Logarithmic de Rham–Witt complexes via the Décalage operator

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Abstract

We provide a new formalism of de Rham–Witt complexes in the logarithmic setting. This construction generalizes a result of Bhatt–Lurie–Mathew, and agrees with those of Hyodo–Kato and Matsuue for log-smooth schemes of log-Cartier type.

Contents

1. Introduction

Recently Bhatt–Lurie–Mathew [BLM18] gives a relatively elementary construction of the de Rham–Witt complex (hence the crystalline cohomology) for smooth varieties over a perfect field of characteristic p. This paper extends their construction to the logarithmic setting, which in turn allows us to study the $\mathrm{A}_{\mathrm{inf}}$-cohomology in the case of semistable reduction in [Yao19].

1.1 Main Results

Let $\underline{k}$ be a perfect log field of characteristic p. Let $\underline{X}$ be a coherent log scheme (with underlying scheme X) over $\underline{k}$, and $\underline{R} = (R, M)$ be a local chart for its log structure. To contextualize our construction, we first define the category $\mathrm{DA}^{\log}$ of log Dieudonné algebras, which consists of commutative differential graded algebras (cdga) A^*, equipped with a Frobenius map F, a log structure L and a log derivation δ in a compatible manner. In particular, the data are required to satisfy $dF = pFd$ and $\delta F = pF\delta$. There is a full subcategory $\mathrm{DA}^{\log}_{\mathfrak{str}}$ of *strict* log Dieudonné algebras, which essentially consists of log Dieudonné algebras that admit the Verschiebung maps V and are complete with respect to the V-filtration.[1] This allows us to give the following primitive

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[1] We will formulate the strictness condition using the log version of décalage operator η_p, which on p-torsion free cochain complexes is given by

$$\eta_p A^i := \{x \in p^i A^i | dx \in p^{i+1} A^{i+1}\}.$$

definition.[2]

Definition 1 *The saturated log de Rham–Witt complex, as a functor from the category* $\mathrm{Alg}_{\underline{k}}$ *of log* $\underline{k}$*-algebras to strict log Dieudonné algebras, is the left adjoint of* $\iota : \mathrm{DA}^{\log}_{\mathfrak{str}} \longrightarrow \mathrm{Alg}_{\underline{k}}$*, which sends* $A^* \mapsto \underline{A}^0/\operatorname{im} V$*. Denote by* $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ *the saturated log de Rham–Witt complex of* $\underline{R}$ *(if exists).*

Our first main result states that

Theorem 2 $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ *exists and glues to a sheaf* $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ *on the* étale *site* $X_{\text{ét}}$ *of* X*.*

Our version of the log de Rham–Witt complexes agrees with the existing ones in [HK94], [LZ04] and [Mat17] under the additional smoothness assumptions.

Theorem 3 *Let* $\underline{X}$ *be a log scheme over* $\underline{k}$ *which is log-smooth of log-Cartier type, then there exists isomorphisms of sheaves of complexes*

$$\mathcal{W}\omega^*_{\underline{X}/\underline{k}} \xrightarrow{\sim} W\Lambda^*_{\underline{X}/\underline{k}} \cong W^{\mathrm{HK}}\omega^*_{\underline{X}/\underline{k}},$$

compatible with the Frobenius operators, where $W\Lambda^*_{\underline{X}/\underline{k}}$ *is the complex constructed by Matsuue [Mat17] (see Section 6), and* $W^{\mathrm{HK}}\omega^*_{\underline{X}/\underline{k}}$ *is the Hyodo–Kato complex constructed via the log crystalline site in [HK94]. In other words, in this case all three versions of "log de Rham–Witt complexes" agree, and compute the log crystalline cohomology* $R\Gamma_{\text{log-cris}}(\underline{X}/W(\underline{k}))$*.*

Remark 4 *In general, the saturated complex* $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ *differs from the crystalline construction of Hyodo–Kato. This distinction is already present in the case of ordinary schemes and has little to do with log structures. See Section 6.3 of [BLM18] for an explicit example of a cusp.*

One notable feature of our construction is that we start with the Frobenius and produce the Verschiebung and Restriction maps "along the way". Another feature is that our formulation makes essential use of the décalage operator η_p. To be more precise, we turn the isomorphism

$$\phi_F : W\omega^*_{\underline{X}/\underline{k}} \cong \eta_p W\omega^*_{\underline{X}/\underline{k}}$$

which is typically an output [3] of previous constructions of the log de Rham–Witt complex, as part of the input. As a result, our complex is characterized by a different universal property, which makes it easier to compare to cdga's equipped with Frobenius.

Remark 5 *This is essentially what makes this formalism suitable for studying the log crystalline comparison of* A_{inf}*-cohomology in [Yao19]. To explain further, let* $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ *be the* A_{inf}*-cohomology of a formal scheme* $\mathfrak{X}$ *over* $\mathcal{O}_C$ *with semistable reduction, where* C *is a completed algebraic closure of* $W(k)[\frac{1}{p}]$*. Write* $\underline{\mathfrak{X}}$ *for the formal log scheme with the divisorial log structure from its mod* p *fiber* $\mathfrak{X}_{\mathcal{O}_C/p}$*. In [Yao19] we apply the main results of this article to prove*

Theorem 6 ([Yao19]) *There is a Frobenius compatible quasi-isomorphism*

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})\widehat{\otimes}^{\mathbf{L}}_{A_{\mathrm{inf}}} W(k) \cong R\Gamma_{\log\text{-cris}}(\underline{\mathfrak{X}_k}/W(k)),$$

relating the specialization of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ *to* $W(k)$ *with the log crystalline cohomology of the special fiber of* $\mathfrak{X}$ *over* k*.*

For each p-torsion free object A^* in $\mathrm{DA}^{\log}$, there is a morphism $\phi_F : A^* \to \eta_p(A^*)$ of log Dieudonné algebras. The existence of V is closely related to ϕ_F being an isomorphism.

[2] Strictly speaking, we need to restrict to subcategories $\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}$ of $\mathrm{DA}^{\log}_{\mathfrak{str}}$, spanned by objects for which the log Frobenius is given by multiplication by p. We ignore this issue in the introduction for the sake of exposition.

[3] This isomorphism is due to Illusie for (ordinary) de Rham–Witt complexes and to Hyodo–Kato in the logarithmic setting. See Definition 2.3 and Lemma 6.4.

1.2 The construction

The construction of the saturated log de Rham–Witt complex is modeled after [BLM18]. Assume for simplicity that R is reduced so $W(R)$ is p-torsion free. We first consider $W(\underline{R}) = (W(R), M)$, where the log structure is obtained from the Teichmuller lift. The log de Rham complex $\omega^*_{W(\underline{R})/\underline{W}}$ admits a canonical structure of a log Dieudonné algebra. We then construct a certain left adjoint $W(\ )_{\mathrm{sat}} : \mathrm{DA}^{\log} \longrightarrow \mathrm{DA}^{\log}_{\mathfrak{str}}$ of the inclusion functor $\mathrm{DA}^{\log}_{\mathfrak{str}} \hookrightarrow \mathrm{DA}^{\log}$, and define the saturated log de Rham–Witt complex of $\underline{R}/\underline{k}$ as

$$\mathcal{W}\omega^*_{\underline{R}/\underline{k}} := W\left((\omega^*_{W(\underline{R})/\underline{W}})_{\mathfrak{sat}}\right).$$

Finally, for a quasi-coherent log scheme $\underline{X}$ over $\underline{k}$, we show that the construction $\mathcal{W}\omega^i_{-/\underline{k}}$ satisfies étale descent, hence globalizes to an étale sheaf $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ of log Dieudonné algebras.

Remark 7 *Although it might be possible to develop the global theory directly, we prefer to work locally, and then globalize only in the last step. This is especially convenient for our application to A_{inf}-cohomology in [Yao19], where local charts are used in an essential way.*

From the construction it is not difficult to prove the following comparison theorems, which ultimately amounts to the existence of the Cartier isomorphism in the logarithmic setting.

Theorem 8 *Suppose that $\underline{R}$ is log-smooth over $\underline{k}$ of log-Cartier type. Then $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ satisfies the following "de Rham comparisons".*

– (mod V). *There is a canonical isomorphism of cochain complexes*

$$\omega^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}}.$$

– (mod p). *There is a canonical quasi-isomorphism*

$$\mathcal{W}\omega^*_{\underline{R}/\underline{k}}/p\mathcal{W}\omega^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}} \cong \omega^*_{\underline{R}/\underline{k}}.$$

– (with Frobenius lifts). *Suppose that there exists a log-Frobenius lift $(\underline{A}, \varphi)$ of $\underline{R}$ over $W(\underline{k})$ in the sense of Subsection 4.2.2, then there is a quasi-isomorphism*

$$\iota_\varphi : \widehat{\omega}^*_{\underline{A}/W(\underline{k})} \xrightarrow{\sim} \mathcal{W}\omega^*_{\underline{R}/\underline{k}}.$$

As notation suggests, the map ι_φ depends on the choice of φ (compare with the canonical isomorphism in Proposition 4.18).

Remark 9 *These comparisons typically fail for general log rings $\underline{R}$ without any additional assumptions. For example, the $\mathbf{F}_p$-algebra $\mathcal{W}_1\omega^0_{\underline{R}/\underline{k}}$ is always reduced by Remark 2.5, but $\omega^0_{\underline{R}/\underline{k}} = R$ could be arbitrary. In particular, this implies the following interesting corollary.*

Corollary 10 *Let $\underline{X}$ be a fine log scheme over $\underline{k}$ that is log-smooth of log-Cartier type, then the underlying scheme X is reduced.*

This recovers a result of Tsuji in [Tsu19]. Note that, unlike the non-logarithmic case, a log-smooth scheme over a log point may fail to be reduced. For a simple example, let n be a positive integer coprime to p. The map

$$(\mathbf{F}_p, \mathbf{N} : 1 \mapsto 0) \longrightarrow (\mathbf{F}_p[x]/x^n, \mathbf{N} : 1 \mapsto x)$$

of fine log algebras is log-smooth, with the map of monoids $\mathbf{N} \to \mathbf{N}$ given by $1 \mapsto n$. The ring $\mathbf{F}_p[x]/x^n$ is obviously non-reduced when $n \geq 2$.

As another application, the formalism of saturated log de Rham–Witt complexes also provides a convenient framework to reconstruct the monodromy operator of Hyodo–Kato on log crystalline cohomology in a slightly more general setup.

Theorem 11 *Let $\underline{X}/\underline{k}$ be a log scheme of generalized semistable type (see Subsection 7) over the standard log point $\underline{k} = (k, \mathbf{N})$. Let $k^\circ = (k, 0)$ be the trivial log point. Denote the saturated log de Rham–Witt complex of $\underline{X}$ over the trivial log point by $\mathcal{W}\widetilde{\omega}^*_{\underline{X}/\underline{k}} := \mathcal{W}\omega^*_{\underline{X}/k^\circ}$. Then there is a short exact sequence of cochain complexes*

$$0 \to \mathcal{W}\omega^*_{\underline{X}/\underline{k}}[-1] \longrightarrow \mathcal{W}\widetilde{\omega}^*_{\underline{X}/\underline{k}} \longrightarrow \mathcal{W}\omega^*_{\underline{X}/\underline{k}} \to 0$$

over $X_{\text{ét}}$. The connecting homomorphism on cohomology

$$N : H^*_{\text{log-cris}}(\underline{X}/W(\underline{k})) = H^*(X, \mathcal{W}\omega^*_{\underline{X}/\underline{k}}) \longrightarrow H^*_{\text{log-cris}}(\underline{X}/W(\underline{k}))$$

satisfies $N\varphi = p\varphi N$, where φ is the functorial Frobenius on the log crystalline cohomology, and agrees with the monodromy operator constructed in [HK94] when $\underline{X}$ is of semistable type.

1.3 Outline of the paper

Section 2 is largely expository, where we summarize relevant results from [BLM18]. In Section 3, we define log Dieudonné algebras, and its subcategories consisting of p-compatible strict objects. The saturated log de Rham–Witt complex lives in the latter subcategory. Section 4 and 5 consist of the technical core of the construction of $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ and its globalization to $X_{\text{ét}}$. In Section 4 we also compare $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ to the (completed) de Rham complex of a log-smooth "log-Frobenius lifting". In section 6, we show that our construction agrees with the existing ones due to Hyodo–Kato and Matsuue in the case when the log scheme is sufficiently smooth. We then construct the monodromy operator on the log crystalline cohomology for semistable log schemes in Section 7. Appendix A is included as a briefly review of the necessary background on log schemes.

1.4 Conventions

We fix a prime p once and for all. We say that a cochain complex M^* is p-torsion free if each M^i is p-torsion free. By a cdga we mean a commutative differential graded aglebra $(A^* = \oplus_{i\geq 0} A^i, d)$. In particular, the differential operator d increases grading by $+1$ and satisfies $\mathrm{d}(ab) = (\mathrm{d}a)b + (-1)^k a(\mathrm{d}b)$ for $a \in A^k$; commutativity requires that $ab = (-1)^{kl} ba$ for $a \in A^k, b \in A^l$ and that $a^2 = 0$ for all $a \in A^{2j+1}$. The last condition is redundant unless $p = 2$.

For log schemes we refer to Appendix A, where we mostly follow [Kat89] except that we denote a log scheme (X, M_X) by $\underline{X}$. In addition, by a log algebra (R, L) we mean an algebra R together with a monoid morphism $L \to R$, this is the same data as giving a pre-log scheme $\underline{Y} = (\operatorname{Spec} R, L)$ with constant pre-log structure $\beta : L_Y \to \mathcal{O}_Y$. We often denote by $\mathcal{L}^a$ the associated log structure of L on $Y = \operatorname{Spec} R$, and denote its global section by $L^{\text{sh}} = \Gamma(Y, \mathcal{L}^a)$.

Throughout the article, k is a perfect field of characteristic p. We denote by $\underline{W}$ a log algebra $(W(k), N)$ where N could be arbitrary, and reserve the notation $W(\underline{k})$ for the log algebra $(W(k), [\alpha] : N \to W(k))$, where the log structure comes from the Teichmuller lift of a log point $\underline{k} = (k, N)$. The reader is welcome to take $\underline{W} = W(\underline{k})$ for convenience.

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2. Saturated de Rham–Witt complexes

In this section, we summarize the relevant results of Bhatt–Lurie–Mathew [BLM18].

2.1 Dieudonné algebras, saturation and V-completion

First recall the décalage operator [4].

Definition 2.1 *Let R be a ring and $\mu \in R$ a nonzero divisor. Let (M^*, d) be a cochain complex of μ-torsion free R-modules, then $(\eta_\mu M)^* \subset M^*[\frac{1}{\mu}]$ is defined to be the sub-complex given by*

$$(\eta_\mu M)^i = \{x \in \mu^i M^i : dx \in \mu^{i+1} M^{i+1}\}.$$

η_μ kills μ-torsion in the cohomology. More precisely, $H^i(\eta_\mu M^*) \cong H^i(M^*)/H^i(M^*)[\mu]$.

