{\mathbf{L}} W(k)/p^r\Big)$$

We have presented the simplest possible case to simplify notations, though the example above easily generalizes to $R^\square$ with more variables and to the case where $s > 1$. For example, if $R = R^\square = \mathcal{O}_C\langle T_0, ..., T_r, T_{r+1}^{\pm 1}, ..., T_d^{\pm 1}\rangle/(T_0 \cdots T_r - \varpi)$, $L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, \mathrm{A}_{\mathrm{inf}}(R))$ is computed by the Koszul complex

$$\mathbf{K}_{A(R)}\Big(\frac{\gamma_1 - 1}{[\epsilon] - 1}, ..., \frac{\gamma_d - 1}{[\epsilon] - 1}\Big) = \Bigg(A(R) \xrightarrow{\frac{\gamma_1-1}{[\epsilon]-1}, ..., \frac{\gamma_d-1}{[\epsilon]-1}} A(R) \cdot e_1 \oplus \cdots \oplus e_d \longrightarrow \cdots \longrightarrow A(R) \cdot \bigoplus_{1 \le i_1 < \cdots < i_k \le d} e_{i_1} \wedge \cdots \wedge e_{i_k} \longrightarrow \cdots\Bigg)$$

4. The log crystalline specialization

We continue to let $\mathfrak{X}$ be a p-adic formal scheme of generalized semistable reduction over $\mathcal{O}_C$, equipped with the divisorial log structure as in Subsection 3.1. By topological invariance of étale sites $\mathfrak{X}_{\text{ét}} \cong \mathfrak{X}_{k,\text{ét}}$, we view $\mathcal{W}\omega^*_{\underline{\mathfrak{X}_k}/\underline{k}}$ as a sheaf on $\mathfrak{X}_{\text{ét,aff}}$. In this section we prove the following more precise version of Theorem 1.

Theorem 4.1 *There exists a canonical,* φ*-compatible quasi-isomorphism*

$$\mathcal{W}\omega^*_{\underline{\mathfrak{X}_k}/\underline{k}} \xrightarrow{\sim} A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k)$$

in $\mathcal{D}(\mathfrak{X}_{\text{ét}}, W(k))$*, which induces a quasi-isomorphism (see Remark 3.3)*

$$R\Gamma_{\text{log-cris}}(\underline{\mathfrak{X}_k}/W(\underline{k})) \cong R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \widehat{\otimes}^{\mathbf{L}}_{A_{\mathrm{inf}}} W(k).$$

The idea of proof is the following. We first restrict to the smaller site $\mathfrak{X}_{\text{ét,aff}}$ and use Theorem 2.8 to equip the right hand side $A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k)$ with a Dieudonné algebra structure. We then appeal to the universal property of saturated log de Rham–Witt complexes to obtain the desired morphism, which is easily checked to be an isomorphism using the Hodge–Tate comparison (cf. Theorem 3.7) and Lemma 2.5. However, as mentioned in the introduction, aside from the complications caused by allowing the scheme to be more general, we face the additional difficulty of equipping the right hand side with suitable log structures. In order to remedy this, we will work locally on small enough charts and produce the log structures using semistable coordinates, and then show that the morphism on the underlying complexes is canonical (independent of coordinates). We carry out this procedure in detail in the remaining of the article.

Recall from Subsection 3.2 that $A\Omega_{\mathfrak{X}}^{\mathrm{pre}}$ is a commutative algebra object in $\mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \mathcal{D}(\mathrm{A}_{\inf}))$. Taking its (completed) base change along $\widetilde{\vartheta} : \mathrm{A}_{\inf} \to W(k)$ we get

$$A\Omega_{\mathfrak{X},W}^{\mathrm{pre}} := A\Omega_{\mathfrak{X}}^{\mathrm{pre}} \widehat{\otimes}^{\mathbf{L}} W(k) \in \mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \widehat{\mathcal{D}}(\mathbf{Z}_p)),$$

which is a commutative algebra object taking the value $A\Omega_R \widehat{\otimes}^{\mathbf{L}} W(k)$ on $\mathfrak{U} = \operatorname{Spf} R \in \mathfrak{X}_{\text{ét,aff}}$. There is another presheaf

$$(A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k))^{\mathrm{pre}} = \iota\big(A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k)\big),$$

where $\iota : \widehat{\mathcal{D}}(\mathfrak{X}_{\text{ét,aff}}) \to \mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \widehat{\mathcal{D}}(\mathbf{Z}_p))$ is the fully faithful embedding as described in Subsection 3.2, which takes the value $R\Gamma(\mathfrak{U}, A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k))$ on $\mathfrak{U} = \operatorname{Spf} R$. Both presheaves sheafify to $A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k)$ after taking the (derived) p-adic completion. In fact, we claim that the following stronger assertion holds:

Lemma 4.2 *The natural map*

$$A\Omega_{\mathfrak{X},W}^{\mathrm{pre}} = A\Omega_{\mathfrak{X}}^{\mathrm{pre}} \widehat{\otimes}^{\mathbf{L}} W(k) \xrightarrow{\sim} \big(A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k)\big)^{\mathrm{pre}}$$

is an isomorphism.

Proof. On each $\mathfrak{U} = \operatorname{Spf} R \in \mathfrak{X}_{\text{ét,aff}}$, we have a canonical map

$$\begin{array}{ccc} R\Gamma(\mathfrak{U}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \widehat{\otimes}^{\mathbf{L}} W(k) & \longrightarrow & R\Gamma(\mathfrak{U}_{\text{ét}}, A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k)) \\ & & \wr\| \\ & & \operatorname{Rlim} R\Gamma(\mathfrak{U}_{\text{ét}}, A\Omega_{\mathfrak{X}} \otimes^{\mathbf{L}} W_n(k)) \end{array}$$

To prove the lemma, it suffices to show that for each n, we have a quasi-isomorphism

$$R\Gamma(\mathfrak{U}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbf{L}} W_n(k) \xrightarrow{\sim} R\Gamma(\mathfrak{U}_{\text{ét}}, A\Omega_{\mathfrak{X}} \otimes^{\mathbf{L}} W_n(k)).$$

By Theorem 4.9 of [BMS18] we know that each $W_n(k)$ is a filtered colimit of perfect $\mathrm{A}_{\inf}$-modules, since higher direct images of qcqs morphisms commute with filtered colimit (see Tag 07U6 or more generally Tag 0739 of [dJea]), it suffices to show that $A\Omega_R \otimes^{\mathbf{L}} B \xrightarrow{\sim} R\Gamma(\mathfrak{U}_{\text{ét}}, A\Omega_{\mathfrak{X}} \otimes^{\mathbf{L}} B)$ for a perfect $\mathrm{A}_{\inf}$-module B. Therefore we are reduced to the case when B is finite projective, in other words, a direct summand of a finite free $\mathrm{A}_{\inf}$-module, the lemma thus follows. ☐

By the same argument, in order to prove Theorem 4.1 it is enough to prove the first statement, for which it suffices to construct a quasi-isomorphism $\mathcal{W}\omega^{*,\mathrm{pre}}_{\underline{\mathfrak{X}_k}/\underline{k}} \xrightarrow{\sim} A\Omega_{\mathfrak{X},W}^{\mathrm{pre}}$. In the subsequent subsections, we enhance the structure on the right hand side to provide such a map.