2.1.1 *Dieudonné algebras and saturated Dieudonné algebras*

Definition 2.2 *A Dieudonné algebra is a triple (A^*, d, F) where $(A^* = \bigoplus_{i \geq 0} A^i, d)$ is a cdga, $F : A^* \to A^*$ is a graded algebra map satisfying the conditions $dF(x) = pF(dx)$ for all $x \in A^*$, and $F(x) \equiv x^p \mod p$ for all $x \in A^0$.*

Definition 2.3 *Let $A^* = (A^*, d, F)$ be a p-torsion free Dieudonné algebra, then F determines a map of cochain complexes $\phi_F : A^* \to \eta_p A^*$ by sending $x \mapsto p^n F(x)$ for $x \in A^n$. A Dieudonné algebra A^* is saturated if it is p-torsion free and ϕ_F is an isomorphism.*

Morphisms between two Dieudonné algebras are morphisms between cdgas compatible with the Frobenius maps. The category of Dieudonné algebras (resp. the full subcategory spanned by saturated algebras) is denoted by DA (resp. $\mathrm{DA}_{\mathfrak{sat}}$)

2.1.2 *Verschiebung* Consider $A^* \in \mathrm{DA}_{\mathfrak{sat}}$. For each $i \in \mathbf{Z}$, the composed map

$$\phi_F : A^i \xrightarrow{F} \{x \in A^i : dx \in pA^{i+1}\} \xrightarrow{\times p^i} (\eta_p A)^i$$

is an isomorphism, hence F is injective and $F(A^*)$ contains pA^*. Therefore, for each $x \in A^n$, there is a unique element Vx such that $F(Vx) = px$. It is straightforward to check that

$$FV = VF = p, \quad FdV = d, \quad Vd = p\, dV, \quad xVy = V(Fx \cdot y).$$

2.1.3 *Saturation* Suppose $A^* \in \mathrm{AD}$ is p-torsion free, then $(\eta_p A)^*$ with its inherited differential and Frobenius structure is again a Dieudonné algebra. The only thing to check is that for any $x \in (\eta_p A)^0$, $F(x) = x^p + py$ for some $y \in (\eta_p A)^0$ (not just in A^0). The map $\phi_F : A^* \to (\eta_p A)^*$

[4] The décalage operator was first introduced in [BO78]. It was later used in [IR83] for the crystalline construction of the de Rham–Witt complex $W\Omega^*_{X/k}$ (and similarly in [HK94] to construct the log de Rham–Witt complexes from the log crystalline site). Recently it appeared in [BMS18] to define $\mathrm{A_{inf}}$-cohomology.

is a morphism of Dieudonné algebras. The inclusion functor $\mathrm{AD}_{\mathfrak{sat}} \hookrightarrow \mathrm{AD}$ admits a left adjoint, $A^* \mapsto A^*_{\mathfrak{sat}}$, which we call the **saturation** of A^*, described as follows. We replace $A^* \in \mathrm{AD}$ by its p-torsion free quotient if necessary and suppose that A^* is p-torsion free, and then dfine $A^*_{\mathfrak{sat}}$ to be the direct limit of

$$A^* \xrightarrow{\phi_F} (\eta_p A)^* \xrightarrow{\eta_p(\phi_F)} (\eta_p \eta_p A)^* \xrightarrow{\eta_p^2(\phi_F)} (\eta_p^3 A)^* \to \cdots$$

By the discussion above, the saturation $A^*_{\mathfrak{sat}}$ inherits the structure of a Dieudonné algebra, and the natural map $A^* \to A^*_{\mathfrak{sat}}$ is a morphism of Dieudonné algebras.

2.1.4 *Strict Dieudonné algebras and V-completion* Let A^* be a saturated Dieudonné algebra. For each $r \geq 1$, we form the quotient

$$W_r(A^*) := A^*/(V^r A^* + dV^r A^*),$$

which is a cdga as $V^r A^* + dV^r A^*$ is a differential graded ideal. Next we define the V-completion of A^* to be the limit $W(A^*) := \varprojlim W_r A^*$ along the natural projection maps $R : W_r A^* \to W_{r-1} A^*$. It is easy to check that the map F on A^* induces $F : W_r(A^*) \to W_{r-1}(A^*)$ on the quotients. Similarly, the Verschiebung map induces maps $V : W_r(A^*) \to W_{r+1}(A^*)$.

Definition 2.4 *A saturated Dieudonné algebra A^* is strict if the canonical map $A^* \longrightarrow W(A^*)$ is an isomorphism. The full subcategory of $\mathrm{DA}_{\mathfrak{sat}}$ spanned by strict algebras is denoted by $\mathrm{DA}_{\mathfrak{str}}$.*

Remark 2.5 *Let A^* be a saturated Dieudonné algebra as above.*

- *A^0/VA^0 is a reduced $\mathbf{F}_p$-algebra by Lemma 3.6.1 of [BLM18].*
- *$W(A^*)$ in the definition above becomes a Dieudonné algebra with F the inverse limit of $F : W_r(A^*) \to W_{r-1}(A^*)$. It is in fact still saturated. One needs to check that (i). $W(A^*)$ is p-torsion free, (ii). if $x \in A^i := (W(A^*))^i$ is an element with $dx \in pW^{i+1}$, then $x \in \mathrm{im}(F)$, and (iii). the inverse limit F on $W(A^*)$ satisfies $F(x) \equiv x^p \mod p$, for the detail we refer to Section 2.6 and 3.5 of [BLM18].*

Remark 2.6 *The V-completion $W(A^*)$ for $A^* \in \mathrm{DA}_{\mathfrak{sat}}$ is strict. More precisely, for each $r \geq 1$, the canonical map $W_r(A^*) \to W_r(W(A^*))$ is an isomorphism (by [BLM18] Proposition 2.7.5). The completion functor $A^* \mapsto W(A^*)$ provides a left adjoint of the inclusion $\mathrm{DA}_{\mathfrak{str}} \hookrightarrow \mathrm{DA}_{\mathfrak{sat}}$.*

Remark 2.7 *For a strict Dieudonné algebra A^*, each A^i is p-complete. Moreover, A^0 is the ring of Witt-vectors of $W_1(A)^0 = A^0/VA^0$. In other words, there is a canonical isomorphism $\mu : A^0 \to W(A^0/VA^0)$ such that the composition $A^0 \to W(A^0/VA^0) \xrightarrow{\mathrm{can}} A^0/VA^0$ is the projection map. The Frobenius F on A^0 corresponds to the Witt vector Frobenius under μ.*

2.2 The Cartier criterion

Let $A \in \mathrm{DA}$ be a p-torsion free Dieudonné algbera. Consider the cochain complex $(H^*(A^*/pA^*), \beta)$ where β is the Bockstein differential induced from $0 \to A^*/p \xrightarrow{p} A^*/p^2 \to A^*/p \to 0$. Then we have the following commutative diagrams where each arrow is a map of cochain complexes:

$$\begin{array}{ccc} (A^*/pA^*, d) & \xrightarrow{\quad F \quad} & (H^*(A^*/pA^*), \beta) \\ & {\scriptstyle \phi_F} \searrow \quad \nearrow {\scriptstyle \gamma} & \\ & (\eta_p(A^*)/p, d) & \end{array}$$

Here γ is defined by sending $x \in (\eta_p A)^i$ to x/p^i. Suppose that A^* is in addition saturated, then the composition F in the diagram factors through the quotient cochain complex $W_1(A^*)$, as in the diagram below:

$$\begin{array}{ccc} & (W_1(A^*), d) & \\ \nearrow^{q} & & \searrow^{F_1} \\ (A^*/pA^*, d) & \xrightarrow{\quad F \quad} & (H^*(A^*/pA^*), \beta) \end{array}$$

Lemma 2.8 ([BLM18] Proposition 2.4.1 & 2.7.1) *Let A^* be a p-torsion free Dieudonné algebra, then*

(1) $\gamma : \eta_p(A^*)/p \to H^*(A^*/p)$ is a quasi-isomorphism of cochain complexes;

(2) Suppose that A^* is saturated, then $F_1 : W_1(A^*) \to H^*(A^*/p)$ is an isomorphism.

Definition 2.9 *We say that a Dieudonné algebra $A^* \in \mathrm{DA}$ is of Cartier type (or satisfies the Cartier criterion) if $F : (A^*/p, d) \to (H^*(A^*/p), \beta)$ is an isomorphism of cochain complexes.*

In the case in the above definition, we denote the map F^{-1} by C.

Corollary 2.10 *Let A^* be a p-torsion free Dieudonné algebra.*

(1) If A^* is of Cartier type, then $\phi_F : A^*/p \to \eta_p(A^*)/p$ is a quasi-isomorphism, therefore we have a quasi-isomorphism

$$A^*/pA^* \longrightarrow A^*_{\mathfrak{sat}}/pA^*_{\mathfrak{sat}}$$

(2) If A^* is saturated, then $q : A^*/p \to W_1(A^*)$ is a quasi-isomorphism of cochain complexes.

(3) Let $f : A^* \to B^*$ be a morphism of saturated Dieudonné algebras, then $f : A^*/p \to B^*/p$ is a quasi-isomorphism if and only if $f : W_1(A^*) \to W_1(B^*)$ is an isomorphism, if and only if $f : W(A^*) \to W(B^*)$ is an isomorphism (see Corollary 2.7.4 in [BLM18]).

Corollary 2.11 *Let A^* be a p-complete and p-torsion free Dieudonné algebra satisfying the Cartier criterion, then the canonical map $A^* \longrightarrow W(A^*_{\mathfrak{sat}})$ is a quasi-isomorphism.*

Proof. It suffices to show that $A^*/p \to A^*_{\mathfrak{sat}}/p \to W(A^*_{\mathfrak{sat}})/p$ is a quasi-isomorphism since each A^i is p-adically complete and p-torsion free. The first map is a quasi-isomorphism by Corollary 2.10 (1), and the second map is a quasi-isomorphism by Remark 2.6 and Corollary 2.10 (3). □

3. Log Dieudonné algebras

In this section we describe an enhancement of Dieudonné algebras that incorporates log structures, which serves as a basis for our theory of saturated log de Rham–Witt complexes on local charts. Some of the definitions in this section are analogous to those in [BLM18], with minor differences. In particular, we prove that, for a strict Dieudonné algebra with a p-compatible log structure, its log structure is "valued in" Teichmuller representatives (Proposition 3.12). For notations on log geometry we refer to Subsection 1.4 and Appendix A. In this section and in the construction of saturated log de Rham–Witt complexes, we do not require the log structures to satisfy additional properties such as being integral or coherent.

3.1 Log Dieudonné algebras

Definition 3.1 *Let $\underline{W} = (W(k), N)$ be a log algebra. A log Dieudonné $\underline{W}$-algebra is a tuple $(A^*, L, d, \delta, F, F_L)$, consisting of the following data:*

- $(A^* = \oplus_{i \geq 0} A^i, d)$ *is a cdga over* W,
- $\underline{A}^0 = (A^0, L)$ *is a log algebra over* $\underline{W}$,
- $\delta : L \to A^1$ *is a map of monoids, and*
- $F : A^* \to A^*$ *a graded algebra homomorphism,*

satisfying the following requirements:

(1) (A^*, d, F) *is a Dieudonné algebra in the sense of Definition 2.2.*

(2) $(d : A^0 \to A^1, \delta : L \to A^1)$ *is a log derivation of* $\underline{A}^0/\underline{W}$, *where we further require that the composition* $L \xrightarrow{\delta} A^1 \xrightarrow{d} A^2$ *is* 0.

(3) δ *is compatible with* $F : A^1 \to A^1$, *in other words it satisfies* $\delta F_L = pF\delta$.

(4) $F_L : L \to L$ *is a monoid homomorphism satisfying* $F \circ \alpha = \alpha \circ F_L$.

Notation. We often suppress notations and simply write A^* for a log Dieudonné algebra.

Remark 3.2 *In contrast to the non-logarithmic setting, for the definition of log Dieudonné algebras we specify a base $\underline{W}$. In the non-logarithmic case, $\Omega_{k/\mathbf{F}_p} = 0$ (and $\Omega_{W(k)/\mathbf{Z}_p} = 0$) for a perfect field k of char p, so the base is to some extent irrelevant (as one can see from the construction in Subsection 4.2, for example). On the other hand, if we have two distinct log structures N_1, N_2 on k, $\Omega_{(k,N_1)/(k,N_2)}$ is typically nonzero.*

Definition 3.3 *A morphism between two log Dieudonné $\underline{W}$-algebras A^* and B^* is a pair (f, ψ), where $f : (A^*, d, F) \to (B^*, d', F')$ is a morphism between Dieudonna algebras over W, and $\psi : L_A \to L_B$ is a monoid morphism over N which is compatible with F_L, α and δ:*

(1) $f \circ d = d' \circ f$ *and* $f \circ F = F' \circ f$ *on each* A^i.

(2) The structure map $N \to L_B$ *agrees with* $N \to L_A \xrightarrow{\psi} L_B$ *(resp.* $W \to B^0$ *agrees with* $W \to A^0 \xrightarrow{f|_{A^0}} B^0$*). Moreover* $f \circ \alpha_A = \alpha_B \circ \psi$ *on* L_A. *This is simply saying that* $(f, \psi)|_{(A^0, L_A)}$ *is a morphism of log* $\underline{W}$*-algebras.*

(3) ψ *satisfies* $\psi \circ F_{L_A} = F_{L_B} \circ \psi$, *and* $f \circ \delta_A = \delta_B \circ \psi$.

Notation. We denote the category of log Dieudonné algebras by $\mathrm{DA}^{\log}_{/\underline{W}}$, and we often suppress notations to write $\mathrm{DA}^{\log} = \mathrm{DA}^{\log}_{/\underline{W}}$ when $\underline{W}$ is understood.

Lemma 3.4 *A morphism $(f, \psi) : A^* \to B^*$ of log Dieudonné algebras over $\underline{W}$ is an isomorphism if and only if both f and ψ are isomorphisms.*

Proof. Suppose that f has an inverse $g : B^* \to A^*$ as Dieudonné algebras over W and that ψ has an inverse $\chi : L_B \to L_A$, the lemma claims that χ satisfies the compatibilities (2) and (3) in Definition 3.3. This is straight-forward. □

3.2 p-compatible log Dieudonné algebras

Definition 3.5 *We say that a log Dieudonné algebra A^* over $\underline{W}$ is p-compatible if the Frobenius on the log structure L is given by $F_L = p$.*

Let $\mathrm{DA}^{\log,\mathrm{p}}$ denote the full subcategory of $\mathrm{DA}^{\log}$ spanned by p-compatible objects. In this article we are mostly interested in p-compatible log Dieudonné algebras (see Proposition 4.5).

Remark 3.6 *The p-compatibility condition imposes some restrictions on what (pre-)log structures are allowed. For example, for a log structure $\mathcal{L}$ on an affine $\mathbf{Z}_p$-scheme $\mathfrak{X} = \operatorname{Spec} A$, the log algebra $(A, \Gamma(\mathfrak{X}, \mathcal{L}))$ is usually not p-compatible. A typical example that arises is $(\mathbf{Z}_p[x,y]/xy, L = \mathbf{N}^2)$ with the pre-log structure $(a,b) \mapsto x^a y^b$ and Frobenius F_L given by multiplication by p. On the (sections of the) associated log structure $L^a = \mathbf{Z}_p^\times \oplus \mathbf{N}^2$, the induced Frobenius is $F_{L^a} = \mathrm{id} \oplus F_L$. For this reason we prefer to work with pre-log structures.*

3.3 Saturated log Dieudonné algebras

In this subsection and next we discuss saturation and V-completion in the logarithmic setting.

Definition 3.7 *Let $(A^*, L \xrightarrow{\alpha} A^0, d, \delta, F, F_L)$ be a p-torsion free log Dieudonné algebra. We define $\eta_p(A^*)$ to be the tuple $\left(\eta_p(A^*), L \xrightarrow{\eta_p(\alpha)} \eta_p A^0, d, \eta_p(\delta), F, F_L\right)$ where*

- *$\eta_p A^i = \{x \in p^i A^i : dx \in p^{i+1} A^{i+1}\}$ as in Definition 2.1,*
- *the log algebra $(\eta_p A^0, L)$ is given by $\eta_p(\alpha) : L \xrightarrow{F_L} F_L(L) \xrightarrow{\alpha} \eta_p A^0$,*
- *the log derivation $\eta_p(\delta)$ is given by $\eta_p(\delta) : L \xrightarrow{F_L} F_L(L) \xrightarrow{\delta} \eta_p A^1$,* [5]

Both $\eta_p(\alpha)$ and $\eta_p(\delta)$ are well-defined by the relations $dF = pFd$ and $\delta F = pF\delta$. It is straightforward to verify that $\eta_p(A^*)$ is in fact a log Dieudonné $\underline{W}$-algebra, with its $\underline{W}$-structure given by $W \to A^0 \to (\eta_p A)^0$ and $N \to L$ (compare with Remark 3.1.7 in [BLM18]). Now let $\mathrm{DA}^{\log}_{p\text{-tf}}$ be the full subcategory of $\mathrm{DA}^{\log}$ spanned by p-torsion free complexes, then η_p induces a logarithmic décalage operator $\mathrm{DA}^{\log}_{p\text{-tf}} \longrightarrow \mathrm{DA}^{\log}_{p\text{-tf}}$. For a morphism $f = (f, \psi) : A^* \to B^*$ between log Dieudonné algebras, one obtains a morphism $\eta_p(f) : \eta_p(A^*) \to \eta_p(B^*)$ in a functorial way. The operator η_p clearly preserves the p-compatible objects. Moreover, for a p-torsion free object A^* in the category $\mathrm{DA}^{\log}$ (resp. in $\mathrm{DA}^{\log,\mathrm{p}}$), we have a natural map

$$\phi_F = (\phi_F, \mathrm{id}) : A^* \longrightarrow \eta_p(A^*)$$

of log Dieudonné algebras, where ϕ_F is the map $p^i F$ on A^i, which by 2.1.3 is a morphism between Dieudonné algebras.

Definition 3.8 *A log Dieudonné algebra $A^* \in \mathrm{DA}^{\log}$ is saturated if A^* is p-torsion free and ϕ_F is an isomorphism, in other words, if the underlying Dieudonné algebra is saturated.*

Notation. The full subcategory of $\mathrm{DA}^{\log}$ spanned by saturated objects (resp. p-compatible saturated objects) is denoted by $\mathrm{DA}^{\log}_{\mathfrak{sat}}$ (resp. $\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{sat}}$).

[5] Equivalently, $\eta_p(\delta)$ is given by the composition $L \xrightarrow{\delta} A^1 \xrightarrow{pF} \eta_p A^1$. Likewise, $\eta_p(\alpha)$ is given by the composition $L \xrightarrow{\alpha} A^0 \xrightarrow{F} \eta_p A^0$.

Remark 3.9 *The definition of being saturated has nothing to do with the log structure L. Moreover, by the same discussion in Subsection 2.1.2, a saturated log Dieudonné algebra admits the Verschiebung operator $V : A^i \to A^i$ satisfying the relations listed there.*

Since the log structure is irrelevant in the condition of being saturated, the construction of the saturation of a log Dieudonné algebra is essentially the same as in the non-logarithmic setup. First we replace $\underline{A}^* = (A^*, L)$ by $(A^*/A^*[p^\infty], L)$ to reduce to the p-torsion free case while keeping the monoid L. For a p-torsion free log Dieudonné algebra A^*, we apply the log décalage operator η_p to the morphism $\phi_F = (\phi_F, \mathrm{id}) : A^* \to \eta_p(A^*)$ repeatedly and obtain

$$(A^*)_{\mathfrak{sat}} := \operatorname{colim}\big(A^* \xrightarrow{\phi_F} \eta_p(A^*) \xrightarrow{\phi_F} \eta_p\eta_p(A^*) \to \cdots\big)$$

More explicitly, we take $L_{\mathfrak{sat}} := \operatorname{colim}\Big(L \xrightarrow{F_L} F_L(L) \xrightarrow{F_L} F_LF_L(L) \to \cdots\Big)$ in the category of monoids (colimit exists in the category of monoids, as both coproducts and coequalizers exist). This is compatible with the colimit $\varinjlim_{n,\phi_F} \eta_p^n(A^*)$, hence we get natural monoid maps $\alpha_{\mathfrak{sat}} : L_{\mathfrak{sat}} \to (A_{\mathfrak{sat}})^0$ and $\delta_{\mathfrak{sat}} : L_{\mathfrak{sat}} \to (A_{\mathfrak{sat}})^1$. The saturation is then

$$\Big((A^*)_{\mathfrak{sat}}, L \to L_{\mathfrak{sat}} \xrightarrow{\alpha_{\mathfrak{sat}}} (A_{\mathfrak{sat}})^0, L \to L_{\mathfrak{sat}} \xrightarrow{\delta_{\mathfrak{sat}}} (A_{\mathfrak{sat}})^1\Big).$$

3.4 Strict log Dieudonné algebras and V-completion

Let $A^* \in \mathrm{DA}^{\log}$ be a saturated object, in Subsection 2.1 we formed the inverse limit $W(A^*) = \varprojlim W_r(A^*)$ along the restriction maps $R : W_r(A^*) \to W_{r-1}(A^*)$, where $W_r(A^*) = A^*/\mathrm{Fil}^r$ is the quotient of A^* by the differential graded ideal $\mathrm{Fil}^r = V^r(A^*) + \mathrm{d}V^r(A^*) \subset A^*$ (which defines the V-filtration on A^*). The Dieudonné algebra $W(A^*)$ inherits a log structure $L \to A^0 \to W(A)^0$ and a log derivation $L \to A^1 \to W(A)^1$ from A^*, which makes it a log Dieudonné algebra. The canonical map $\rho : A^* \to W(A^*)$ of saturated Dieudonné algebras upgrades to (ρ, id) as a morphism in $\mathrm{DA}^{\log}$, which we still denote by ρ. By Remark 2.5, we know that, if $A^* \in \mathrm{DA}^{\log}$ is saturated (resp. saturated and p-compatible), then $W(A^*)$ is again saturated (resp. saturated and p-compatible).

Definition 3.10 *A saturated log Dieudonné algebra A^* is strict if $\rho : A^* \to W(A^*)$ is an isomorphism, in other words, if its underlying Dieudonné algebra is strict.*

Notation. The full subcategory of $\mathrm{DA}^{\log}_{\mathfrak{sat}}$ (resp. $\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{sat}}$) spanned by strict objects is denoted by $\mathrm{DA}^{\log}_{\mathfrak{str}}$ (resp. $\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}$).

Lemma 3.11 *(1) The saturation functor $\mathfrak{sat} : \mathrm{DA}^{\log} \to \mathrm{DA}^{\log}_{\mathfrak{sat}}$ is the left adjoint of the inclusion $\mathrm{DA}^{\log}_{\mathfrak{sat}} \subset \mathrm{DA}^{\log}$. The same is true for $\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{sat}}$.*

(2) The completion $W(A^*)$ of a saturated log Dieudonné algebra is strict. Moreover the completion functor $W : \mathrm{DA}^{\log}_{\mathfrak{sat}} \to \mathrm{DA}^{\log}_{\mathfrak{str}}$ provides a left adjoint of the inclusion $\mathrm{DA}^{\log}_{\mathfrak{str}} \subset \mathrm{DA}^{\log}_{\mathfrak{sat}}$, similarly for the p-compatible subcategory.

Proof. Immediate from definitions and Subsection 2.1. □

The condition of being strict and p-compatible imposes some rather strict restrictions on the log structures. For the setup, let $A^* \in \mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}$, and recall from Section 2 that we have a canonical isomorphism $\mu : A^0 \xrightarrow{\sim} W(A^0/VA^0)$ respecting Frobenii on both sides. We use μ to identify A^0 with the Witt vectors of A^0/VA^0 in the following proposition, which is important for our construction of log de Rham–Witt complexes in Section 4.

Proposition 3.12 *Let $A^* \in \mathrm{DA}_{\mathfrak{str}}$ be a strict Dieudonné algebra. Let $\alpha : L \to A^0$ be a log structure on A^0, such that $F(\alpha(m)) = \alpha(m)^p$. Then the image of α consists of Teichmuller lifts of elements of A^0/VA^0. In other words, consider the log algebra $\overline{\alpha} : L \to A^0 \to A^0/VA^0$ with log structure inherited from α, then*

$$\alpha(m) = [\overline{\alpha}(m)]$$

for any $m \in L$. This in particular applies to the case of a strict p-compatible log Dieudonné algebra.

Proof. It suffices to show that $\alpha(m) - [\overline{\alpha}(m)] \in \mathrm{Fil}^s(A^0) = V^s(A^0)$ for every $s \in \mathbf{Z}_{\geq 1}$ since A^0 is complete with respect to the V-filtration. We proceed by induction. By assumption $\alpha(m) - [\overline{\alpha}(m)] \in \mathrm{Fil}^1$. We claim that if $\alpha(m) - [\overline{\alpha}(m)] \in \mathrm{Fil}^r$ for $r \geq 1$, then $\alpha(m) - [\overline{\alpha}(m)] \in \mathrm{Fil}^{2r}$. Suppose that $\alpha(m) - [\overline{\alpha}(m)] = V^r b$, then

$$F^r(\alpha(m)) - F^r([\overline{\alpha}(m)]) = \alpha(m)^{p^r} - [\overline{\alpha}(m)]^{p^r} = p^r \cdot b.$$

In the first equality we use that A^* is p-compatible. On the other hand

$$\begin{aligned}\alpha(m)^{p^r} &= \Big([\overline{\alpha}(m)] + V^r b\Big)^{p^r} \\ &= [\overline{\alpha}(m)]^{p^r} + p^r[\overline{\alpha}(m)]^{p^r-1} \cdot V^r b + \sum_{k=2}^{p^r} \binom{p^r}{k} [\overline{\alpha}(m)]^{p^r-k} \cdot \big(V^r b\big)^k\end{aligned}$$

Note that since $xV^r y = V^r((F^r x)y)$, we have

$$(V^r b)^2 = V^r((F^r V^r b)b) = V^r(p^r b^2) = p^r V^r(b^2).$$

Therefore

$$p^r b = p^r[\overline{\alpha}(m)]^{p^r-1} \cdot V^r b + p^r V^r(b^2) \cdot \sum_{k=2}^{p^r} \binom{p^r}{k} [\overline{\alpha}(m)]^{p^r-k} \cdot \big(V^r b\big)^{k-2}$$

Since A^0 is p-torsion free, we conclude that $b \in V^r(A^0)$. □

Corollary 3.13 *Retain notations from above. Let $\underline{A}^0/V\underline{A}^0$ be the log algebra $(A^0/VA^0, \overline{\alpha} : L \to A^0/V\underline{A}^0)$, and $W(\underline{A}^0/V\underline{A}^0)$ be the log algebra obtained via the Teichmuller lifts. Then $\mu : A^0 \xrightarrow{\sim} W(A^0/VA^0)$ induces an isomorphism of log algebras*

$$\underline{\mu} = (\mu, id) : \underline{A}^0 = (A^0, L) \to W(\underline{A}^0/V\underline{A}^0).$$

Proof. This is immediate from Proposition 3.12. □

4. Saturated log de Rham–Witt complexes

The goal of this section is to construct the saturated log de Rham–Witt complexes $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ for a log scheme $\underline{X}$ over $\underline{k}$, which is a sheaf on $X_{\text{ét}}$ valued in log Dieudonné algebras. We construct this object locally on charts in Subsection 4.1, and compare it with de Rham complexes in the subsequent subsections. In Subsection 5 we glue the local construction on the étale site of X.

For this section, we fix the base log algebra $\underline{W} = W(\underline{k})$ where $\underline{k} = (k, N)$ as before[6]. In

[6]This is less general than the previous section, namely we require that the log structure on $W(k)$ comes from the Teichmuller lift from log structure on k. Examples include the trivial log structure on $W(k)$, or $\mathbf{N} \xrightarrow{1 \mapsto 0} W(k)$, but excludes $\mathbf{N} \xrightarrow{1 \mapsto p} W(k)$.

Subsection 4.1, we do not impose any condition on the log structures, but later on we gradually impose restrictions on the log algebra to prove various forms of comparison theorems.

4.1 Saturated log de Rham–Witt complexes for log algebras

Let $\mathrm{Alg}^{\log}_{\underline{k}}$ be the category of log algebras over $\underline{k}$. An element $\underline{R} = (R, M) \in \mathrm{Alg}^{\log}_{\underline{k}}$ can be equipped with a pair of Frobenius (F, F_M) where $F(x) = x^p$ for all $x \in R$ and $F_M(m) = pm$ for all $m \in M$. From the previous section, we have a functor

$$\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}} \longrightarrow \mathrm{Alg}^{\log}_{\underline{k}} \qquad \text{given by} \quad A^* \mapsto \underline{A}^0/V(A^0),$$

where $\underline{A}^0/V(A^0)$ consists of the data $(A^0/V(A^0), \alpha : L \to A^0 \twoheadrightarrow A^0/V(A^0))$. Now we are ready to give the definition of saturated log de Rham–Witt complexes for log algebras (Definition 1 in the introduction).

Definition 4.1 *Let $\underline{R} = (R, M) \in \mathrm{Alg}^{\log}_{\underline{k}}$ be a log algebra over $\underline{k}$. The saturated log de Rham–Witt complex $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ of $\underline{R}$ is a strict p-compatible log Dieudonné $W(\underline{k})$-algebra equipped with an isomorphism $e : \underline{R} \xrightarrow{\sim} \mathcal{W}_1\omega^0_{\underline{R}/\underline{k}}$, such that for any $A^* \in \mathrm{DA}^{\log,p}_{\mathfrak{str}}$, the natural map*

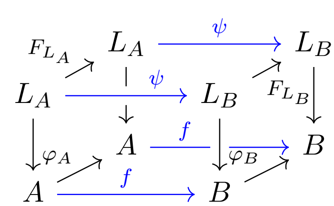
$$\mathrm{Hom}_{\mathrm{DA}^{\log}_{\mathfrak{str}}}(\mathcal{W}\omega^*_{\underline{R}/\underline{k}}, A^*) \longrightarrow \mathrm{Hom}_{\mathrm{Alg}^{\log}_{\underline{k}}}(\underline{R}, \underline{A}^0/V(A^0))$$

is an isomorphism. In particular, the association $\underline{R} \mapsto \mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ (if exists) provides a left adjoint of the functor $\mathrm{DA}^{\log,p}_{\mathfrak{str}/\underline{W}} \longrightarrow \mathrm{Alg}^{\log}_{\underline{k}}$ sending $A^* \longmapsto (A^0/V(A^0), L)$.

Remark 4.2 *The restriction to the subcategory of p-compatible objects seems necessary to obtain the desired universal property. In particular, we do not obtain a left adjoint of the functor $\mathrm{DA}^{\log}_{\mathfrak{str}} \to \mathrm{Alg}^{\log}_{\underline{k}}$ from the larger category $\mathrm{DA}^{\log}_{\mathfrak{str}}$.*

4.1.1 *Preliminaries* I Let $\underline{A} = (A, L_A)$ and $\underline{B} = (B, L_B)$ be two log $\underline{W}$-algebras equipped with Frobenius maps (φ_A, F_{L_A}) and (φ_B, F_{L_B}), where φ_A (resp. φ_B) lifts the p-power Frobenius on A/p (resp. B/p).

Notation 4.3 *We denote by $\mathrm{Hom}_F(\underline{A}, \underline{B})$ the set of Frobenius compatible $\underline{W}$-morphisms between $\underline{A}$ and $\underline{B}$, which are pairs (f, ψ) of morphisms making every square in the cube below commute.*



The following setup is of particular interest. Let $\underline{R}, \underline{R}' \in \mathrm{Alg}^{\log}_{\underline{k}}$ be two log algebras over $\underline{k}$. Let $W(\underline{R}) = (W(R), [\alpha])$, where $[\alpha] = [\ ] \circ \alpha : M \to R \to W(R)$ is the Teichmuller lift of the log structure on R. $W(\underline{R})$ is equipped with the Frobenius map (F, p) where F is the Witt vector Frobenius. Similarly define $W(\underline{R}')$. Then $\mathrm{Hom}_F(W(\underline{R}), W(\underline{R}'))$ consists precisely of the homomorphisms $(f, \psi) : W(\underline{R}) \to W(\underline{R}')$ of log $\underline{W}$-algebras, such that $f \circ F = F \circ f$.