4.1 The Dieudonné algebra structure

In this subsection we prove

Proposition 4.3 *There exists a canonical isomorphism*

$$A\Omega_{\mathfrak{X},W}^{\mathrm{pre}} \xrightarrow{\sim} L\eta_p A\Omega_{\mathfrak{X},W}^{\mathrm{pre}},$$

which puts $A\Omega^{\mathrm{pre}}_{\mathfrak{X},W} \in \mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \widehat{\mathcal{D}}(\mathbf{Z}_p))^{L\eta_p}$. *Denote by* $A^{\mathrm{pre}}_{\mathfrak{X}}$ *the image of* $A\Omega^{\mathrm{pre}}_{\mathfrak{X},W}$ *in the following sequence of identifications*

$$\mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \widehat{\mathcal{D}}(\mathbf{Z}_p))^{L\eta_p} \cong \mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \widehat{\mathcal{D}}(\mathbf{Z}_p)^{L\eta_p}) \cong \mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \mathrm{DC}_{\mathfrak{str}}),$$

then $A^{\mathrm{pre}}_{\mathfrak{X}}$ *takes values in the category of strict Dieudonné algebras.*

For this we need a sequence of lemmas, starting with

Lemma 4.4 *There exists a canonical isomorphism* $\left(\mathrm{colim}\, A_{\inf}/\varphi^{-r}(\mu)\right)^{\widehat{}}_p \xrightarrow{\sim} W(k)$ *from the (classical) p-adic completion of the filtered colimit of* $A_{\inf}/\varphi^{-r}(\mu)$ *to* $W(k)$.

Proof. The map $\widetilde{\vartheta}$ sends $\mu = [\epsilon]-1$ to $0 \in W(k)$, therefore we have the canonical map as above. As both sides are p-complete and $W(k)$ is p-torsion free, we can check the isomorphism by reducing mod p. Note that taking filtered colimit is exact, so $\mathrm{colim}\, \mathrm{A}_{\inf}/\varphi^{-r}(\mu) \cong \mathrm{A}_{\inf}/\bigcup \varphi^{-r}(\mu)$, therefore it suffices to show that the canonical map (induced by $\widetilde{\vartheta}$)

$$\mathrm{A}_{\inf}/\Big(\bigcup_r \varphi^{-r}(\mu), p\Big) \cong \mathcal{O}^\flat_C/\bigcup_r \left([\epsilon]^{\frac{1}{p^r}}-1\right) \longrightarrow k$$

is an isomorphism, which follows by considering the $p^\flat$-adic valuation of each $[\epsilon]^{1/p^r}-1 \in \mathcal{O}^\flat_C$. ☐

Also recall that the décalage operators behaves well under completions.

Lemma 4.5 ([BMS18] 6.20) *For any object* $C \in D(W(k))$, *the natural maps*

$$\widehat{L\eta_p C} \xrightarrow{\sim} L\eta_p \widehat{C} \xrightarrow{\sim} \mathrm{R}\varprojlim L\eta_p(C \otimes^{\mathbf{L}} W_n(k))$$

are quasi-isomorphisms, where all completions are derived p-adic completions.

Corollary 4.6 *For any* $\mathrm{Spf}\, R \in \mathfrak{X}_{\text{ét,aff}}$, *there is a natural Frobenius compatible quasi-isomorphism:*

$$\big(L\eta_{\widetilde{\xi}} A\Omega_R\big)\widehat{\otimes}^{\mathbf{L}} W(k) \cong L\eta_p\big(A\Omega_R \widehat{\otimes}^{\mathbf{L}} W(k)\big).$$

Therefore, we get an isomorphism of presheaves (valued in $\widehat{\mathcal{D}}(\mathbf{Z}_p)$*):*

$$\big(L\eta_{\widetilde{\xi}} A\Omega^{\mathrm{pre}}_{\mathfrak{X}}\big)\widehat{\otimes}^{\mathbf{L}} W(k) \xrightarrow{\sim} L\eta_p\big(A\Omega^{\mathrm{pre}}_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k)\big) = L\eta_p A\Omega^{\mathrm{pre}}_{\mathfrak{X},W}.$$

Proof. The proof is similar to that of 10.3.10 of [BLM18]. It suffices to show that

$$\Big(L\eta_{\widetilde{\xi}} A\Omega_R\Big) \otimes^{\mathbf{L}} \mathrm{A}_{\inf}/\varphi^{-r}(\mu) \xrightarrow{\sim} L\eta_p\Big(A\Omega_R \otimes^{\mathbf{L}} \mathrm{A}_{\inf}/\varphi^{-r}(\mu)\Big) \qquad (\star\star)$$

is a quasi-isomorphism. For this it is enough to show that $H^i(A\Omega_R \otimes^{\mathbf{L}} \mathrm{A}_{\inf}/\widetilde{\xi})$ is flat over $\mathcal{O}_C$ (hence $\varphi^{-r}(\mu)$-torsion free) by Lemma 3.6. Now apply Theorem 3.7, we have $H^i(A\Omega_R \otimes^{\mathbf{L}} \mathrm{A}_{\inf}/\widetilde{\xi}) \cong \omega^i_{\underline{R}/\underline{\mathcal{O}_C}}$, which is flat over $\mathcal{O}_C$, since by assumption $\underline{R}$ is essentially log smooth over $\underline{\mathcal{O}_C}$ as discussed in Remark 3.2. For completeness we finish the argument by the following sequence of quasi-isomorphisms:

$$\begin{aligned}
\big(L\eta_{\widetilde{\xi}} A\Omega_R\big)\widehat{\otimes}^{\mathbf{L}} W(k) &\cong \left[\big(L\eta_{\widetilde{\xi}} A\Omega_R\big) \otimes^{\mathbf{L}} \varinjlim \mathrm{A}_{\inf}/\varphi^{-r}(\mu)\right]^{\widehat{}}_p && \text{by Lemma 4.4}\\
&\cong \left[\varinjlim \Big(L\eta_{\widetilde{\xi}} A\Omega_R \otimes^{\mathbf{L}} \mathrm{A}_{\inf}/\varphi^{-r}(\mu)\Big)\right]^{\widehat{}}_p && (\star)\\
&\cong \left[\varinjlim L\eta_p\Big(A\Omega_R \otimes^{\mathbf{L}} \mathrm{A}_{\inf}/\varphi^{-r}(\mu)\Big)\right]^{\widehat{}}_p && \text{by } (\star\star) \text{ above}\\
&\cong \left[L\eta_p\Big(A\Omega_R \otimes^{\mathbf{L}} \big(\varinjlim \mathrm{A}_{\inf}/\varphi^{-r}(\mu)\big)\Big)\right]^{\widehat{}}_p && (\star)\\
&\cong L\eta_p\Big(A\Omega_R \widehat{\otimes}^{\mathbf{L}} W(k)\Big) && \text{by Lemma 4.5}
\end{aligned}$$

where $(\star)$ uses the fact that both $L\eta$ and tensor commute with filtered colimit. $\square$

This proves the first part of Proposition 4.3:

Corollary 4.7 *There is a canonical isomorphism $A\Omega^{\mathrm{pre}}_{\mathfrak{X},W} \xrightarrow{\sim} \varphi_* L\eta_p A\Omega^{\mathrm{pre}}_{\mathfrak{X},W}$ of commutative algebra objects in* $\mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \widehat{\mathcal{D}}(\mathbf{Z}_p))$.

Proof. Recall from Subsection 3.2 that on each $\mathrm{Spf}\, R \in \mathfrak{X}_{\text{ét,aff}}$ we have the functorial quasi-isomorphism induced from the Frobenius automorphism on $\mathbf{A}_{\inf}$:

$$L\eta_\mu R\Gamma(U, \mathbf{A}_{\inf}) \xrightarrow{\sim} \varphi_* L\eta_{\varphi(\mu)} R\Gamma(U, \mathbf{A}_{\inf}) \cong \varphi_* L\eta_{\widetilde{\xi}} L\eta_\mu R\Gamma(U, \mathbf{A}_{\inf}),$$

which gives an isomorphism of commutative algebra objects $A\Omega^{\mathrm{psh}}_{\mathfrak{X}} \xrightarrow{\sim} \varphi_* L\eta_{\widetilde{\xi}} A\Omega^{\mathrm{psh}}_{\mathfrak{X}}$. By the last assertion in Theorem 3.4, this transports to an isomorphism [15]

$$A\Omega^{\mathrm{pre}}_{\mathfrak{X}} \xrightarrow{\sim} \varphi_* L\eta_{\widetilde{\xi}} A\Omega^{\mathrm{pre}}_{\mathfrak{X}}.$$