Lemma 4.4 *Let $\underline{R}, \underline{R}' \in \mathrm{Alg}^{\log}_{\underline{k}}$ be as above. Suppose that both R and R' are reduced, then*

$$\mathrm{Hom}_F(W(\underline{R}), W(\underline{R}')) \cong \mathrm{Hom}_{\mathrm{Alg}^{\log}_{\underline{k}}}(\underline{R}, \underline{R}')$$

Proof. Let $(\overline{f}, \psi) : \underline{R} \to \underline{R}'$ be a map over $\underline{k}$. We observe that, since $W(R)$ is p-torsion free (since R is reduced), and $F : W(R) \to W(R)$ lifts the Frobenius on R, the map $W(R) \to R \xrightarrow{\overline{f}} R'$ lifts uniquely [7] to a map $f : W(R) \to W(R')$ which is compatible with Frobenius on both $W(R)$ and $W(R')$. Similarly we have structure morphisms $W(k) \to W(R)$ and $W(k) \to W(R')$ such that $f : W(R) \to W(R')$ is a map of $W(k)$-algebras (by uniqueness of the lifting).

Now we show the compatibility between f and ψ. This is implied by the compatibility between f and taking Teichmuller lifts: namely that $f([x]) = [\overline{f}(x)]$ for every $x \in R$. Since $W(R')$ is complete with respect to the V-filtration, it suffices to show that $f([x]) - [\overline{f}(x)] \in \mathrm{Fil}^s = V^s W(R')$ for every $s \in \mathbf{N}$. We proceed by induction. Clearly $f([x]) - [\overline{f}(x)] \in \mathrm{Fil}^1$, now suppose that $f([x]) - [\overline{f}(x)] \in \mathrm{Fil}^r$, then $f([x]) = [\overline{f}(x)] + V^r(y)$ for some $y \in W(R')$, so we have

$$F(f[x]) = f([x])^p = \Big([\overline{f}(x)] + V^r(y)\Big)^p \equiv F[\overline{f}(x)] \mod pV^r W(R')$$

which implies that there exists some $y' \in W(R')$ such that

$$F(f[x]) = F[\overline{f}(x)] + pV^r(y') = F\Big([\overline{f}(x)] + V^{r+1}(y')\Big).$$

But F is injective on $W(R')$ since R' is reduced, hence $f[x] = [\overline{f}(x)]$.

Applying the same argument again, we see that the map (f, ψ) is a morphism over $W(\underline{k})$, where the latter is equipped with the log structure $N \to k \xrightarrow{[\,]} W(k)$. □

4.1.2 *A key example of log Dieudonné algebra* The construction of log de Rham–Witt complexes for $\underline{R} \in \mathrm{Alg}^{\log}_{\underline{k}}$ relies on the following proposition.

Proposition 4.5 *Let $(A, \alpha : L \to A)$ be a log algebra over $\underline{W}$. Suppose that A is p-torsion free and that the pair (A, L) is equipped with morphisms $\varphi : A \to A$ and $F_L : L \to L$ such that*

- *$\varphi(x) \equiv x^p \mod p$ is a lifting of the absolute Frobenius mod p;*
- *$F_L = p$ on L and is compatible with φ, that is, $\varphi \circ \alpha = \alpha \circ F_L$.*

Then there exists a unique graded ring homomorphism $F : \omega^*_{\underline{A}/\underline{W}} \longrightarrow \omega^*_{\underline{A}/\underline{W}}$ which extends φ on A, such that

$$F(dx) = x^{p-1}dx + d\Big(\frac{\varphi(x) - x^p}{p}\Big) \quad \textit{for all } x \in A.$$

Moreover, this makes $\underline{\omega^*} := (\omega^*_{\underline{A}/W(\underline{k})}, L, d, \delta, F, F_L)$ a p-compatible log Dieudonné $\underline{W}$-algebra.

Proof. For the first claim we need to show the existence of $F : \omega^1_{\underline{A}/W(\underline{k})} \to \omega^1_{\underline{A}/W(\underline{k})}$. We regard the second copy of $\omega^1_{\underline{A}/W(\underline{k})}$ as an A-module where A acts by a twist of φ, namely $a \cdot \lambda = \varphi(a)\lambda$ for all $\lambda \in \omega^1_{\underline{A}/W(\underline{k})}$. By the universal property of $\omega^1_{\underline{A}/W(\underline{k})}$ and Remark A.5, it suffices to supply the structure of a log derivation from (A, L) to $\omega^1_{\underline{A}/W(\underline{k})}$ with this twisted A-module structure. For this we define

$$\widetilde{\delta} : L \to \omega^1_{\underline{A}/W(\underline{k})}, \quad \text{and} \quad \widetilde{d} : A \to \omega^1_{\underline{A}/W(\underline{k})}$$

respectively as follows:

- we set $\widetilde{\delta}(l) := \delta(l)$, and

[7] This is sometimes called Cartier's Dieudonné-Dwork lemma. Another way to produce this lifting is to observe that the Witt vectors provide a right adjoint to the forgetful functor from the category of δ-rings to rings, in the sense of [Joy85].

– $\widetilde{d}(x) := x^{p-1}dx + d\theta(x)$ where $\theta(x) = \dfrac{\varphi(x) - x^p}{p}$. [8]

To proceed we need to show that $\widetilde{\delta}$ and $\widetilde{d}$ form a log derivation of $\underline{A}$ over $W(\underline{k})$, which requires

(1) $\widetilde{d}(\gamma x) = \gamma \cdot \widetilde{d}x := F(\gamma)\widetilde{d}x$ for all $\gamma \in W(k), x \in A$;
(2) $\widetilde{d}(x + y) = \widetilde{d}(x) + \widetilde{d}(y)$ for all $x, y \in A$;
(3) $\widetilde{d}(xy) = x \cdot \widetilde{d}y + y \cdot \widetilde{d}x := \varphi(x)\widetilde{d}y + \varphi(y)\widetilde{d}x$ for all $x, y \in A$;
(4) $\alpha(l) \cdot \widetilde{\delta}(l) := \varphi(\alpha(l))\widetilde{\delta}(l) = \widetilde{d}(\alpha(l))$ for all $l \in L$;
(5) $\widetilde{\delta}(\iota(n)) = 0$ for all $n \in N$, where $\iota : N \to L$ is map of monoids in $W(\underline{k}) \to \underline{A}$.

(1) follows from (3) since $p\widetilde{d}(\lambda) = d(F(\lambda)) = 0$, where $F : W(k) \to W(k)$ is the Frobenius map on Witt vectors. (2) and (3) follow from the explicit description of $\theta(x)$ (for example see [BLM18] Proposition 3.2.1). For (4), since $\theta(\alpha(l)) = 0$, we have

$$\varphi(\alpha(l))\widetilde{\delta}(l) = \alpha(l)^p\delta(l) = \alpha(l)^{p-1}d(\alpha(l)) = \widetilde{d}(\alpha(l)).$$

(5) is clear since $\delta(\iota(n)) = 0$ for all $n \in N$. We thus get the desired map $F : \omega^1_{\underline{A}/W(\underline{k})} \to \omega^1_{\underline{A}/W(\underline{k})}$ which satisfies $F(\delta(l)) = \widetilde{\delta}(l) = \delta(l)$ and $Fd(x) = \widetilde{d}(x) = x^{p-1}dx + d\theta(x)$.

To finish the proof of the proposition, we need to check that (i). $dF = pFd$ on all of $\omega^*_{\underline{A}/W(\underline{k})}$ and (ii). $\delta F_L = pF\delta$ on L. For (i), note that $\omega^*_{\underline{A}/W(\underline{k})}$ is generated over A by $\omega^1_{\underline{A}/W(\underline{k})}$, and it is clear that we only need to check the relation on x, dx for all $x \in A$, and on $\delta(l)$ for all $l \in L$. The fact that $dF(x) = pFd(x)$ follows directly from the construction of F on $\omega^1_{\underline{A}/W(\underline{k})}$, since $Fd = \widetilde{d}$ by construction; on dx and $\delta(l)$, both dF and pFd evaluate to 0 (since $d\delta = 0$ on L). Part (ii) is automatic by construction of F again, since $F\delta = \delta$. □

We record a variant of the proposition above, which allows slightly more flexible Frobenius F_L on the monoid L.

Proposition 4.6 *Let $(A, \alpha : L \to A)$ be a log algebra over $\underline{W}$. Suppose that A is p-torsion free and p-separated, and that $\omega^1_{\underline{A}/\underline{W}}$ is p-torsion free. Also suppose that the pair (A, L) is equipped with Frobenius morphisms $\varphi : A \to A$ and $F_L : L \to L$ such that*

– *$\varphi(x) \equiv x^p \mod p$ is a lifting of the absolute Frobenius mod p;*
– *F_L is compatible with φ.*

Then the same conclusions of Proposition 4.5 hold except for p-compatibility.

Proof. First we need the following sublemma:

Sublemma 4.7 *Retain the notation and assumption of the proposition. For any $k \in \mathbf{N}$, if $p^k|\varphi(x)$, then $p^k|\big(x^{d-1}dx + d\theta(x)\big)$.*

Proof of the sublemma. The case $k = 0$ is tautological. We proceed by induction. Suppose that $k \geq 1$ and that the sublemma has been verified up to $k - 1$. Now let $\varphi(x) = p^k y = x^p + p\theta(x)$, since $p|\varphi(x)$, we know that $p|x$, so $x = pz$ for some $z \in A$. Since A is p-torsion free, $\varphi(x) = p\varphi(z) = p^k y$ implies that $\varphi(z) = p^{k-1}y$, so $p^{k-1}|\big(z^{p-1}dz + d\theta(z)\big)$ by induction hypothesis. Note that $x^{p-1}dx + d\theta(x) = (pz)^{p-1}d(pz) + d\theta(pz) = p\big(z^{p-1}dz + d\theta(z)\big)$, so the lemma follows. □

[8] Remark on notation: $\theta(x)$ here defines what is usually called a δ-structure in literature, which features for example in the theory of prismatic cohomology. In this article we reserve the letter δ for log derivations.

Now we proceed as in the proof of Proposition 4.5, but replace $\widetilde{\delta}$ by

$$\widetilde{\delta}(l) := \delta(F_L(l))/p.$$

We need to show that $p|\delta(F_L(l))$ for any $l \in L$ for $\widetilde{\delta}$ to be well-defined. For this we need the assumption that A is p-separated: write $x = \alpha(l)$, then there exists some $k \in \mathbf{N}$ such that $p^k|\varphi(x)$ but $p^{k+1} \nmid \varphi(x)$. By the sublemma above we know that $p^{k+1}|d(\varphi(x))$. Now since

$$\varphi(x)\ \delta(F_L(l)) = \alpha(F_L(l))\ \delta(F_L(l)) = d\alpha(F_L(l)) = d\varphi(x)$$

and that $p^{k+1} \nmid \varphi(x)$, $p^{k+1}|d\varphi(x)$, we know that $p|\delta(F_L(l))$ (here we use the assumption that $\omega^1_{\underline{A}/W(\underline{k})}$ is p-torsion free). The rest of the proof of part (1) is the same, except that to check $\alpha(l) \cdot \widetilde{\delta}(l) = \widetilde{d}(\alpha(l))$, we need to use

$$\begin{aligned} p\widetilde{d}(\alpha(l)) &= d\varphi(\alpha(l)) \\ &= \alpha(F_L(l))\delta(F_L(l)) \\ &= p\varphi(\alpha(l))\widetilde{\delta}(l) \quad = p\alpha(l) \cdot \widetilde{\delta}(l). \end{aligned}$$

The rest of the proof is identical to that of Proposition 4.5. □

4.1.3 *An interlude: Frobenius on finite level* Proposition 4.5 will be applied to $A = W(R)$ where R is a reduced $\mathbf{F}_p$-algebra. For applications later, we also record the following remark, which the reader may skip in first reading.

Remark 4.8 *The graded ring homomorphism F can be constructed on the finite length Witt vectors, namely there is a natural graded ring homomorphism*

$$F : \omega^*_{W_n(\underline{R})/W_n(\underline{k})} \to \omega^*_{W_{n-1}(\underline{R})/W_{n-1}(\underline{k})}$$

extending the Frobenius $F : W_n(R) \to W_{n-1}(R)$. We prove the claim using the same strategy as Proposition 4.5, by constructing a log derivation $(\widetilde{d}, \widetilde{\delta})$ of $W_n(\underline{R})/W_n(\underline{k})$ into $F_*\omega^1_{W_{n-1}(\underline{R})}$, where $\omega^1_{W_{n-1}(\underline{R})}$ is regarded as a $W_n(R)$ structure via $F : W_n(R) \to W_{n-1}(R)$. We define $\widetilde{d}$ as follows: write $x \in W_n(R)$ uniquely as $x = [x_0] + Vx'$ with $x' \in W_{n-1}(R)$ and $x_0 \in R$, then define

$$\widetilde{d}x := [x_0]^{p-1}d[x_0] + dx'$$

where $[x_0]$ now lives in $W_{n-1}(R)$. We then define $\widetilde{\delta}(m) := \delta(m)$.

• *Let us first check that $\widetilde{d}(x+y) = \widetilde{d}(x)+\widetilde{d}(y)$, for which it suffices to show that if $[x_0]+[y_0] = [x_0 + y_0] + Vz'$, then*

$$[x_0]^{p-1}d[x_0] + [y_0]^{p-1}d[y_0] = [x_0 + y_0]^{p-1}d[x_0 + y_0] + dz'.$$

We check this equality in $\omega^1_{W(\underline{R})/W(\underline{k})}$. For any element $z = [x] + Vy \in W(R)$, we claim that

$$[x]^{p-1}d[x] + dy = z^{p-1}dz + d\Big(\frac{F(z) - z^p}{p}\Big).$$

This is straightforward by expanding the terms on the right hand side:

$$\big([x] + Vy\big)^{p-1}d\big([x] + Vy\big) + d\Big(\frac{[x]^p + py - \big([x] + Vy\big)^p}{p}\Big).$$

Now the desired additivity follows from the proof of Proposition 4.5.

- *Next we check that*

$$\begin{aligned}\widetilde{d}(xy) &= \Big([x_0]^p + px'\Big)\Big([y_0]^{p-1}d[y_0] + dy'\Big) + \Big([y_0]^p + py'\Big)\Big([x_0]^{p-1}d[x_0] + dx'\Big) \\ &= F(x)\widetilde{d}y + F(y)\widetilde{d}x = x \cdot \widetilde{d}y + y \cdot \widetilde{d}x\end{aligned}$$

As $(\widetilde{d}, \widetilde{\delta})$ forms a log $W_n(\underline{k})$-derivation from $W_n(\underline{R})$ into $\omega^1_{W_{n-1}(\underline{R})/W_{n-1}(\underline{k})}$, we have the desired map $F : \omega^1_{W_n(\underline{R})/W_n(\underline{k})} \to \omega^1_{W_{n-1}(\underline{R})/W_{n-1}(\underline{k})}$. Finally, this extends to a graded ring homomorphism on $\omega^*_{W_n(\underline{R})/W_n(\underline{k})}$.

4.1.4 *Preliminaries* II Now the final ingredient for the construction of the saturated log de Rham–Witt complexes.

Lemma 4.9 *Let $\underline{A}$ be a log algebra over $\underline{W} = W(\underline{k})$ satisfying the conditions in either Proposition 4.5 or 4.6, and $\underline{\omega^*} = \underline{\omega}^*_{\underline{A}/\underline{W}}$ be the corresponding log Dieudonné $\underline{W}$-algebra constructed there. Let B^* be a p-torsion free log Dieudonné $\underline{W}$-algebra. There is a canonical bijection between*

$$\mathrm{Hom}_{\mathrm{DA}^{\log}}(\underline{\omega^*}, B^*) \cong \mathrm{Hom}_F(\underline{A}, \underline{B}^0),$$

where Hom_F is as defined in Notation 4.3.