The corollary then follows by taking base change to $W(k)$ and applying Corollary 4.6, as in the following diagram

$$\begin{array}{ccc} A\Omega^{\mathrm{pre}}_{\mathfrak{X},W} = A\Omega^{\mathrm{pre}}_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k) & \xrightarrow{\sim} & (\varphi_* L\eta_{\widetilde{\xi}} A\Omega^{\mathrm{pre}}_{\mathfrak{X}}) \widehat{\otimes}^{\mathbf{L}} W(k) \\ & \searrow^{\psi} & \downarrow \wr \\ & & \varphi_* L\eta_p A\Omega^{\mathrm{pre}}_{\mathfrak{X},W} = \varphi_* L\eta_p \big(A\Omega^{\mathrm{pre}}_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k)\big). \end{array}$$

Note that the vertical isomorphism preserves commutative algebra objects, since $L\eta_p$ is lax symmetric monoidal by [BMS18] 6.7. $\square$

Ignoring the $W(k)$-linear structure, the isomorphism $A\Omega^{\mathrm{pre}}_{\mathfrak{X},W} \xrightarrow{\sim} L\eta_p A\Omega^{\mathrm{pre}}_{\mathfrak{X},W}$ puts the presheaf $A\Omega^{\mathrm{pre}}_{\mathfrak{X},W}$ in $\mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \widehat{\mathcal{D}}(\mathbf{Z}_p)^{L\eta_p}) \cong \mathrm{Psh}(\mathfrak{X}_{\text{ét,aff}}, \mathrm{DC}_{\mathfrak{str}})$, where the isomorphism follows from Theorem 2.8. We denote the image of $A\Omega^{\mathrm{pre}}_{\mathfrak{X},W}$ in the latter category by $A^{\mathrm{pre}}_{\mathfrak{X}}$ to emphasize that it takes value in strict Dieudonné complexes. Note that $A\Omega^{\mathrm{pre}}_{\mathfrak{X},W}$ (hence $A^{\mathrm{pre}}_{\mathfrak{X}}$) takes value in the subcategory of commutative algebra objects in $\mathcal{D}(\mathbf{Z}_p)^{L\eta_p}$ (resp. in $\mathrm{DC}_{\mathfrak{str}}$). Next we prove the remaining of Proposition 4.3.

Lemma 4.8 *For each $\mathrm{Spf}\, R \in \mathfrak{X}_{\text{ét,aff}}$, the commutative algebra object in strict Dieudonné complexes $A^*_R := R\Gamma_{\text{ét}}(\mathrm{Spf}\, R, A^{\mathrm{pre}}_{\mathfrak{X}})$ is a strict Dieudonné algebra.*

Proof. By Remark 3.1.5 in [BLM18], we need to show that A^*_R is concentrated in degree ≥ 0 and that F is the p-power Frobenius on $W_1(A^0_R) = A^0_R / V(A^0_R)$. By Remark 2.9 we have

$$A^*_R = \varprojlim_{R_r} H^*(A\Omega_R \otimes^{\mathbf{L}} W(k)/p^r).$$

For the first assertion, it suffices to show that

$$A\Omega_R \otimes^{\mathbf{L}} W(k)/p = \widetilde{\Omega}_R \otimes^{\mathbf{L}}_{\mathcal{O}_C} \mathcal{O}_C/\mathfrak{m}$$

is coconnective (therefore by induction each $A\Omega_R \otimes^{\mathbf{L}} W(k)/p^r$ is coconnective). This is follows from Theorem 3.7 (note that since $H^0(\widetilde{\Omega}_R) \cong R$ is $\mathcal{O}_C$-flat, we have $H^0(\widetilde{\Omega}_R) \otimes k = H^0(\widetilde{\Omega}_R \otimes^{\mathbf{L}} k)$).

[15] Note that this crucially uses the definition of $\mathfrak{X}_{\text{ét,aff}}$, which only consists of small enough affine opens.

Now it remains to show that the morphism induced by F on $H^0(\widetilde{\Omega}_R \otimes^{\mathbf{L}}_{\mathcal{O}_C} k)$ is the p-power Frobenius. The proof given here mimics the proof of 10.3.15 of [BLM18] but differs slightly towards the end. We first choose a semistale coordinate $\square : R^{\square} \to R$ and (hence) a formally étale morphism of perfectoid algebras $R^{\square}_{\infty} \to R_{\infty}$ as in Subsection 3.2. From Proposition 3.4 we obtain a φ-equivariant map $A\Omega_R \to \mathrm{A}_{\mathrm{inf}}(R_\infty)$, since $A\Omega_R \cong L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, \mathrm{A}_{\mathrm{inf}}(R_\infty))$. Note that since R_∞ is perfectoid we have a φ-equivariant identification $\mathrm{A}_{\mathrm{inf}}(R_\infty) \otimes^{\mathbf{L}} W(k) = W(R_{\infty,k})$, where φ equals to the Witt-vector Frobenius on $W(R_{\infty,k})$, by Lemma 3.13 of [BMS18]. Therefore, after base change to $k = \mathcal{O}_C/\mathfrak{m}$ we get a morphism $\widetilde{\Omega}_R \otimes^{\mathbf{L}}_{\mathcal{O}_C} k \to R_{\infty,k}$. Now taking H^0 we get the following φ-equivariant morphism

$$R_k = H^0(\widetilde{\Omega}_R \otimes^{\mathbf{L}}_{\mathcal{O}_C} k) \longrightarrow R_{\infty,k}$$

where φ agrees with F on the left and identities with the usual Frobenius on the right. It suffices to show that the map $R_k \to R_{\infty,k}$ is injective. For this note that the following diagram

$$\begin{array}{ccc} R_k^{\square} & \longrightarrow & R^{\square}_{\infty,k} \\ \downarrow & & \downarrow \\ R_k & \longrightarrow & R_{\infty,k} \end{array}$$

is cocartesian. Since $R_k^{\square} \to R_k$ is étale, and $R_k^{\square} \to R^{\square}_{\infty,k}$ is injective, the proposition therefore follows. Note that unlike the situation in [BLM18], here $R/p \to R_\infty/p$ is not flat (at the singular point of the special fiber). $\square$

4.2 A log Dieudonné algebra structure (after fixing coordinates)

Now we have two presheaves $\mathcal{W}\omega^{*,\mathrm{pre}}_{\underline{\mathfrak{X}_k}/\underline{k}}$ and $A^{\mathrm{pre}}_{\mathfrak{X}}$ on $\mathfrak{X}_{\text{ét},\mathrm{aff}}$, both valued in the category $\mathrm{DA}_{\mathfrak{str}}$ of strict Dieudonné algebras. In order to obtain a functorial map between them, we want to enhance the right hand side with log structures. As mentioned in the beginning of this section, we will do this locally on small enough charts. For this subsection we let $\mathrm{Spf}\, R \in \mathfrak{X}_{\text{ét},\mathrm{aff}}$ and fix a choice $\square : R^{\square} \longrightarrow R$ of a semistable coordinate as in Subsection 3.1. Our goal is to use this coordinate and the local analysis in Subsection 3.2 and 3.4, to equip A^*_R with a log structure and log derivation, hence to upgrade A^*_R to a log Dieudonné algebra.

4.2.1 *The log structure* Recall from Subsection 3.1 that we have the log algebra $(R, M^{\square})$ over $(\mathcal{O}_C, N^{\square})$ where $N^{\square} = \mathbf{N}^s$. Let $A(R^{\square}) \to A(R)$ be the (formally) étale morphism given by $\square : R^{\square} \to R$ as in Subsection 3.3, this determines the vertical map in the following diagram

$$\begin{array}{ccc} H^0(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^{\square})) \otimes^{\mathbf{L}} W_r) & \overset{3.9}{=\!=} & H^0\Big(\Big(\bigotimes_{1\le h\le s} \mathbf{K}_{A(R^{\square})}(\frac{\gamma_{h,i}-1}{[\epsilon]-1})\Big) \otimes W_r\Big) \\ \downarrow & \searrow^{\kappa_r} & \\ H^0(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R)) \otimes^{\mathbf{L}} W_r) & \longrightarrow & H^0(A\Omega_R \otimes^{\mathbf{L}} W_r) \end{array}$$

where we have abbreviated notation by using W_r for $W_r(k)$ and using $\mathbf{K}_{A(R^{\square})}(\frac{\gamma_{h,i}-1}{[\epsilon]-1})$ to denote the Koszul complex

$$\mathbf{K}_{A(R^{\square})}(\frac{\gamma_{h,1}-1}{[\epsilon]-1}, ..., \frac{\gamma_{h,d_h}-1}{[\epsilon]-1}).$$

The natural composition is denoted by κ_r.

Definition 4.9 *Let $\mathfrak{b}_{h,i} = (...,0,1,0,...) \in M^{\square} = \oplus_h \mathbf{N}^{r_h+1}$ be the element with 1 at the (h,i)-entry and 0 otherwise. For each $r \geq 1$, we define the monoid maps*

$$\alpha_r = \alpha_r^{\square} : M^{\square} \to H^0(A\Omega_R \otimes^{\mathbf{L}} W_r)$$

by sending each $\mathfrak{b}_{\mathfrak{h},\mathrm{i}}$ (where $1 \leq h \leq s, 0 \leq i \leq r_h$) to $\kappa_r(T_{h,i}^{p^{r-1}})$, where κ_r is the composition as defined above. Note that $T_{h,i}^{p^{r-1}}$ lives in $H^0(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^{\square})) \otimes^{\mathbf{L}} W_r)$ by the discussion in Example 3.9.

Remark 4.10 *Since $\widetilde{\xi} \overset{\widetilde{\vartheta}}{\mapsto} p \in W(k)$, the map $\widetilde{\vartheta}$ induces $\widetilde{\vartheta} : A_{\inf}/\widetilde{\xi}^r \to W(k)/p^r$. This gives rise to the following natural commutative diagram*

$$\begin{array}{ccc} H^0(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^{\square})) \otimes^{\mathbf{L}}_{\widetilde{\theta}} A_{\inf}/\widetilde{\xi}^r) & \longrightarrow & H^0(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^{\square})) \otimes^{\mathbf{L}} W_r) \\ \downarrow & & \downarrow \\ H^0(A\Omega_R \otimes^{\mathbf{L}}_{\widetilde{\theta}} A_{\inf}/\widetilde{\xi}^r) & \longrightarrow & H^0(A\Omega_R \otimes^{\mathbf{L}} W_r) \end{array}$$

The map α_r can be equivalently described as follows: $\alpha_r(\mathfrak{b}_{\mathfrak{h},\mathrm{i}})$ is the image of

$$T_{h,i}^{p^{r-1}} = \widetilde{\theta}(X_{h,i}^{p^r}) \in H^0(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^{\square})) \otimes^{\mathbf{L}}_{\widetilde{\theta}} A_{\inf}/\widetilde{\xi}^r)$$

in $H^0(A\Omega_R \otimes^{\mathbf{L}} W_r)$, where $\widetilde{\theta} : A_{\inf}(R_\infty^{\square,\flat}) \to R_\infty^{\square}$ is defined similarly as before. [16]

From the next lemma we use the same notation from Remark 2.9, it is convenient to also recall the diagram

$$\begin{array}{ccc} H^i(A\Omega_R \otimes^{\mathbf{L}} W_r) & \xrightarrow{\mu_r} & H^i(L\eta_{\widetilde{\xi}} A\Omega_R \otimes^{\mathbf{L}} W_{r-1}) \\ & \searrow^{R_r} & \downarrow \psi^{-1} \\ & & H^i(A\Omega_R \otimes^{\mathbf{L}} W_{r-1}) \end{array}$$

that defines the restriction map R_r, where μ_r sends $[y]$ to $[p^i y]$. Note that in the top right entry we have used the identification $L\eta_p(A\Omega_R \widehat{\otimes}^{\mathbf{L}} W) \cong (L\eta_\xi A\Omega_R)\widehat{\otimes}^{\mathbf{L}} W$ as proven in Corollary 4.6.

Lemma 4.11 *The maps $\{\alpha_r\}$ is compatible with the restriction maps R_r, and satisfies $F(\alpha_r(m)) = \alpha_r(m)^p$. In other words the following diagrams commute*

$$\begin{array}{ccc} M^{\square} & \xrightarrow{\alpha_r} & H^0(A\Omega_R \otimes^{\mathbf{L}} W_r) \\ & \searrow^{\alpha_{r-1}} & \downarrow R_r \\ & & H^0(A\Omega_R \otimes^{\mathbf{L}} W_{r-1}) \end{array} \qquad \begin{array}{ccc} M^{\square} & \xrightarrow{\alpha_r} & H^0(A\Omega_R \otimes^{\mathbf{L}} W_r) \\ \downarrow \times p & & \downarrow F \\ M^{\square} & \xrightarrow{\alpha_{r-1}} & H^0(A\Omega_R \otimes^{\mathbf{L}} W_{r-1}) \end{array}$$

Consequently, we obtain a monoid morphism

$$\alpha = \alpha^{\square} : M^{\square} \to A_R^0 = \varprojlim_{R_r} H^0(A\Omega_R \otimes^{\mathbf{L}} W_r)$$

which satisfies $F(\alpha(m)) = \alpha(m)^p$ for all $m \in M^{\square}$.

---

[16]From Example 3.9, it is clear that $X_{h,i}^{p^r}$ is indeed an element in $H^0(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^{\square}))\widehat{\otimes}^{\mathbf{L}}_{\widetilde{\theta}} \mathrm{A}_{\inf}/\widetilde{\xi}^r)$.

Proof. The commutativity of the second diagram follows directly from the definitions. For the commutativity of the first diagram, we reduce to the case of $R^\square$ by functoriality. To simply notation, write

$$A\Omega^{\mathrm{gp}}_{R^\square} := L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, \mathrm{A_{inf}}(R^\square_\infty))$$

By Proposition 3.4 we know that $A\Omega^{\mathrm{gp}}_{R^\square} \xrightarrow{\sim} A\Omega_{R^\square}$ is a quasi-isomorphism, and by the same proof of 4.6, we have a quasi-isomorphism

$$\psi : A\Omega^{\mathrm{gp}}_{R^\square}\widehat{\otimes}^{\mathbf{L}} W \longrightarrow L\eta_p\big(A\Omega^{\mathrm{gp}}_{R^\square}\widehat{\otimes}^{\mathbf{L}} W\big).$$

Now consider the following commutative diagram

$$\begin{array}{ccccc}
H^0(A\Omega^{\mathrm{gp}}_{R^\square}\widehat{\otimes}^{\mathbf{L}} W_r) & \xrightarrow{\mu_r} & H^0(L\eta_p\Big(A\Omega^{\mathrm{gp}}_{R^\square}\widehat{\otimes}^{\mathbf{L}} W\Big)/p^{r-1}) & \xrightarrow{\psi^{-1}} & H^0(A\Omega^{\mathrm{gp}}_{R^\square}\widehat{\otimes}^{\mathbf{L}} W_{r-1}) \\
\downarrow & & \downarrow & & \downarrow \\
H^0(A\Omega_{R^\square}\widehat{\otimes}^{\mathbf{L}} W_r) & \xrightarrow{\mu_r} & H^0(L\eta_p\Big(A\Omega_{R^\square}\widehat{\otimes}^{\mathbf{L}} W\Big)/p^{r-1}) & \xrightarrow{\psi^{-1}} & H^0(A\Omega_{R^\square}\widehat{\otimes}^{\mathbf{L}} W_{r-1})
\end{array}$$