Proof. Given a morphism $(f, \psi) \in \mathrm{Hom}_F(\underline{A}, \underline{B}^0)$, we need to extend it to a morphism $f : \underline{\omega^*} \to \underline{B}^*$ of log Dieudonné $\underline{W}$-algebras. As in the proof of Proposition 4.5, we first construct a map $f : \omega^1_{\underline{A}/\underline{W}} \to B^1$ that fits into the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{d} & \omega^1_{\underline{A}/W(\underline{k})} \\ \big\downarrow f & & \big\downarrow f \\ B^0 & \xrightarrow{d_B} & B^1 \end{array}$$

Define $d_f : A \longrightarrow B^1$ by $d_f = d_B \circ f$ and $\delta_f : L \longrightarrow B^1$ by $\delta_f = \delta_B \circ \psi$. One easily verifies that d_f is a derivation of $A/W(k)$ into B^1. To show that (d_f, δ_f) is a log derivation of $\underline{A}/\underline{W}$, we check that $\delta_f(\iota(n)) = 0$ and that

$$\begin{aligned}\alpha(l) \cdot \delta_f(l) &= \alpha_B(\psi(l))\, \delta_B(\psi(l)) \\ &= d_B\, \alpha_B(\psi(l)) = d_B\, f(\alpha(l)) = \delta_f(\alpha(l)).\end{aligned}$$

Therefore, we have the dotted map $f : \omega^1_{\underline{A}/W(\underline{k})} \to B^1$ in the diagram above.

Next we extend f to $\omega^*_{\underline{A}/\underline{W}}$ as a differential graded morphism (as $\omega^*_{\underline{A}/\underline{W}}$ is generated over A by $\omega^1_{\underline{A}/\underline{W}}$). Note that f is compatible with δ by construction, so it remains to show its compatibility with F. For this we proceed as the proof of Proposition 4.5, namely we check it on x, dx for $x \in A$ and $\delta(l)$ for $l \in L$ since both F and f are algebra morphisms. We know that $Ff(x) = fF(x)$ by assumption. For dx (where $x \in A^0$), we have

$$\begin{aligned}pfF(dx) &= fdF(x) = d(fF(x)) \\ &= dF(f(x)) = pF(df(x)) = pFf(dx)\end{aligned}$$

and since B^* is p-torsion free, we get $fF(dx) = Ff(dx)$. The proof for $\delta(l)$ is similar. □

4.1.5 *The construction* Now we prove our first main result (Theorem 2 in the introduction). We continue to assume that $\underline{k}$ has the form (k, N) where $N\backslash\{0\} \mapsto 0 \in k$.

Theorem 4.10 *Let* $\underline{R} = (R, M) \in \mathrm{Alg}^{\log}_{\underline{k}}$ *be a log algebra over* $\underline{k}$*. The saturated log de Rham–Witt complex* $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ *of* $\underline{R}/\underline{k}$ *exists. Moreover, the association of* $\underline{R} \mapsto \mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ *is functorial.*

Proof. Without loss of generality assume that R is reduced by replacing it with its reduced quotient if necessary (for an object $B^* \in \mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}$, $B^0/V(B^0)$ is reduced by Remark 2.5). Let $W(\underline{R}) = (W(R), L = M \xrightarrow{[\alpha]} W(R))$ be the Witt vector of $\underline{R}$. Since R is reduced, $W(R)$ is p-torsion free. [9] By Proposition 4.5, there exits a unique log Dieudonné $\underline{W}$-algebra on the relative log de Rham complexes:

$$\underline{\omega^*} = (\omega^*_{W(\underline{R})/W(\underline{k})}, L, d, \delta, F, F_L)$$

where $L = M, F_L = p$ and $F(dx) = x^{p-1}dx + d(\frac{\varphi(x)-x^p}{p})$. Define

$$\mathcal{W}\omega^*_{\underline{R}/\underline{k}} = W(\underline{\omega^*})_{\mathfrak{sat}}.$$

The construction is clearly functorial in $\underline{R}$. We claim that $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ is the log de Rham–Witt complex of $\underline{R}/\underline{k}$. To prove the claim, consider any strict p-compatible log Dieudonné $W(\underline{k})$-algebra $B^* \in \mathrm{DA}^{\log,p}_{\mathfrak{str}}$, then we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}}(\mathcal{W}\omega^*_{\underline{R}/\underline{k}}, B^*) &= \mathrm{Hom}_{\mathrm{DA}^{\log,\mathrm{p}}}(\underline{\omega^*}, B^*) && \text{by Lemma 3.11}\\
&= \mathrm{Hom}_F(W(\underline{R}), \underline{B}^0) && \text{by Lemma 4.9}\\
&= \mathrm{Hom}_F(W(\underline{R}), W(\underline{B}^0/V\underline{B}^0)) && \text{by Cor 3.13}\\
&= \mathrm{Hom}_{\mathrm{Alg}^{\log}_{\underline{k}}}(\underline{R}, \underline{B}^0/V\underline{B}^0) && \text{by Lemma 4.4}
\end{aligned}$$

□

Notation. We denote $\mathcal{W}_n\omega^*_{\underline{R}/\underline{k}} := W_n\big((\underline{\omega^*})_{\mathfrak{sat}}\big)$.

Remark 4.11 *For each* $n \geq 1$*, write* $\omega^*_n := \omega^*_{W_n(\underline{R})/W_n(\underline{k})}$*, then in the construction above we may use* $\varprojlim \omega^*_n$ *instead of* $\omega^*_{W(\underline{R})/W(\underline{k})}$*. In other words, consider* $\varprojlim \omega^*_n$ *as a* p*-compatible log Dieudonné algebra (see for example Remark 4.8), then we may define* $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ *as* $W(\varprojlim \omega^*_n)_{\mathfrak{sat}}$*. It suffices to show that the canonical map* $\omega^*_{W(\underline{R})/W(\underline{k})} \to \varprojlim \omega^* n$ *induces a bijection* [10]

$$\mathrm{Hom}_{\mathrm{DA}^{\log,\mathrm{p}}}(\underline{\omega^*}, B^*) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DA}^{\log,\mathrm{p}}}(\varprojlim \omega^*_n, B^*).$$

For this, we show that any map $\omega^* \to B^*$ *of log Dieudonné algebras induces a map* $\omega^*_n \to W_n(B^*)$*, which is compatible with transition maps as well as Frobenius on both sides, hence giving rise to* $\varprojlim \omega^*_n \to B^*$ *on the inverse limit. Note that we have* $\omega^0_n = W_n(R) \to W_n B^0 = B^0/V^n(B^0)$*. Similar to the proof of Lemma 4.9, we claim that the map* $\omega^1_{W(\underline{R})/W(\underline{k})} \to B^1$ *factors through* $\omega^1_n \to W_n B^1$*. This follows from the description of* $\omega^1_{W(\underline{R})/W(\underline{k})}$*, which is given by the quotient (see Subsection A.2.2)*

$$\Omega^1_{W(R)/W(k)} \oplus (W(R) \otimes M^{\mathrm{gp}})/\sim.$$

This tells us that the kernel of $\omega^1_{W(\underline{R})/W(\underline{k})} \to \omega^1_n$ *is generated by* $\mathrm{im}(V^n) + \mathrm{im}(dV^n)$ *as desired.*

Remark 4.12 (Choice of charts) *In forming the saturated log de Rham–Witt complexes for a saturated log algebra* (R, M)*, we might have different log structures on* R *which give rise to*

[9] and in fact p-separated (since $W(R) \hookrightarrow W(K) \hookrightarrow W(K^{\mathrm{perf}})$ where $K = \mathrm{Fr}(R)$), however $\omega^1_{W(R)/W(k)}$ is not p-torsion free.

[10] This is slightly different from the setup in Lemma 4.17, where $\widehat{\omega}^*$ there denotes the p-adic completion, while our limit here is the "V-completion."

the same affine log scheme $(\operatorname{Spec} R, \mathcal{M}^a)$*, which amounts to different choices of charts for* $\mathcal{M}^a$*. This choice is almost irrelevant in our discussion, since the underlying Dieudonné algebras will be the same. More precisely, let* $(R, \alpha : M \to R)$ *be a log algebra over* $\underline{k}$*, and* $\mathcal{M}^a$ *be the log structure associated to the constant pre-log structure* M *on* $\operatorname{Spec} R$ *(following our convention in Introduction 1.4). Let* $M^{\mathrm{sh}} = \Gamma(\operatorname{Spec} R, \mathcal{M}^a)$ *and denote the log algebra* (R, M^{sh}) *by* $\underline{R}^{\mathrm{sh}}$*.*

Lemma 4.13 *The induced map* $\mathcal{W}\omega^*_{\underline{R}/\underline{k}} \to \mathcal{W}\omega^*_{\underline{R}^{\mathrm{sh}}/\underline{k}}$ *of the saturated log de Rham–Witt complexes from* $\underline{R} \to \underline{R}^{\mathrm{sh}}$ *is an isomorphism on the underlying log Dieudonné algebras. In particular, if* (R, M) *and* (R, M') *are two charts for the affine log scheme* $(\operatorname{Spec} R, \mathcal{M}^a)$*, then the underlying Dieudonné algebra of their saturated log de Rham–Witt complexes are isomorphic.*

Proof. It suffices to show that for each $n \geq 1$, the pre-log structures $\beta : M \to W_n(R)$ and $\beta^{\mathrm{sh}} : M^{\mathrm{sh}} \to W_n(R)$ induce isomorphic log structures over $\operatorname{Spec} W_n(R)$, thus we have

$$\omega^*_{\left(W_n(R), M\right)/W_n(\underline{k})} \xrightarrow{\sim} \omega^*_{\left(W_n(R), M^{\mathrm{sh}}\right)/W_n(\underline{k})}.$$

The proposition then follows from Remark 4.11. Now write $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} W_n(R)$. Let $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) be the structure sheaf of $X_{\text{ét}}$ (resp. $Y_{\text{ét}}$), and $W_n(\mathcal{O}_X)$ be the sheaf sending $U \mapsto W_n(\Gamma(U, \mathcal{O}_X))$. [11] We need to show that the following two pushouts

$$\begin{array}{ccc} \beta^{-1}(\mathcal{O}_Y^\times) & \longrightarrow & \mathcal{O}_Y^\times \\ \downarrow & & \\ M & & \end{array} \qquad\qquad \begin{array}{ccc} (\beta^{\mathrm{sh}})^{-1}(\mathcal{O}_Y^\times) & \longrightarrow & \mathcal{O}_Y^\times \\ \downarrow & & \\ M^{\mathrm{sh}} & & \end{array}$$

of étale sheaves of monoids over $Y_{\text{ét}}$ are isomorphic. By topological invariance of étale sites, we may compute the pushout in $X^\sim_{\text{ét}}$. Over X, the morphism β corresponds to $M \xrightarrow{\alpha} \mathcal{O}_X \xrightarrow{\gamma} W_n(\mathcal{O}_X)$, where γ is given by Teichmuller liftings. Note that $\gamma^{-1}(W_n(\mathcal{O}_X)^\times) = \mathcal{O}_X^\times$ since for an algebra R we have $W_n(R)^\times \cap [R] = [R^\times]$, so we have $\beta^{-1}(W_n(\mathcal{O}_X)^\times) = \alpha^{-1}(\mathcal{O}_X^\times)$. Thus the first pushout can be computed via two steps, first take the pushout of $\alpha^{-1}(\mathcal{O}_X^\times) \to M$ along $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$, then along $\mathcal{O}_X^\times \to W_n(\mathcal{O}_X)^\times$. The first step precisely computes $\mathcal{M}^a$, which is isomorphic to $(\mathcal{M}^{\mathrm{sh}})^a$ by Lemma A.1, therefore M and M^{sh} indeed induces the same log structures on $Y = \operatorname{Spec} W_n(R)$. ∎

Remark 4.14 *The proof we give of Lemma 4.13 is somewhat indirect (for example it involves Remark 4.11). One issue is that it seems more involved to show that* M *and* M^{sh} *give rise to the same log structures over* $\operatorname{Spec} W(R)$*. However, if we assume that* $\underline{R}/\underline{k}$ *is log-smooth of log-Cartier type, then the proposition immediately follows from Corollary 2.10 and Proposition 4.22, since we have an isomorphism* $\omega^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} \omega^*_{\underline{R}^{\mathrm{sh}}/\underline{k}}$ *by Lemma A.1.*

4.2 Construction via log Frobenius liftings

In this subsection we discuss an alternative construction of saturated log de Rham Witt complexes for log algebras which admit liftings to $W(k)$ together with lifts of Frobenius.

4.2.1 *Log Frobenius liftings* We record a (variant of a) definition from [LZ04].

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[11] It is straightforward to directly check that the pre-sheaf $W_n(\mathcal{O}_X)$ is a Zariski sheaf, and satisfies the sheaf condition for an étale cover $R \to R'$. It corresponds to $\mathcal{O}_Y$ on $Y_{\text{ét}} = (\operatorname{Spec} W_n(R))_{\text{ét}}$ under the identification of étale topos $X^\sim_{\text{ét}} \cong Y^\sim_{\text{ét}}$.

Definition 4.15 *Let* $(f, \psi) : \operatorname{Spec} \underline{R} = (\operatorname{Spec} R, M) \to \operatorname{Spec} \underline{k}$ *be a morphism of affine pre-log schemes with integral pre-log structures.*

(1) A Witt-lifting of f *consists of a system* $\left(\underline{A_n} = (A_n, M_n),\ \delta_n : W_n(\operatorname{Spec} \underline{R}) \to \operatorname{Spec} \underline{A_n}\right)_{n \geq 1}$, *where* $(A_1, M_1) = (R, M)$, *and satisfies the following list of conditions:*

- $\operatorname{Spec} \underline{A_n} = (\operatorname{Spec} A_n, M_n)$ *is log-smooth over* $W_n(\operatorname{Spec} \underline{k})$*;*
- *For each* n*, the following diagram is cartesian*

$$\begin{array}{ccc} \operatorname{Spec} \underline{A_n} & \xrightarrow{R} & \operatorname{Spec} \underline{A_{n+1}} \\ \downarrow & \square & \downarrow \\ W_n(\operatorname{Spec} \underline{k}) & \xrightarrow{R} & W_{n+1}(\operatorname{Spec} \underline{k}) \end{array}$$

  where $R : A_{n+1} \to A_n$ *is the transition map of the inverse system.*
- *The maps* δ_n *are compatible with the system* $(\operatorname{Spec} \underline{A_n})$ *under the transition maps.*

(2) A log Frobenius lifting of f *is a projective system* $(\underline{A_n}, \delta_n, \varphi_n)_{n \geq 1}$ *where* $(\underline{A_n}, \delta_n)$ *is a Witt-lifting of* f*, and* $\varphi_n : \operatorname{Spec} \underline{A_n} \to \operatorname{Spec} \underline{A_{n+1}}$ *is a morphism of pre-log schemes, such that*

- *the maps* φ_n *are compatible with the Frobenius on* $\operatorname{Spec} \underline{R}$ *given by the* p^{th} *power Frobenius on* R *and multiplication by* p *on* M*.*
- φ_n *are compatible with the Frobenius of Witt vectors on both* $\operatorname{Spec} \underline{k}$ *and* $\operatorname{Spec} \underline{R}$*.*

(3) A log Frobenius lifting $(\underline{A_n} = (A_n, M_n), \delta_n, \varphi_n)$ *of* f *is called* p*-compatible, if the log-restriction map* $R : M_{n+1} \to M_n$ *is identity for all* $n \geq 1$ *(so in particular* $M_n = M$ *for all* n*) and the log Frobenius map* $\varphi_n : M \to M$ *is multiplication by* p*.*

The requirements on φ_n in the definition of log Frobenius liftings can be summarized in the following commutative diagrams:

$$\begin{array}{ccc} \operatorname{Spec} \underline{A_{n+1}} & \xrightarrow{R} & W_{n+1}(\operatorname{Spec} \underline{R}) \\ {\scriptstyle \varphi_n} \uparrow & & \uparrow {\scriptstyle F_n} \\ \operatorname{Spec} \underline{A_n} & \xrightarrow{R} & W_n(\operatorname{Spec} \underline{R}) \end{array} \qquad \begin{array}{ccc} \operatorname{Spec} \underline{A_n} & \xrightarrow{\varphi_n} & \operatorname{Spec} \underline{A_{n+1}} \\ \downarrow & & \downarrow \\ W_n(\operatorname{Spec} \underline{k}) & \xrightarrow{F_n} & W_{n+1}(\operatorname{Spec} \underline{k}) \end{array}$$

Also note that, though we stated the definitions on log algebras following [LZ04], it is clear that one can globalize and extend the definitions to pre-log schemes in general.