The top right arrow ψ^{-1} comes from the quasi-isomorphism ψ described above, and can be represented by

$$\phi_F : \big(\otimes_h \mathbf{K}_{\mathrm{A_{inf}}(R^\square_\infty)}(\frac{\gamma_{h,i}-1}{[\epsilon]-1})\big) \otimes W \longrightarrow \eta_p\Big(\big(\otimes_h \mathbf{K}_{\mathrm{A_{inf}}(R^\square_\infty)}(\frac{\gamma_{h,i}-1}{[\epsilon]-1})\big) \otimes W\Big),$$

given by $p^k F$ in degree k. In particular, in degree 0, the map sends $T_{h,i} \xrightarrow{\psi=F} T^p_{h,i}$. Therefore, unwinding definitions, we see that for each $r \geq 2$ the top arrow in the digram above sends

$$T^{p^{r-1}}_{h,i} \overset{\mu}{\longmapsto} T^{p^{r-1}}_{h,i} \overset{\psi^{-1}}{\longmapsto} T^{p^{r-2}}_{h,i}.$$

This proves the commutativity of the triangle, hence the lemma. $\square$

Remark 4.12 *Note that we have to use $A\Omega^{\mathrm{gp}}_{R^\square}$ instead of $L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^\square))$, for otherwise ψ^{-1} is not well-defined. However, keeping track of the element $T^{p^{r-1}}_{h,i}$ only involves $L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^\square))$.*

Lemma 4.13 *$(A^0_R, M^\square \xrightarrow{\alpha} A^0_R)$ is a log algebra over $W(\underline{k}^\square)$. Here $\underline{k}^\square = (k, N^\square)$ is the log point with 0 maps to 1 and everything else maps to 0.*

Proof. We need to show that for every h such that $1 \leq h \leq s$, we have

$$\alpha(\mathfrak{b}_{\mathfrak{h},\mathfrak{o}} + \cdots + \mathfrak{b}_{\mathfrak{h},\mathfrak{r}_\mathfrak{h}}) = 0.$$

Again it suffices to assume that $R = R^\square$, and suffices to show that for each $r \geq 1$, $\alpha_r(\mathfrak{b}_{\mathfrak{h},\mathfrak{o}} + \cdots + \mathfrak{b}_{\mathfrak{h},\mathfrak{r}_\mathfrak{h}}) = \alpha_r(\mathfrak{b}_{\mathfrak{h},\mathfrak{o}}) \cdots \alpha_r(\mathfrak{b}_{\mathfrak{h},\mathfrak{r}_\mathfrak{h}}) = 0$, but $L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^\square))\widehat{\otimes}^{\mathbf{L}} W$ is computed by the product (over h) of

$$\Big(\mathbf{K}_{A(R^\square_h)}(\frac{\gamma_{h,i}}{[\epsilon]-1})\Big) \otimes W$$

(cf. Example 3.9), and the degree 0 term is (a product of)

$$W(k)\langle T^{1/p}_{h,0}, ..., T^{1/p}_{h,r_h}, ..., T^{\pm 1/p}_{h,d_h}\rangle/(T^{1/p}_{h,0} \cdots T^{1/p}_{h,r_h}).$$

In particular, in $H^0(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^\square)) \otimes^{\mathbf{L}} W_r)$,

$$\alpha_r(\mathfrak{b}_{\mathfrak{h},\mathfrak{o}}) \cdots \alpha_r(\mathfrak{b}_{\mathfrak{h},\mathfrak{r}_\mathfrak{h}}) = T^{p^{r-1}}_{h,0} \cdots T^{p^{r-1}}_{h,r_h} = 0.$$

$\square$

4.2.2 *The log derivation* The log derivation is defined in a similar fashion. We start with the commutative diagram

$$\begin{array}{ccc} H^1(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R^\square)) \otimes^{\mathbf{L}} W_r) & = & H^1\Big(\Big(\bigotimes_{1\le h\le s} \mathbf{K}_{A(R^\square)}(\frac{\gamma_{h,i}-1}{[\epsilon]-1})\Big) \otimes W_r\Big) \\ \downarrow & \searrow^{\lambda_r} & \\ H^1(L\eta_\mu R\Gamma_{\mathrm{ct}}(\Delta, A(R)) \otimes^{\mathbf{L}} W_r) & \longrightarrow & H^1(A\Omega_R \otimes^{\mathbf{L}} W_r) \end{array}$$

As in Example 3.9, we specify a collection of dummy basis $\{e_{h,i}\}_{\substack{1\le h\le s \\ 1\le i\le d_h}}$ in degree 1 for each

$$\mathbf{K}_{A(R^\square)}(\frac{\gamma_{h,i}-1}{[\epsilon]-1}) = A(R^\square) \xrightarrow{\frac{\gamma_{h,1}-1}{[\epsilon]-1}, \dots, \frac{\gamma_{h,d_h}-1}{[\epsilon]-1}} A(R^\square)\cdot e_{h,1} \oplus \cdots \oplus A(R^\square)\cdot e_{h,d_h} \longrightarrow \cdots$$

such that for each $i \ge 1$, $X_{h,i}^m \mapsto (1+[\epsilon]+\cdots+[\epsilon]^{m-1})X_{h,i}^m \cdot e_{h,i}$. We denote its image in H^1 of the (base change of the) Koszul complex again by $e_{h,i}$.

Definition 4.14 *For $r \ge 1$, we define a monoid morphism*

$$\delta_r = \delta_r^\square : M^\square \to H^1(A\Omega_R \otimes^{\mathbf{L}} W_r)$$

by sending each $\mathfrak{b}_{h,i}$ for $1 \le h \le s, 0 \le i \le r_h$ (cf. Definition 4.9) to

$$\delta_r(\mathfrak{b}_{h,i}) = \begin{cases} \gamma_r(e_{h,i}) & 1 \le i \le r_h \\ -(\gamma_{e_{h,1}} + \cdots + \gamma_{e_{h,r_h}}) & i = 0 \end{cases}.$$

Lemma 4.15 *For every $r > 1$, every $m \in M^\square$, we have*

$$R_r(\delta_r(m)) = F(\delta_r(m)) = \delta_{r-1}(m).$$

In other words, the maps $\{\delta_r\}$ are compatible with R_r. This gives rise to a map

$$\delta = \delta^\square : M^\square \to A_R^1 := \varprojlim_{R_r} H^1(A\Omega_R \otimes^{\mathbf{L}} W_r)$$

of monoids, which satisfies $F\delta = \delta$.

Proof. Similar to the proof of Lemma 4.11, we need to show that the map

$$H^1(A\Omega_{R^\square}^{\mathrm{gp}} \otimes^{\mathbf{L}} W_r) \xrightarrow{\mu_r} H^1(L\eta_p\Big(A\Omega_{R^\square}^{\mathrm{gp}} \widehat{\otimes}^{\mathbf{L}} W\Big)/p^{r-1}) \xrightarrow{\psi^{-1}} H^1(A\Omega_{R^\square}^{\mathrm{gp}} \otimes^{\mathbf{L}} W_{r-1})$$

sends $e_{h,i}$ to $e_{h,i}$ for each h and $1 \le i \le r_h$. As before, $A\Omega_{R^\square}^{\mathrm{gp}}$ can be computed by the complex $\otimes_h \mathbf{K}_{\mathrm{A_{inf}}(R_\infty^\square)}(\frac{\gamma_{h,i}-1}{[\epsilon]-1})$. Unwinding definitions, the map μ_r is given by $[x] \mapsto [px]$ for a cocycle $[x] \in H^1(A\Omega_{R^\square}^{\mathrm{gp}} \otimes^{\mathbf{L}} W_r)$. On the other hand, the map ψ is given by

$$\phi_F : \Big(\otimes_h \mathbf{K}_{\mathrm{A_{inf}}(R_\infty^\square)}(\frac{\gamma_{h,i}-1}{[\epsilon]-1})\Big) \otimes W \longrightarrow \eta_p\Big(\Big(\otimes_h \mathbf{K}_{\mathrm{A_{inf}}(R_\infty^\square)}(\frac{\gamma_{h,i}-1}{[\epsilon]-1})\Big) \otimes W\Big),$$

which is $p \cdot F$ in degree 1, and sends $T_{h,i}^m \cdot e_{h,i} \mapsto pT_{h,i}^m \cdot e_{h,i}$. In particular it sends $e_{h,i} \mapsto p \cdot e_{h,i}$, therefore $\psi^{-1}(\mu_r(e_{h,i})) = e_{h,i}$ as desired. □

Lemma 4.16 *The monoid maps α and δ constructed above satisfy*

- *$\alpha(m)\delta(m) = d(\alpha(m))$ for any $m \in M^\square$.*

– $\delta(\mathrm{diag}(n)) = 0$ *for any* $n \in N^{\square}$.
– $d\delta(m) = 0$ *for* $m \in M^{\square}$.

Proof. The lemma follows from the corresponding statements for each α_r and δ_r, and the last two claims directly follow from definitions. We need to show that for each $r \geq 1$, each h and $1 \leq i \leq r_h$, $\alpha_r(\mathfrak{b}_{h,i})\delta_r(\mathfrak{b}_{h,i}) = \beta(\alpha_r(\mathfrak{b}_{h,i}))$ where β stands for the Bockstein differential.

Again by functoriality, it suffices to show that the Bockstein differential

$$H^0(A\Omega^{\mathrm{gp}}_{R^{\square}} \otimes^{\mathbf{L}} W_r) \xrightarrow{\beta_r} H^1(A\Omega^{\mathrm{gp}}_{R^{\square}} \otimes^{\mathbf{L}} W_r)$$

sends $T_{h,i}^{p^{r-1}} \overset{\beta_r}{\longmapsto} T_{h,i}^{p^{r-1}} \cdot e_{h,i}$. By Example 3.9 (more precisely the computation of the base change to $W(k)$), we know that the differential β on

$$\Big(\otimes_h \mathbf{K}_{A(R^{\square})}(\frac{\gamma_{h,i} - 1}{[\epsilon] - 1}) \Big) \otimes W(k)$$

sends $T_{h,i}^{p^{r-1}} = (T_{h,i}^{1/p})^{p^r}$ to $p^r T_{h,i}^{p^{r-1}} \cdot e_{h,i}$ for every h and every $i \geq 1$. Therefore

$$\beta_r(T_{h,i}^{p^{r-1}}) = \frac{\beta(T_{h,i}^{p^{r-1}})}{p^r} = T_{h,i}^{p^{r-1}} \cdot e_{h,i}.$$

□

Let us summarize what we have shown so far.

Proposition 4.17 *Upon choosing a semistable coordinate* $\square : R \to R^{\square}$*, the strict Dieudonné algebra* A_R^* *equipped with* $M^{\square} \xrightarrow{\alpha} A_R^0$ *and* $M^{\square} \xrightarrow{\delta} A_R^1$ *is a strict* p*-compatible log Dieudonné algebra over* $W(\underline{k}^{\square})$*. We denote this log Dieudonné algebra by* $A_R^{*,\square}$ *(the log data depend on coordinates).*

4.3 The local comparison

We are ready to prove a local version of log crystalline comparison (upon fixing coordinates). Recall that $\widetilde{\Omega}_R \cong A\Omega_R \widehat{\otimes}^{\mathbf{L}}_{\tilde{\theta}} \mathcal{O}_C$. We first observe that there is a natural commutative diagram

$$\begin{array}{ccccc} M^{\square} & \longrightarrow & R^{\square} & \longrightarrow & R \\ & \searrow^{\alpha_1} & \downarrow & & \downarrow \\ & & H^0(\widetilde{\Omega}_{R^{\square}}) & \longrightarrow & H^0(\widetilde{\Omega}_R) \end{array}$$

where the monoid morphism α_1 comes from Remark 4.10 (and gives rise to the map $\alpha_1 : M^{\square} \to H^0(A\Omega_{R^{\square}}) \otimes^{\mathbf{L}} k$ via base change), while the map $R^{\square} \xrightarrow{\sim} H^0(\widetilde{\Omega}_{R^{\square}})$ (resp. $R \xrightarrow{\sim} H^0(\widetilde{\Omega}_R)$) is the isomorphism given by part (2) of Theorem 3.7, which sends each $T_{h,i}$ to $T_{h,i}$. Since we have $H^0(\widetilde{\Omega}_R) \otimes k = H^0(\widetilde{\Omega}_R \otimes^{\mathbf{L}} k)$, the base change of the diagram above induces an isomorphism of log algebras over $\underline{k}^{\square} = (k, N)$:

$$\tau_1^{\square,0} : (R_k, M^{\square}) \xrightarrow{\sim} (H^0(A\Omega_R \otimes^{\mathbf{L}} k), M^{\square}).$$

By the construction of A_R^*, we have $W_1(A_R^0) = H^0(A\Omega_R \otimes^{\mathbf{L}} k)$, therefore this isomorphism induces a map

$$\tau^{\square} : \mathcal{W}\omega^*_{(R_k, M^{\square})/\underline{k}^{\square}} \to A_R^{*,\square}$$

of strict p-compatible log Dieudonné algebras over $W(\underline{k}^{\square})$.

Lemma 4.18 *$\tau^\square$ is an isomorphism of log Dieudonné algebras.*

Proof. By Lemma 2.5, it suffices to show that the dotted arrow in the diagram

$$\begin{array}{ccc} \mathcal{W}_1\omega^*_{(R_k,M^\square)/\underline{k}^\square} & \longrightarrow & W_1(A_R^{*,\square}) \\ \uparrow \wr & & \downarrow \wr \\ \omega^*_{(R_k,M^\square)/\underline{k}^\square} & \overset{\tau_1^\square}{\dashrightarrow} & \left(H^*(\widetilde{\Omega}_R \otimes^{\mathbf{L}} k), \beta\right) \end{array}$$

is an isomorphism of cdga's, where $W_1(A_R^{*,\square})$ is equipped with the Bockstein differential β. By Subsection 3.2 we have isomorphisms of graded algebras

$$H^*(\widetilde{\Omega}_R \otimes^{\mathbf{L}} k) \cong H^*(\widetilde{\Omega}_R) \otimes k \cong \omega^*_{\underline{R}/\underline{\mathcal{O}_C}} \otimes k \cong \omega^*_{\underline{R_k}/\underline{k}}.$$

The target of $\tau_1^\square$ can be identified with $\left(\omega^*_{\underline{R_k}/\underline{k}}, \beta\right)$ as a cdga, where the differentials come from the Bockstein differentials on $H^*(\widetilde{\Omega}_R \otimes^{\mathbf{L}} k)$, associated to $0 \to \mathbf{Z}/p \xrightarrow{p} \mathbf{Z}/p^2 \to \mathbf{Z}/p \to 0$; and is equipped with a log structure coming from $\alpha_1^\square : M^\square \to H^0(\widetilde{\Omega}_R \otimes^{\mathbf{L}} k)$ as in Definition 4.9 (and similarly for $\delta_1^\square$).