Lemma 4.16 *Suppose that* $\underline{R}$ *is log-smooth over* $\underline{k}$*, then there exists a* p*-compatible log Frobenius lifting* $(\underline{A}_n, \delta_n, \varphi_n)$ *of* $\underline{R}$*. More generally, let* $f : \underline{X} \to \operatorname{Spec} \underline{k}$ *be a log-smooth morphism of fine (pre-)log schemes, then a* p*-compatible log Frobenius liftings exist* étale *locally.*

Proof. This follows from Proposition 3.2 in [LZ04] and Lemma 5.5 of [Mat17]. The morphism $(k, N) \to (R, M)$ factors through the morphism $(k, N) \to (k \otimes_{\mathbf{Z}[N]} \mathbf{Z}[M], M) \to (R, M)$, where the second map is étale on the underlying rings. The first arrow admits a p-compatible log Frobenius lifting given by (T_n, M) where $T_n := W_n(k) \otimes_{\mathbf{Z}[N]} \mathbf{Z}[M]$ and φ_n is given by $a \otimes b \mapsto F(a) \otimes b^p \in T_{n-1}$. Then we need to lift the étale morphism $T_1 \to R$ along $\cdots T_n \to T_{n-1} \to \cdots \to T_1$, which exists by [LZ04] Proposition 3.2. Note that the log structure on each A_n is given by $M \to T_n \to A_n$ (while $\varphi_n | M$ is still multiplication by p), so in particular we have constructed a p-compatible log Frobenius lifting. □

4.2.2 *A p-complete de Rham complex* For the rest of this Subsection we make the following

Assumption. $\underline{R}/\underline{k}$ is integral and admits a p-compatible log Frobenius lifting $(\underline{A}_n, \delta_n, \varphi_n)$.

Examples include log algebras $\underline{R}$ that are log-smooth and integral over $\underline{k}$ (Lemma 4.16). [12]

Write $\underline{\widehat{A}} = (\widehat{A}, M)$ where $\widehat{A} = \varprojlim A_n$, which is equipped with the log structure from the filtered inverse limit. We then form the completed log de Rham complex

$$\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})} (= \widehat{\omega}^*) := \varprojlim_n \omega^*_{\underline{A}_n/W_n(\underline{k})} \cong \varprojlim_n \big(\omega^*_{\underline{\widehat{A}}/W(\underline{k})}/p^n\big).$$

From Subsection A.3.1, f being integral means that the map $\mathbf{Z}[N] \to \mathbf{Z}[M]$ induced from the map of monoids is flat, therefore for each n, $\operatorname{Spec} \underline{A}_n \to \operatorname{Spec} W(\underline{k})$ is log-smooth and integral, hence flat. Now, since the restriction map $R : A_{n+1} \to A_n$ is surjective, so the inverse limit $\widehat{A}$ is also flat over $W(k)$, hence p-torsion free. This allows us to apply Proposition 4.5 and get a log Dieudonné algebra $\omega^*_{\underline{\widehat{A}}/W(\underline{k})}$. The completed log de Rham complex $\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}$ has a unique cdga structure such that the canonical map $\omega^*_{\underline{\widehat{A}}/W(\underline{k})} \to \widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}$ is a map of cdga's. Moreover, $\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}$ inherits the Frobenius structure from $\omega^*_{\underline{\widehat{A}}/W(\underline{k})}$ which makes it a p-compatible log Dieudonné algebra.

4.2.3 *Preliminaries* III The following lemma is a variant of Lemma 4.9. In this lemma we do not need to assume that $\underline{A}$ comes from log Frobenius liftings.

Lemma 4.17 *Let $\underline{A}$ be a log algebra over $W(\underline{k})$ satisfying the conditions in either Proposition 4.5 or Proposition 4.6, and let $\omega^* = \underline{\omega}^*_{\underline{A}/W(\underline{k})}$ be the log Deiudonne algebra constructed there. Let $\widehat{\omega}^*$ be the term-wise p-completion of ω^*. Then for any p-torsion free and p-complete log Dieudonné $W(\underline{k})$-algebra B^*, the canonical map*

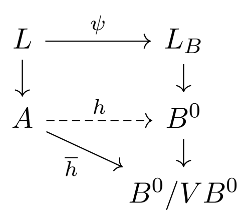
$$\operatorname{Hom}_{\mathrm{DA}^{\log}}(\widehat{\omega}^*, B^*) \cong \operatorname{Hom}_F(\underline{A}, \underline{B}^0)$$

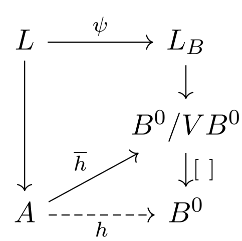
is a bijection. Suppose that B^* is in addition strict and p-compatible, then in fact

$$\operatorname{Hom}_{\mathrm{DA}^{\log}}(\widehat{\omega}^*, B^*) \cong \operatorname{Hom}(\underline{A}/p, \underline{B}^0/V\underline{B}^0)$$

where the second set denotes homomorphisms of log algebras over $\underline{k}$.

Proof. The first assertion follows directly from Lemma 4.9. For the second assertion, we appeal to the Cartier-Dieudonné-Dwork lemma, which in our setup says that, since A is equipped with a lift of Frobenius φ satisfying $\varphi(a) \equiv a^p$ for all $a \in A$, a homomorphism $\overline{h} : A \to B^0/VB^0$ has a unique lift to a homomorphism $h : A \to W(B^0/VB^0) \xrightarrow{\mu^{-1}} B^0$ such that $F \circ h = h \circ \varphi$. We still need to check that, given a morphism of log algebras $(\overline{h}, \psi)$, its lift $(h, \psi) : \underline{A} \to \underline{B}^0$ is a morphism of log algebras, namely the top square in the left diagram commutes:





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[12]Note that our definition of log-smooth requires the monoid M to be integral and coherent, but this is different from requiring the morphism $\operatorname{Spec} \underline{R} \to \operatorname{Spec} \underline{k}$ to be integral, see Appendix A.3.1.

This is indeed the case, since the log structure on $\overline{B}^0$ factors through the Teichmuller lifts by Corollary 3.13, as the diagram on the right indicates. Moreover, this also shows that any map $(h,\psi) : \underline{A} \to \underline{B}^0$ of log algebras comes from the lifting of a pair $(\overline{h},\psi)$. Finally, since p is 0 in B^0/VB^0, we get $\mathrm{Hom}_F(\underline{A},\underline{B}^0) = \mathrm{Hom}(\underline{A}/p,\underline{B}^0/V\underline{B}^0)$ as desired. □

4.2.4 *Comparison with log de Rham–Witt complexes* Now let us apply the lemma above to the setup in Subsection 4.2.2, where $\widehat{\underline{A}}$ comes from a log Frobenius lifting of $\underline{R}$. By Lemma 4.17 there is a canonical bijection

$$\mathrm{Hom}_{\mathrm{DA}^{\log,p}}(\widehat{\omega}^*_{\widehat{\underline{A}}/W(\underline{k})}, \mathcal{W}\omega^*_{\underline{R}/\underline{k}}) \cong \mathrm{Hom}(\underline{R}, \mathcal{W}_1\omega^0_{\underline{R}/\underline{k}})$$

Let $e : R \to \mathcal{W}_1\omega^0_{\underline{R}/\underline{k}}$ be the tautological map from the definition of the log de Rham–Witt complex $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$. Then $(e,\mathrm{id}) : \underline{R} \to \mathcal{W}_1\omega^0_{\underline{R}/\underline{k}}$ gives rise to a map $\upsilon : \widehat{\omega}^*_{\widehat{\underline{A}}/W(\underline{k})} \to \mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ in the category $\mathrm{DA}^{\log,\mathrm{p}}$ of p-compatible log Dieudonné $\underline{W}$-algebras.

Proposition 4.18 *The map υ induces a canonical isomorphism (again denoted by υ)*

$$\upsilon : W\big(\widehat{\omega}^*_{\widehat{\underline{A}}/W(\underline{k})}\big)_{\mathfrak{sat}} \xrightarrow{\sim} \mathcal{W}\omega^*_{\underline{R}/\underline{k}}.$$

Proof. By construction $W\big(\widehat{\omega}^*_{\widehat{\underline{A}}/W(\underline{k})}\big)_{\mathfrak{sat}} \in \mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}$, so it suffices to show that for any $B^* \in \mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}$, the map $\upsilon : W\big(\widehat{\omega}^*_{\widehat{\underline{A}}/W(\underline{k})}\big)_{\mathfrak{sat}} \to \mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ induces a bijection between

$$\mathrm{Hom}_{\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}}(\mathcal{W}\omega^*_{\underline{R}/\underline{k}}, B^*) \overset{?}{=} \mathrm{Hom}_{\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}}(W\big(\widehat{\omega}^*_{\widehat{\underline{A}}/W(\underline{k})}\big)_{\mathfrak{sat}}, B^*).$$

This follows from the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}}(\mathcal{W}\omega^*_{\underline{R}/\underline{k}}, B^*) & \xrightarrow{\upsilon} & \mathrm{Hom}_{\mathrm{DA}^{\log,\mathrm{p}}_{\mathfrak{str}}}(W\big(\widehat{\omega}^*_{\widehat{\underline{A}}/W(\underline{k})}\big)_{\mathfrak{sat}}, B^*) \\
\Big\| \text{Thm.4.10} & & \Big\| \\
 & & \mathrm{Hom}_{\mathrm{DA}^{\log,p}}(\widehat{\omega}^*_{\widehat{\underline{A}}/W(\underline{k})}, B^*) \\
 & & \Big\| \text{Lemma 4.17} \\
\mathrm{Hom}_{\mathrm{Alg}^{\log}_{\underline{k}}}(\underline{R}, \underline{B}^0/V\underline{B}^0) & = & \mathrm{Hom}_{\mathrm{Alg}^{\log}_{\underline{k}}}(\underline{R}, \underline{B}^0/V\underline{B}^0)
\end{array}$$

□

Therefore, if $\underline{R}/\underline{k}$ is integral and admits a p-compatible log Frobenius lifting, the saturated the log de Rham–Witt complex of $\underline{R}/\underline{k}$ can be constructed by taking $\widehat{\omega}^*_{\widehat{\underline{A}}/W(\underline{k})}$, and then taking its associated V-complete saturation (cf. Subsection 3.4).

4.3 Comparison with more general log Frobenius liftings

In this Subsection we prove a more general form of Theorem 8 (3). In general, we cannot construct $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ from a non-p-compatible Frobenius lifting of $\underline{R}/\underline{k}$ as in Subsection 4.2. However, in some cases we can still compare the completed log de Rham complexes of non-p-compatible Frobenius liftings with $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ in the derived category. This crucially relies on the notion of log-Cartier type (cf. Subsection A.3.2). In contrast, the Cartier criterion is not needed in Subsection 4.2.

Assumption. $\underline{R}/\underline{k}$ is log-smooth of log-Cartier type (which is in particular integral), and $(\underline{A}_n, \varphi_n, \delta_n)$ is a log Frobenius lifting such that $\widehat{A} = \varprojlim A_n$ is p-torsion free.

Each $\omega^1_{\underline{A}_n/W_n(\underline{k})}$ is a finitely generated locally free module over $A_n = \widehat{A}/p^n$, by log smoothness of $\underline{A}_n/W_n(\underline{k})$, so $\widehat{\omega}^1_{\underline{\widehat{A}}/W(\underline{k})} = \varprojlim \omega^1_{\underline{A}_n/W_n(\underline{k})}$ is locally free over $\widehat{A}$, hence p-torsion free. Then $\underline{\widehat{A}}$ satisfies the condition in Proposition 4.6 with $\widehat{\omega}^1_{\underline{\widehat{A}}/W(\underline{k})}$ replacing $\omega^1_{\underline{\widehat{A}}/W(\underline{k})}$. The conclusion of Proposition 4.6 holds by the same proof. To summarize, $\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}$ is equipped with a Frobenius extending $\varphi = \varprojlim \varphi_n$, which makes it a log Dieudonné $W(\underline{k})$-algebra,[13] and there is a bijection

$$\mathrm{Hom}_{\mathrm{DA}^{\log}}(\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}, \mathcal{W}\omega^*_{\underline{R}/\underline{k}}) \cong \mathrm{Hom}(\underline{R}, \mathcal{W}_1\omega^0_{\underline{R}/\underline{k}})$$

by Lemma 4.17. The tautological map $\underline{R} \to \mathcal{W}_1\omega^0_{\underline{R}/\underline{k}}$ again induces a map $\upsilon : \widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})} \to \mathcal{W}\omega^*_{\underline{R}/\underline{k}}$.

Remark 4.19 *Unlike previous subsection, in general υ will not induce an isomorphism from the complete saturation $W\big((\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})})_{\mathfrak{sat}}\big)$ of $\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}$ to the saturated log de Rham–Witt complex.*

Theorem 4.20 *Let $\underline{R}$ and $\widehat{A}$ be as in the setup above, then the map*

$$\upsilon : \widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})} \longrightarrow \mathcal{W}\omega^*_{\underline{R}/\underline{k}}$$

induces a quasi-isomorphism on the underlying cochain complexes.

Proof. (1). First suppose that the log Frobenius lift $(\underline{A}_n, \varphi_n, \delta_n)$ in the Assumption above is p-compatible, then by Proposition 4.18, we have

$$\upsilon : \widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})} \to W(\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})})_{\mathfrak{sat}} \xrightarrow{\sim} \mathcal{W}\omega^*_{\underline{R}/\underline{k}}.$$

Since both $\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}$ and $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ are p-torsion free and p-complete, it suffices to show that υ is a quasi-isomorphism after mod p. In other words, we need to show that

$$\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}/p \to (\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})})_{\mathfrak{sat}}/p \to W(\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})})_{\mathfrak{sat}}/p \xrightarrow{\sim} \mathcal{W}\omega^*_{\underline{R}/\underline{k}}/p$$

is a quasi-isomorphism. The rest of the proof is similar to the proof of Corollary 2.11. The key point is that the underlying Dieudonné algebra $\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}$ satisfies the Cartier criterion (cf. Definition 2.9) by Proposition A.10. In other words, we have

$$F : \widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}/p \cong \omega^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} H^*(\omega^\bullet_{\underline{R}/\underline{k}}) \cong H^*(\widehat{\omega}^\bullet_{\underline{\widehat{A}}/W(\underline{k})}/p).$$

To finish the proof, note that both of the maps of $(\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})})_{\mathfrak{sat}}/p \to W(\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})})_{\mathfrak{sat}}/p$ and $\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}/p \to (\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})})_{\mathfrak{sat}}/p$ are quasi-isomorphisms by Corollary 2.10.