Note that $\omega^0_{(R_k,M^\square)/\underline{k}^\square} = R_k$ and receives the log structure $M^\square \to R_k$. Unwinding definitions (of the construction of $\tau^\square$), the morphism $\tau_1^\square$ on the log algebra $(R_k, M^\square)$ is given by $\mathrm{id} : (R_k, M^\square) \to (R_k, M^\square)$. Therefore, to prove the lemma, it suffices to identify the Bockstein differential β with the de Rham differential on $\omega^*_{(R_k,M^\square)/\underline{k}^\square} \cong \omega^*_{\underline{R_k}/\underline{k}}$. By the commutativity of

$$\begin{array}{ccccc} \widetilde{\Omega}_R = A\Omega_R/\widetilde{\xi} & \xrightarrow{\widetilde{\xi}} & A\Omega_R/\widetilde{\xi}^2 & \longrightarrow & A\Omega_R/\widetilde{\xi} \\ \downarrow & & \downarrow & & \downarrow \\ A\Omega_R \otimes^{\mathbf{L}} W/p & \xrightarrow{p} & A\Omega_R \otimes^{\mathbf{L}} W/p^2 & \longrightarrow & A\Omega_R \otimes^{\mathbf{L}} W/p \end{array}$$

We are reduced to show that the Bockstein differentials on $H^*(\widetilde{\Omega}_R)$ agree with the de Rham differentials via $H^*(\widetilde{\Omega}_R) \cong \omega^*_{\underline{R}/\underline{\mathcal{O}_C}}$, but this is part of Theorem 3.7. $\square$

Remark 4.19 *On the underlying rings, the map $\tau_1^{\square,0}$ created in the paragraph above the lemma is given by the canonical isomorphism $R_k \xrightarrow{\sim} H^0(\widetilde{\Omega}_R \otimes^{\mathbf{L}} k)$, and has nothing to do with the choice of coordinates. In particular, as a map of cdga's, the isomorphism $\tau_1^\square$ in the proof above is independent of choice of $\square$.*

4.4 Independence of coordinates

We freely use notations from Subsection 3.1. Let be $\mathfrak{U} = \operatorname{Spf} R \in \mathfrak{X}_{\text{ét,aff}}$ as in previous subsections. Let $M_{\mathfrak{U}} = M_{\mathfrak{X}}|_{\mathfrak{U}}$ be the log structure on $\mathfrak{U}_{\text{ét}}$ restricted from $\mathfrak{X}$, which is equal to the log structure associated to the monoid $M = \Gamma(\operatorname{Spf} R, M_{\mathfrak{X}})$. We write $\underline{R_k}$ for the log algebra (R_k, M). From Remark 3.2, for each choice of coordinates $\square$, we have an isomorphism of *strict Dieudonné algebras*

$$\lambda^\square : \mathcal{W}\omega^*_{(R_k,M^\square)/\underline{k}^\square} \xrightarrow{\sim} \mathcal{W}\omega^*_{\underline{R_k}/\underline{k}}$$

induced by $M^\square \to M$ on the log structures on R_k.

Proposition 4.20 *If we ignore the logarithmic structures, then the isomorphism*

$$\tau : \mathcal{W}\omega^*_{\underline{R_k}/\underline{k}} \xrightarrow{(\lambda^\square)^{-1}} \mathcal{W}\omega^*_{(R_k, M^\square)/\underline{k}^\square} \xrightarrow{\tau^\square} A^*_R$$

on strict Dieudonné algebras is independent of choice of coordinates.

Proof. In the proof we use $\lambda^\square$ to identify the underlying Dieudonné algebras of $\mathcal{W}\omega^*_{(R_k,M^\square)/\underline{k}^\square}$ and of $\mathcal{W}\omega^*_{\underline{R_k}/\underline{k}}$ for any coordinate $\square$. Let $R^\square \to R$ and $R^{\square\prime} \to R$ be two choices of coordinates. We need to show that, the morphism of Dieudonné algebras $\tau^\square : W\omega^*_{(R_k,M^\square)/\underline{k}^\square} \xrightarrow{\sim} A^{*,\square}_R$ and $\tau^{\square\prime} : W\omega^*_{(R_k,M^{\square\prime})/\underline{k}^{\square\prime}} \xrightarrow{\sim} A^{*,\square\prime}_R$ coincide. Note that for each n, both pre-log structures $M^\square$ and $M^{\square\prime}$ on $\operatorname{Spec} W_n(R_k)$ factor through the pre-log structure $\alpha : M \to R_k \xrightarrow{[\cdot]} W_n(R_k)$. From Remark 4.11 in [Yao19b] (see also Lemma 4.9 and the proof of Theorem 4.10 thereof), it suffices to show that, after we identify

$$\lambda_1 : \varprojlim \omega^1_{W_n(R_k,M^\square)/W_n(\underline{k}^\square)} \xrightarrow{\sim} \varprojlim \omega^1_{W_n(\underline{R_k})/W_n(\underline{k})}$$
$$\text{and} \quad \lambda_2 : \varprojlim \omega^1_{W_n(R_k,M^{\square\prime})/W_n(\underline{k}^{\square\prime})} \xrightarrow{\sim} \varprojlim \omega^1_{W_n(\underline{R_k})/W_n(\underline{k})},$$

the two maps $\tau^\square, \tau^{\square\prime}$ from $\varprojlim \omega^1_{W_n(\underline{R_k})/W_n(\underline{k})}$ to A^1_R, induced respectively from the log structures $M^\square$ and $M^{\square\prime}$, agree with each other (see Lemma 4.9 in [Yao19b] for their constructions). More precisely, the lemma will follow from the following

Claim. Consider the log differential $\operatorname{dlog}_1 : M^\square \longrightarrow \varprojlim \omega^1_{W_n(R_k,M^\square)/W_n(\underline{k}^\square)}$. For every $m \in M^\square$, let $\operatorname{dlog}(m) := \lambda_1(\operatorname{dlog}_1(m)) \in \varprojlim \omega^1_{W_n(\underline{R_k})/W_n(\underline{k})}$. The following element in A^1_R defined by

$$\delta = \delta(m) := \tau^\square(\operatorname{dlog}(m)) - \tau^{\square\prime}(\operatorname{dlog}(m))$$

is equal to 0.

Indeed, $\omega^1_{W_n(\underline{R_k})/W_n(\underline{k})}$ is generated by elements dx for $x \in W_n(R_k)$ and $\operatorname{dlog}(m)$ for $m \in M^\square$, as all maps in discussion preserve differentials, the claim implies the lemma by Remark 4.11 and Lemma 4.9 (and its proof) in *loc.cit.*

It remains to justify the claim. We first observe that for any $m \in M^\square$, both $\tau^\square(\operatorname{dlog}(m))$ and $\tau^{\square\prime}(\operatorname{dlog}(m))$ are fixed by the Frobenius F on A^1_R.[17] Therefore, if we use τ to identify $\mathcal{W}\omega^1_{\underline{R_k}/\underline{k}} \xrightarrow{\sim} A^1_R$, we have that

$$\delta \in \ker(\mathcal{W}\omega^1_{\underline{R_k}/\underline{k}} \xrightarrow{F-1} \mathcal{W}\omega^1_{\underline{R_k}/\underline{k}}).$$

By Corollary 2.14 of [Lor02], the Frobenius fixed points of $\mathcal{W}\omega^1_{\underline{R_k}/\underline{k}}$ are precisely the Hodge–Witt forms $W\omega^1_{\log}$, consisting of sections which are étale locally sums of sections $\operatorname{dlog}(m_i)$ for $m_i \in (\mathcal{M}^a)^{\mathrm{gp}}$, where $\mathcal{M}^a$ is the log structure on $\operatorname{Spec} R_k$ associated to the constant pre-log structure $M^\square$. Now from the relations

$$\alpha(m) \cdot \tau^\square(\operatorname{dlog}(m)) = \alpha(m) \cdot \tau^{\square\prime}(\operatorname{dlog}(m)) = d\alpha(m),$$

we know that δ satisfies $\alpha(m) \cdot \delta = 0$. The claim, therefore the proposition, follows from the next lemma. □

[17] For example, for $\tau^\square$, note that M is generated by $M^\square$ and $R^\times$, so the image of dlog is generated by $\operatorname{dlog}(m_0)$ for $m_0 \in M^\square$ and dx/x for $x \in R^\times$ as a monoid, both fixed by F.

We retain the same setup from Proposition 4.20 in the following lemma. For convenience we also denote $Y = \operatorname{Spec} R_k$ and $\underline{Y} = (Y, M_Y)$ which the log scheme associated to the log algebra (R_k, M).