(2). For the general case, again we need to show that $\overline{\upsilon} : \widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}/p \cong \omega^*_{\underline{R}/\underline{k}} \xrightarrow{\upsilon \bmod p} \mathcal{W}\omega^*_{\underline{R}/\underline{k}}/p$ is a quasi-isomorphism. The map $\overline{\upsilon}$ depends on the lifting $\widehat{A}$ of R and φ of Frobenius, but the induced map in the derived category does not. More precisely, let $\underline{\widehat{A}'}/\underline{W}$ be a p-compatible formal

[13] Though we construct the log Dieudonné algebra $\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}$ directly using universal properties of $\widehat{\omega}^1_{\underline{\widehat{A}}/W(\underline{k})}$, Lemma 4.17 still holds by inspecting its proof (as well as the proof of Lemma 4.9).

lift of $\underline{R}/\underline{k}$ (which exists thanks to Lemma 4.16), and consider the following diagram

$$\begin{array}{ccccccc} \widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})} & \longrightarrow & \omega^*_{\underline{R}/\underline{k}} & \xrightarrow{\overline{\upsilon}} & \mathcal{W}\omega^*_{\underline{R}/\underline{k}}/p & \xrightarrow{\mathrm{pr}} & \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}} \\ & & \| & & & & \| \\ \widehat{\omega}^*_{\underline{\widehat{A}'}/W(\underline{k})} & \longrightarrow & \omega^*_{\underline{R}/\underline{k}} & \xrightarrow{\overline{\upsilon}'} & \mathcal{W}\omega^*_{\underline{R}/\underline{k}}/p & \xrightarrow{\mathrm{pr}} & \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}} \end{array}$$

where $\mathrm{pr} : \mathcal{W}\omega^*_{\underline{R}/\underline{k}}/p \twoheadrightarrow \mathcal{W}\omega^*_{\underline{R}/\underline{k}}/\mathrm{Fil}^1 = \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}}$ is the canonical projection. Unwinding definitions, we see that $\mathrm{pr}\circ\overline{\upsilon} = \mathrm{pr}\circ\overline{\upsilon}'$ (the square commutes). From step (1), we know that $\overline{\upsilon}'$ is a quasi-isomorphism. By Proposition 4.22, $\mathrm{pr}\circ\overline{\upsilon}'$ is an isomorphism, hence pr is a quasi-isomorphism. This in turn (using Proposition 4.22 again) implies that $\overline{\upsilon}$ is a quasi-isomorphism. ☐

Remark 4.21 *(1) The proof of the general case (part (2)) uses Proposition 4.22, whose proof builds on the conclusion of the special case (part (1)) of the theorem, but does not use part (2) of this proof. Therefore there is no circular argument.*

(2) In general $\overline{\upsilon}$ does not agree with $\overline{\upsilon}'$ if R is non-perfect. Note that υ comes from the map $f_\varphi : \widehat{A} \to W(\widehat{A}) \to W(R)$, where the first map is determined (on ghost coordinates) by

$$\widehat{A} \longrightarrow W(\widehat{A}) \xhookrightarrow{\mathrm{gh}} \widehat{A}^{\mathbf{N}}, \qquad \widetilde{x} \mapsto (\widetilde{x}, \varphi(\widetilde{x}), \varphi^2(\widetilde{x}), \ldots).$$

On Witt coordinates, $f_\varphi(\widetilde{x}) = (\widetilde{x}, \theta(\widetilde{x}), \ldots)$ where $\theta(\widetilde{x}) = (\varphi(\widetilde{x}) - \widetilde{x}^p)/p \mod p \in R$. The reduction mod p of f_φ gives $\overline{\upsilon}|_R : R = \widehat{A}/p \to W(R)/p$, which depends on $\widehat{A}$ and φ. This is not surprising: without the liftings, we cannot produce a map from $R = W(R)/V \to W(R)/p$ when R is non-perfect.

4.4 Comparison with de Rham complexes in characteristic p

Let $\underline{R}/\underline{k}$ be a log algebra and let $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ be its saturated log de Rham–Witt complex. We have a map $(e, \mathrm{id}) : \underline{R} \to \mathcal{W}_1\omega^0_{\underline{R}/\underline{k}}$, which extends uniquely to a map $\nu : \omega^*_{\underline{R}/\underline{k}} \to \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}}$ of cdga's that is compatible with log structures and log differentials (In fact, $\nu = \mathrm{pr}\circ\overline{\upsilon} = \mathrm{pr}\circ\overline{\upsilon}'$ in the proof of Theorem 4.20). Our first goal of this subsection is to prove the following proposition.

Proposition 4.22 *Suppose that $\underline{R}/\underline{k}$ is log-smooth of log-Cartier type, then*

$$\nu : \omega^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}}$$

is an isomorphism.

Proof. The proof is similar to that of Proposition 4.3.2 of [BLM18]. Let $(\widehat{A}, M)$ be a p-compatible formal lift of $\underline{R}$ over $\underline{W}$, which exists by Lemma 4.16. Then we have a commutative diagram of cochain complexes

$$\begin{array}{ccccc} \omega^*_{\underline{R}/\underline{k}} & & \xrightarrow{\quad C^{-1}\quad} & & H^*(\omega^*_{\underline{R}/\underline{k}}) = H^*(\widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})}/p) \\ \downarrow{\scriptstyle \overline{\upsilon}} & \searrow^{\nu} & & & \downarrow{\scriptstyle H(\overline{\upsilon})} \\ & {}^{q}\nearrow & \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}} & \searrow^{F_1} & \\ \mathcal{W}\omega^*_{\underline{R}/\underline{k}}/p & & \xrightarrow{\quad F\quad} & & H^*(\mathcal{W}\omega^*_{\underline{R}/\underline{k}}/p) \end{array}$$

where both complexes on the right are equipped with the Bockstein differentials (see Subsection 2.2). The vertical maps are both induced by $\upsilon : \widehat{\omega}^*_{\underline{\widehat{A}}/W(\underline{k})} \to \mathcal{W}\omega^*_{\underline{R}/\underline{k}}$, while the bottom triangle is the triangle above Lemma 2.8 applied to the saturated log de Rham–Witt complex. From the proof of part (1) of Theorem 4.20, $\overline{v}$ is a quasi-isomorphism, so $H(\overline{v})$ is an isomorphism. The map F_1 is an isomorphism by part (2) of Lemma 2.8. Finally, by the assumption that $\underline{R}/\underline{k}$ is log-smooth of log-Cartier type, C^{-1} is an isomorphism. Therefore, ν is an isomorphism. □

Corollary 4.23 *We continue to assume that $\underline{R}/\underline{k}$ is log-smooth of log-Cartier type. The projection $\mathcal{W}\omega^*_{\underline{R}/\underline{k}} \to \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}} \xrightarrow{\nu^{-1}} \omega^*_{\underline{R}/\underline{k}}$ induces a quasi-isomorphism of cochain complexes*

$$\mathrm{pr} : \mathcal{W}\omega^*_{\underline{R}/\underline{k}}/p\mathcal{W}\omega^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} \omega^*_{\underline{R}/\underline{k}}.$$

In other words, $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ is a deformation of $\omega^*_{\underline{R}/\underline{k}}$ in the derived category.

Proof. Immediate from the commutative diagram above, since $\overline{v}$ is an isomorphism. □

We now turn to the following interesting corollary (already mentioned in the introduction).

Corollary 4.24 *Let $\underline{X}$ be a fine log scheme over $\underline{k}$ that is log-smooth of log-Cartier type, then the underlying scheme X is reduced.*

Proof. It suffices to work locally on charts. Under the assumption in Proposition 4.22, we have $R \xrightarrow{\sim} \mathcal{W}_1\omega^0_{\underline{R}/\underline{k}}$, the claim then follows from Remark 2.5. □

Remark 4.25 *This is related to a result of Tsuji in log geometry. Suppose in addition that $\underline{X}$ and* $\mathrm{Spec}\,\underline{k}$ *are saturated log schemes as defined in [Tsu19]. From loc.cit. II.2.14 and II.3.1 (attributed to Kato) it follows that $\underline{X}$ is of log-Cartier type if and only if the morphism $\underline{X} \to \underline{k}$ is saturated, and by Theorem II.4.2 of loc.cit. the latter condition is equivalent to $\underline{X}$ being reduced.*

Our next remark is helpful for the discussion on monodromy operator and for [Yao19].

Remark 4.26 *In the proof of Theorem 4.20 and Proposition 4.22, the assumption that $\underline{R}/\underline{k}$ is log-smooth of log-Cartier type is only used to guarantee that the following two criterions hold:*

(1) There is a lift $\underline{A}/W(\underline{k})$ of $\underline{R}$ with a lift of Frobenius such that $\omega^*_{\underline{A}/W(\underline{k})}$ is p-torsion free.

(2) The Cartier isomorphism $C^{-1} : \omega^i_{\underline{R}/\underline{k}} \xrightarrow{\sim} H^i(\omega^*_{\underline{R}/\underline{k}})$ holds.

Thus the conclusions of Theorem 4.20 and Proposition 4.22 continue to hold for log algebras $\underline{R}/\underline{k}$ that meet the two criterions above.

5. Étale base change

In this section, we show that the saturated log de Rham–Witt complex $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ behaves well with respect to étale base change, therefore, for any quasi-coherent log scheme $\underline{X}$ over $\underline{k}$, we obtain a sheaf $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ of log Dieudonné algebras on the étale site $X_{\text{ét}}$.

To formulate the étale base change we consider the category of log cdga's, consisting of $\underline{A}^* = (A^*, M, d, \delta)$ where (A^*, d) is a cdga, $M \xrightarrow{\alpha} A^0$ a log algebra and $\delta : M \to A^1$ a monoid morphism (with the additive monoid structure on A^1) satisfying $d \circ \delta = 0$ and $\alpha(m)\delta(m) = d(\alpha(m))$ for any $m \in M$. Morphisms between $\underline{A}^*, \underline{B}^*$ are pairs of maps (f, ψ) where $f : A^* \to B^*$ is a morphism of cdga's and $\psi : M_A \to M_B$ is a morphism of monoids, compatible with α and δ.

Definition 5.1 – *A morphism $(f, \psi) : (A, L_A) \to (B, L_B)$ of log algebras is naively* étale *if $f : A \to B$ is* étale *and ψ is an isomorphism of monoids.*

– *A morphism* $(f,\psi) : \underline{A}^* \to \underline{B}^*$ *between log cdga's is* étale *if its restriction on the log algebra* $(f,\psi) : (A^0, M_A) \to (B^0, M_B)$ *is naively* étale *and that* f *induces an isomorphism* $A^* \otimes_{A^0} B^0 \to B^*$ *of graded algebras.*
– *If* $\underline{A}^*, \underline{B}^* \in \mathrm{DA}^{\log}_{\mathfrak{str}}$ *are strict log Dieudonné algebras, then* $(f,\psi) : \underline{A}^* \to \underline{B}^*$ *is* V*-adically* étale *if for each* $n \geq 1$*, the induced map of log cdga's* $W_n(A^*) \to W_n(B^*)$ *is* étale.

If $\underline{A}^*$ is a log cdga, we denote by $\text{Ét}_{\underline{A}^*}$ the category of log cdga's étale over $\underline{A}^*$. Similarly, for $\underline{A}^* \in \mathrm{DA}^{\log}_{\mathfrak{str}}$, we denote by $\mathrm{V\acute{E}t}_{\underline{A}^*}$ the category of V-adically étale $\underline{A}^*$-algebras in $\mathrm{DA}^{\log}_{\mathfrak{str}}$.

Proposition 5.2 *(1) The functor* $\underline{B}^* \mapsto B^0$ *induces an equivalence of categories*

$$\text{Ét}_{\underline{A}^*} \xrightarrow{\sim} \text{Ét}_{A^0} = \{\text{étale } A^0\text{-algebras}\}.$$

(2) Let $\underline{A}^* \in \mathrm{DA}^{\log}_{\mathfrak{str}}$*. The functor that sends a* V*-adically* étale $\underline{A}^*$*-algebra* $\underline{B}^* \in \mathrm{DA}^{\log}_{\mathfrak{str}}$ *to the* $A^0/V(A^0)$*-algebra* $B^0/V(B^0)$ *induces an equivalence of categories*

$$\mathrm{V\acute{E}t}_{\underline{A}^*} \xrightarrow{\sim} \text{Ét}_{A^0/V(A^0)}\,.$$

Proof. The functor $\underline{B}^* \mapsto B^*$ forgetting the log structure on $\underline{B}^*$ is an equivalence of categories between étale $\underline{A}^*$-algebras and étale A^*-algebras. Likewise $\underline{B}^* \mapsto B^*$ induces an equivalence between $\mathrm{V\acute{E}t}_{\underline{A}^*}$ and $\mathrm{V\acute{E}t}_{A^*}$. Therefore claim (1) and (2) follow from Proposition 5.3.2 and Theorem 5.3.4 in [BLM18] respectively. ☐

By the same proof of Corollary 5.3.5 of [BLM18], we arrive at

Proposition 5.3 (étale base change) *Let* $\underline{R} = (R, M)$ *be a log algebra over* $\underline{k}$*, and* $R \to S$ *be an étale morphism of* k*-algebras. Let* $\underline{S} = (S, M)$ *be the (naively étale) log algebra over* $\underline{R}$*. For any* $n \geq 1$*, the map* $\mathcal{W}_n\omega^*_{\underline{R}/\underline{k}} \to \mathcal{W}_n\omega^*_{\underline{S}/\underline{k}}$ *is* étale*. In other words,* $\mathcal{W}\omega^*_{\underline{R}/\underline{k}} \to \mathcal{W}\omega^*_{\underline{S}/\underline{k}}$ *is* V*-adically* étale*. Moreover, we have a natural isomorphism of graded algebras*

$$\mathcal{W}_n\omega^*_{\underline{R}/\underline{k}} \otimes_{W_n(R)} W_n(S) \xrightarrow{\sim} \mathcal{W}_n\omega^*_{\underline{S}/\underline{k}}.$$

Remark 5.4 *When* $\underline{R}$ *is log-smooth over* $\underline{k}$ *of log-Cartier type, we may also deduce the étale base change* $\mathcal{W}_n\omega^*_{\underline{R}/\underline{k}} \otimes_{W_n(R)} W_n(S) \xrightarrow{\sim} \mathcal{W}_n\omega^*_{\underline{S}/\underline{k}}$ *by comparing to the log de Rham–Witt complexes constructed by Hyodo–Kato (which is defined globally on the étale site) or Matsuue (using Proposition 3.6 in [Mat17]), once we prove Propositions 6.6 and 6.9.*

Now let $\underline{X}$ be a quasi-coherent log scheme over $\operatorname{Spec} \underline{k}$. We construct the sheaf of log de Rham–Witt complexes $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ on $X_{\text{ét}}$.

Notation 5.5 *For a log scheme* $\underline{X} = (X, M_X)$ *over* $\underline{k}$*, let* $X_{\text{ét,aff}}$ *be the site of "small enough" affine étale opens where the log structure of* $\underline{X}$ *admits charts. The objects of* $X_{\text{ét,aff}}$ *are étale morphisms* $U \xrightarrow{h} X$ *over* k *where* $U = \operatorname{Spec} R$ *is affine, such that there exists a constant log structure* $L \to R$ *and an isomorphism* $(\mathcal{L}_U)^a \xrightarrow{\sim} M_X|_U := h^* M_X$*. Here* $(\mathcal{L}_U)^a$ *is the associated log structure of the pre-log structure* $L_U \to \mathcal{O}_U$*. We give* $X_{\text{ét,aff}}$ *the étale topology.*

It is clear that $X_{\text{ét,aff}}$ indeed forms a site. Moreover, if the log structure M_X admits charts étale locally, then these "small enough" affine opens form a basis for the étale topology on X.