Lemma 4.21 *Let $m \in M^\square$, and $\delta \in W\omega^1_{\log}$ be a section of the Hodge–Witt forms, such that $\alpha(m) \cdot \delta = 0$, then $\delta = 0$.*

Proof. We have the following exact sequences of sheaves from [Lor02] (2.13 – 2.14)

$$\begin{array}{ccccccccc}
 & & pW\omega^1_{\underline{Y}/\underline{k},\log} & \hookrightarrow & \operatorname{im}(V + \mathrm{d}V) & \longrightarrow & \operatorname{im}(V+\mathrm{d}V) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W\omega^1_{\underline{Y}/\underline{k},\log} & \longrightarrow & \mathcal{W}\omega^1_{\underline{Y}/\underline{k}} & \xrightarrow{1-F} & \mathcal{W}\omega^1_{\underline{Y}/\underline{k}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \omega^1_{\underline{Y}/\underline{k},\log} & \longrightarrow & \omega^1_{\underline{Y}/\underline{k}} & \xrightarrow{1-F} & \omega^1_{\underline{Y}/\underline{k}}/\mathrm{d}\mathcal{O}_Y & \longrightarrow & 0
\end{array}$$

where $\mathrm{d}\mathcal{O}_Y$ denotes the subsheaf of $\Omega^1_{Y/k}$ generated by dy for $y \in \mathcal{O}_Y$. Note that the induced V-filtration (given by $V + \mathrm{d}V$) on $W\omega^1_{\underline{Y}/\underline{k},\log}$ agrees with the p-filtration since $F = \mathrm{id}$. Write $W\omega^1_{\log}$ for the (étale) sections, for any $\sigma \in W\omega^1_{\log}$, we write $\overline{\sigma}$ for its image in $\omega^1_{\log}$ (reducing mod p).

Now for contradiction we assume that $\delta \neq 0$. If $\overline{\delta} = 0$ then we can write $\delta = p\delta'$ for some $\delta' \in W\omega^1_{\log}$, which still satisfies $\alpha(m) \cdot \delta' = 0$ since $W\omega^1_{\log} \subset \mathcal{W}\omega^1_{\underline{R_k}/\underline{k}}$ is p-torsion free. Note that $W\omega^1_{\log}$ is p-adically separated, so we may assume without loss of generality that $\overline{\delta} \neq 0$. Now we have that $\overline{\delta} \neq 0$ but $\overline{\alpha}(m) \cdot \overline{\delta} = 0$, where $\overline{\alpha}$ denotes the monoid map $M^\square \to R_k^\square \to R_k$. This is impossible from the explicit description of (the Frobenius fixed points of)

$$\omega^1_{\underline{R_k}/\underline{k}} \cong \omega^1_{(R_k, M^\square)/\underline{k}^\square} \cong \omega^1_{(R_k^\square, M^\square)/\underline{k}^\square} \otimes_{R_k^\square} R_k$$

which is isomorphic to the free-R_k modules with basis

$$\Big\{ \operatorname{dlog}(T_{h,i}), \operatorname{dlog}(T_{h,j}) : 0 \leq i \leq r_h - 1, r_h + 1 \leq j \leq d_h \Big\}_{1 \leq h \leq s}$$

for $R^\square$ as described in Subsection 3.1. Note that in the indices above we omit r_h as $\operatorname{dlog}(T_{h,0}) + \cdots + \operatorname{dlog}(T_{h,r_h}) = 0$. $\square$

4.5 The log crystalline comparison

Now we are finally ready to prove the main theorem of this section.

Proof of Theorem 4.1. For each $\mathfrak{U} = \operatorname{Spf} R \in \mathfrak{X}_{\text{ét,aff}}$, we write $\tau_R : \mathcal{W}\omega^*_{\underline{R_k}/\underline{k}} \xrightarrow{\sim} A^*_R$ for the isomorphism obtained from Proposition 4.20. We claim that this supplies a natural map

$$\tau^{\text{pre}} : \mathcal{W}\omega^{*,\text{pre}}_{\underline{\mathfrak{X}_k}/\underline{k}} \xrightarrow{\sim} A\Omega^{\text{pre}}_{\mathfrak{X},W}$$

of presheaves as desired in the beginning of this section.

Lemma 4.22 *Let $f : R \to S$ be a morphism in $\mathfrak{X}_{\text{ét,aff}}$, then the following diagram of maps of*

Dieudonné algebras commutes

$$\begin{array}{ccc} \mathcal{W}\omega^*_{\underline{R_k}/\underline{k}} & \xrightarrow{\tau_R} & A^*_R \\ \downarrow{\scriptstyle \mathcal{W}_f} & & \downarrow{\scriptstyle A_f} \\ \mathcal{W}\omega^*_{\underline{S_k}/\underline{k}} & \xrightarrow{\tau_S} & A^*_S \end{array}$$

Proof. Note that all maps have already been specified (with the vertical arrows obtained from functoriality). We now choose a coordinate $R^\square \to R$, which also serves as a coordinate for S via $R \xrightarrow{f} S$. This equips R and S with pre-log structures $M^\square$ as in Subsection 3.1. By Proposition 4.20, τ_R does not depend on $\square$ and in particular $\tau_R = \tau^\square \circ (\lambda^\square)^{-1}$ (and likewise for τ_S). Therefore, it suffices to show that the right square in the following diagram commutes (as the left square commutes by functoriality of saturated log de Rham–Witt complexes)

$$\begin{array}{ccccc} \mathcal{W}\omega^*_{\underline{R_k}/\underline{k}} & \xrightarrow{(\lambda^\square)^{-1}} & \mathcal{W}\omega^*_{(R_k,M^\square)/\underline{k}^\square} & \xrightarrow{\tau^\square_R} & A^{*,\square}_R \\ \downarrow{\scriptstyle \mathcal{W}_f} & & \downarrow{\scriptstyle \mathcal{W}^\square_f} & & \downarrow{\scriptstyle A_f} \\ \mathcal{W}\omega^*_{\underline{S_k}/\underline{k}} & \xrightarrow{(\lambda^\square)^{-1}} & \mathcal{W}\omega^*_{(S_k,M^\square)/\underline{k}^\square} & \xrightarrow{\tau^\square_S} & A^{*,\square}_S \end{array}$$

Note the maps in the right square are enhanced to maps of log Dieudonné algebras, where the log structures on $A^{*,\square}_R$ and $A^{*,\square}_S$ are constructed in Subsection 4.2 from the coordinate $\square$, and the maps on the monoid $M^\square$ are all given by identify. Now, both maps $A_f \circ \tau^\square_R$ and $\tau^\square_S \circ W^\square_f$ are morphisms $\mathcal{W}\omega^*_{(R_k,M^\square)/\underline{k}^\square} \longrightarrow A^{*,\square}_S$ between log Dieudonné algebras, therefore, by the definition (and construction) of the saturated log de Rham–Witt complex, it suffices to show that they correspond to the same map under the bijection

$$\mathrm{Hom}_{\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}}(\mathcal{W}\omega^*_{(R_k,M^\square)/\underline{k}^\square}, A^{*,\square}_S) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Alg}^{\log}_{\underline{k}}}(\underline{R^\square_k}, A^{0,\square}_S/\operatorname{im} V).$$

This follows from the commutative diagram

$$\begin{array}{ccc} R_k & \xrightarrow{\sim} & H^0(\widetilde{\Omega}_R \otimes^{\mathbf{L}} k) \\ \downarrow & & \downarrow \\ S_k & \xrightarrow{\sim} & H^0(\widetilde{\Omega}_S \otimes^{\mathbf{L}} k) \end{array}$$

□

Theorem 4.1 now follows, as after sheafification, we get an isomorphism of sheaves valued in the derived ∞-categories

$$\tau : \mathcal{W}\omega^*_{\underline{\mathfrak{X}_k}/\underline{k}} \xrightarrow{\sim} A\Omega_{\mathfrak{X},W} \xrightarrow{\sim} A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbf{L}} W(k)$$

in $\mathcal{D}(\mathfrak{X}_{\text{ét},\text{aff}}, W(k)) \cong \mathcal{D}(\mathfrak{X}_{\text{ét}}, W(k))$. The quasi-isomorphism on the derived global sections follows from the same proof of Lemma 4.2. □

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