Lemma 5.6 *Further assume that* $\underline{X}$ *is a quasi-coherent log scheme, then the restriction of an étale sheaf from* $X_{\text{ét}}$ *to* $X_{\text{ét,aff}}$ *induces an equivalence of topoi.*

Proof. The canonical functor $u : X_{\text{ét,aff}} \longrightarrow X_{\text{ét}}$ is continuous, cocontinuous, and satisfies the property that any étale open $Y \in X_{\text{ét}}$ admits a cover by $\{U_i\}$ in $X_{\text{ét,aff}}$, as the log structure M_X is quasi-coherent. Thus u induces an equivalence $\text{Sh}(X_{\text{ét,aff}}) \xrightarrow{\sim} \text{Sh}(X_{\text{ét}})$ by [dJea] Tag 03A0. □

Theorem 5.7 *Let $\underline{X} = (X, M_X)$ be a quasi-coherent log scheme over $\underline{k}$. There exists a unique sheaf $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ on $X_{\text{ét}}$ valued in log Dieudonné algebras, such that on each étale local chart $(U, L) = \text{Spec}(R, L)$ of $\underline{X}$, there is a canonical map of log Dieudonné algebras*

$$\mathcal{W}\omega^*_{\underline{R}/\underline{k}} \longrightarrow \Gamma(U, \mathcal{W}\omega^*_{\underline{X}/\underline{k}})$$

which is an isomorphism on the underlying Dieudonné algebras, and is given by $L \to \Gamma(U, M_X)$ on the log structures.

Proof. It suffices to construct the sheaf of abelian groups $\mathcal{W}_n\omega^i_{\underline{X}/\underline{k}}$ for each $n \geq 1, i \geq 0$ and then take the inverse limit as n varies. By Lemma 5.6 it suffices to construct $\mathcal{W}_n\omega^i_{\underline{X}/\underline{k}}$ on $X_{\text{ét,aff}}$. To this end we define a presheaf of abelian groups $\mathcal{W}_n\omega^i_{\underline{X}/\underline{k}} : X_{\text{ét,aff}} \to \text{Ab}$ as follows. For each $U = \text{Spec}\, R \to X$ in $X_{\text{ét,aff}}$, we write $M(U) := \Gamma(U, M_X)$, and define [14]

$$\mathcal{W}_n\omega^i_{\underline{X}/\underline{k}}(U \to X) := \mathcal{W}_n\omega^i_{(R,M(U))/\underline{k}}.$$

To show that this presheaf is a sheaf on $X_{\text{ét,aff}}$, it suffices to check that, on each étale opens $U = \text{Spec}\, R \in X_{\text{ét,aff}}$, the restriction $W_n\omega^i_{\underline{X}/\underline{k}}|_{U_{\text{ét,aff}}}$ defines a sheaf on $U_{\text{ét,aff}} = X_{\text{ét,aff}}/U$. The latter category consists precisely of all étale R-algebras. In other words, we may assume that $X = \text{Spec}\, R$ is a "small enough" affine log scheme which admits a chart for its log structure. Let $R \to S$ be an étale map, write $V = \text{Spec}\, S$, then we have morphisms

$$\mathcal{W}_n\omega^i_{(R,M(X))/\underline{k}} \xrightarrow{\gamma_1} \mathcal{W}_n\omega^i_{(S,M(X))/\underline{k}} \xrightarrow{\gamma_2} \mathcal{W}_n\omega^i_{(S,M(V))/\underline{k}}$$

by functoriality. γ_2 is an isomorphism by Lemma A.2 and 4.13. We know that

$$\mathcal{W}_n\omega^i_{(S,M(V))/\underline{k}} \cong \mathcal{W}_n\omega^i_{(R,M(X))/\underline{k}} \otimes_{W_n(R)} W_n(S)$$

by Proposition 5.3. From the topological invariance of étale sites and Theorem 5.4.1 of [BLM18], we know that the association $S \mapsto W_n(S)$ identifies the étale algebras over R with étale algebras over $W_n(R)$. Therefore, on the affine étale site $\big(\text{Spec}\, W_n(R)\big)_{\text{ét,affine}}$, the presheaf $\mathcal{W}_n\omega^i_{\underline{X}/\underline{k}}$ is in fact a sheaf associated to the $W_n(R)$-module $\mathcal{W}_n\omega^i_{(R,M(X))/\underline{k}}$, by the usual étale descent. It is straightforward to keep track of differentials, Frobenius operators, as well as the monoid morphisms $M_X \to \mathcal{W}\omega^0_{\underline{X}/\underline{k}}$ and $M_X \xrightarrow{\delta} \mathcal{W}\omega^1_{\underline{X}/\underline{k}}$, hence the theorem follows. □

Remark 5.8 *It might be possible to develop the global theory directly by working with sheaves of log Dieudonné algebras over $X_{\text{ét}}$, mimicking the steps used in the proof of Theorem 4.10. However, we prefer to work locally and only globalize in the last step (which avoids sheafifying at each step). For our application to A_{inf}-cohomology in [Yao19], it is more convenient to construct the saturated log de Rham–Witt complexes in local charts. For example, for a p-adic formal scheme $\mathfrak{X}$ with semistable reduction over $\mathcal{O}_C$, the log structures do not appear so transparently on the complex of A_{inf}-modules $A\Omega_{\mathfrak{X}} \otimes^{\mathbf{L}} W(k)$. We could get around this problem by choosing a certain coordinate locally. Another (less serious) issue is that the formal log scheme $\underline{\mathfrak{X}}$ (with the divisorial log structure from its mod p fiber) is in fact not log-smooth over the base $\underline{\mathcal{O}_C}$, since its log structure is not coherent. This issue again goes away if we restrict to small enough charts. We refer interested reader to [Yao19] for the detail of this discussion.*

[14] The value of $\mathcal{W}_n\omega^i_{\underline{X}/\underline{k}}(U \to X)$ is independent of the choice of chart on U, by Lemma 4.13.

6. Comparison theorems

In this section we show that the saturated log de Rham–Witt complexes agree with existing constructions for log-smooth schemes of log-Cartier type. In this section, our log schemes (for example $\underline{X}$ and $\underline{Y}$) are assumed to be log-smooth of log-Cartier type over $\underline{k} = (k, N)$, where $N\backslash\{0\} \mapsto 0$. In particular, all log structures are assumed to be fine (and hence quasi-coherent).

6.1 Comparison with the constructions of Hyodo–Kato

In [HK94], Hyodo–Kato constructs a log de Rham–Witt complex $W^{\mathrm{HK}}\omega^*_{\underline{X}/\underline{k}}$ using the log crystalline site, as a generalization of [IR83] Let $(\underline{X}/\underline{W_n})_{\log\text{-cris}}$ be the log crystalline site of $\underline{X}$ over $\underline{W_n}$, where $\underline{W_n} = W_n(\underline{k})$ is equipped with the standard PD structure. Let

$$u^{\log}_{\underline{X}/W_n} : (\underline{X}/\underline{W_n})_{\log\text{-cris}} \to X_{\text{ét}}$$

be the canonical projection to the étale site. Hyodo–Kato then defines

$$W_n^{\mathrm{HK}}\omega^i_{\underline{X}/\underline{k}} := R^i u^{\log}_{\underline{X}/W_n,*}(\mathcal{O}_{\underline{X}/\underline{W_n}}),$$

where $\mathcal{O}_{\underline{X}/\underline{W_n}}$ is the structure sheaf on the log crystalline site. These objects are equipped with (collections of) operators d, F, V, and the canonical projections R, which we now briefly recall (see 4.1 and 4.2 of [HK94]). Let C_n^* be a crystalline complex (for a chosen embedding system, cf. 2.18-2.19 of *loc.cit*), which computes $Ru^{\log}_{\underline{X}/W_n,*}(\mathcal{O}_{\underline{X}/\underline{W_n}})$ (cf. Proposition 2.20 of *loc.cit*),

- Define the differential $d : W_n^{\mathrm{HK}}\omega^i_{\underline{X}/\underline{k}} \to W_n^{\mathrm{HK}}\omega^{i+1}_{\underline{X}/\underline{k}}$ to be the "Bockstein differential" induced from the exact sequence $0 \to C_n^* \xrightarrow{p^n} C_{2n}^* \to C_n^* \to 0$.
- Let $F : W_n^{\mathrm{HK}}\omega^i_{\underline{X}/\underline{k}} \to W_{n-1}^{\mathrm{HK}}\omega^i_{\underline{X}/\underline{k}}$ be the map induced by $C_n^* \to C_{n-1}^*$.
- Let $V : W_{n-1}^{\mathrm{HK}}\omega^i_{\underline{X}/\underline{k}} \to W_n^{\mathrm{HK}}\omega^i_{\underline{X}/\underline{k}}$ be the map induced by $C_{n-1}^* \xrightarrow{p} C_n^*$.
- Define $R_n : W_n^{\mathrm{HK}}\omega^i_{\underline{X}/\underline{k}} \to W_{n-1}^{\mathrm{HK}}\omega^i_{\underline{X}/\underline{k}}$ as follows. Let $\mu_n : H^i(C_n^*) \mapsto H^i(\eta_p C_{n-1}^*)$ be the map induced by $x \mapsto p^i x$ on the level of complexes [15], and let $\psi : H^i(C_{n-1}^*) \xrightarrow{\sim} H^i(\eta_p C_{n-1}^*)$ be the isomorphism given by Lemma 2.25 of [HK94], then define $R_n := \mu_n \circ \psi^{-1}$.

We then take the inverse limit along the restriction maps to form the Hyodo–Kato complex $W^{\mathrm{HK}}\omega^*_{\underline{X}/\underline{k}} := \varprojlim_R W_n^{\mathrm{HK}}\omega^*_{\underline{X}/\underline{k}}$.

Theorem 6.1 *Let $\underline{X}$ be a log scheme over $\underline{k}$ that is log-smooth of log-Cartier type. There is a natural isomorphism of sheaves of log Dieudonné algebras*

$$\mathcal{W}\omega^*_{\underline{X}/\underline{k}} \xrightarrow{\sim} W^{\mathrm{HK}}\omega^*_{\underline{X}/\underline{k}}.$$

Remark 6.2 *In particular, $\mathcal{W}\omega^*_{\underline{X}/\underline{k}}$ computes the log crystalline cohomology for quasi-coherent log-smooth schemes of log-Cartier type.*

To prove the theorem it suffices to work locally. The main point is to show that $W^{\mathrm{HK}}\omega^*_{\underline{X}/\underline{k}}$ indeed takes values in log Dieudonné algebras. This is straightforward but quite tedious to carry

[15] The complex "$\eta_p C_{n-1}^*$" needs justification, since C_{n-1}^* is p^{n-1}-torsion. This is in fact the complex denoted by E_{n-1}^* in [HK94], where E_{n-1}^i is defined to be "$(\eta_p C_m)^i/p^{n-1}$" for m sufficiently large, where $\eta_p C_m^i = \{x \in p^i C_m^i : dx \in p^{i+1} C_m^{i+1}\}$ as usual. For $m > n + i + 1$, the issue of p-torsion goes away and the quotient $(\eta_p C_m)^i/p^{n-1}$ is well-defined.

out, which we prove in Lemma 6.3 and 6.4. Now we restrict our attention to the affine case and let $\underline{Y} = (\operatorname{Spec} R, M)$ be an affine log scheme over $\underline{k}$, and consider the Hyodo–Kato complex

$$W^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}} := \Gamma(Y, W^{\mathrm{HK}}\omega^*_{\underline{Y}/\underline{k}}) = \varprojlim_{R_n} \Gamma(Y, W_n^{\mathrm{HK}}\omega^*_{\underline{Y}/\underline{k}}).$$

The canonical projections R_n commute with operators d, F and V on each finite level from the definitions, hence we obtain a cochain complex in the inverse limit, equipped with F and V. By relations given in 4.1.1 of [HK94], $W^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}}$ is a Dieudonné complex.

Lemma 6.3 *The Frobenius F on $W^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}}$ is a graded ring homomorphism.*

This is of course of no surprise and probably well-known to experts, but we cannot track down an explicit proof, so we record the proof here.

Proof. For notational simplicity we write $W^{\mathrm{HK},*}$ (resp. $W_n^{\mathrm{HK},*}$) for the complex $W^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}}$ (resp. $W_n^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}} := \Gamma(Y, W_n^{\mathrm{HK}}\omega^*_{\underline{Y}/\underline{k}})$) in this proof. By Proposition 4.7 of [HK94] and its proof, there is a canonical surjection

$$\psi : \omega^*_{W_n(\underline{R})/W_n(\underline{k})} \twoheadrightarrow W_n^{\mathrm{HK},*}$$

with kernel $I_n \subset \omega^*_{W_n(\underline{R})/W_n(\underline{k})}$ a graded differential ideal generated by elements of the form $\{\eta_{i,j,a,m}, d\eta_{i,j,a,m}\}$ for $0 \leq j \leq i < n, a \in R, m \in M$, where

$$\eta_{i,j,a,m} := V^i\big([a]\big) dV^j\big([\alpha(m)]\big) - V^i\big([a\alpha(m)^{p^{i-j}}]\big) \mathrm{d}\log(m).$$

To further ease notation, we write $\omega^*_{W_n(\underline{R})} := \omega^*_{W_n(\underline{R})/W_n(\underline{k})}$. The map ψ is compatible with the canonical projection maps. By (the corrected version in Section 7 in [Nak05] of) 4.9.1 of [HK94], we have the following isomorphism of cochain complexes

$$s = \psi \circ t : (W_n^{\mathrm{HK},*})'' \xrightarrow{t} \omega^*_{W_n(\underline{R})}/I_n \xrightarrow{\psi} W_n^{\mathrm{HK},*}$$

where $(W^{\mathrm{HK},i})''$ is defined as a quotient of [16]

$$(W_n(R) \otimes \wedge^i M^{\mathrm{gp}}/N^{\mathrm{gp}}) \oplus (W_n(R) \otimes \wedge^i M^{\mathrm{gp}}/N^{\mathrm{gp}}).$$

By Remark 4.8, we have a graded ring homomorphism $F : \omega^*_{W_n(\underline{R})} \to \omega^*_{W_{n-1}(\underline{R})}$. Unwinding its construction, F sends the ideal I_n into I_{n-1}, thus inducing a graded ring homomorphism

$$F : \omega^*_{W_n(\underline{R})}/I_n \to \omega^*_{W_{n-1}(\underline{R})}/I_{n-1}.$$

We claim that the following diagram (on the right) commutes

$$\begin{array}{ccccc} (W_n^{\mathrm{HK},*})'' & \xrightarrow{t} & \omega^*_{W_n(\underline{R})}/I_n & \xrightarrow{\psi} & W_n^{\mathrm{HK},*} \\ \downarrow F & & \downarrow F & & \downarrow F \\ (W_{n-1}^{\mathrm{HK},*})'' & \xrightarrow{t} & \omega^*_{W_{n-1}(\underline{R})}/I_{n-1} & \xrightarrow{\psi} & W_{n-1}^{\mathrm{HK},*} \end{array}$$

[16]The complex $(W^{\mathrm{HK},i})'$ which appears in Proposition 4.6 of [HK94] is incorrect (and the proof given there is incomplete), instead we should take the modification $(W^{\mathrm{HK},i})''$ given in Section 7 of [Nak05]. With this correction, the map s is indeed an isomorphism, by Theorem 7.5 of *loc. cit.*

Where $F : (W_n^{\mathrm{HK},i})'' \to (W_{n-1}^{\mathrm{HK},i})''$ is defined as follows: for any $x \in W_n(R)$ and $y \in W_n(R)^\times$, let us write $y = [y_0] + Vy'$, and send

$$(x \otimes m_1 \wedge \cdots m_i, 0) \mapsto (F(x) \otimes m_1 \wedge \cdots m_i, 0)$$
$$(0, y \otimes m_2 \wedge \cdots m_i) \mapsto ([y_0]^p \otimes \beta^{-1}([y_0]) \wedge m_2 \wedge \cdots \wedge m_i, 0) + (0, y' \otimes m_2 \wedge \cdots \wedge m_i).$$

β here denotes the log structure $M \xrightarrow{\alpha} R \xrightarrow{[\,]} W(R)$, where the first arrow α is a log structure by assumption, so there exists a unique element $\beta^{-1}([y_0]) \in M$ which is sent to $[y_0]$.

The desired commutativity follows from Remark 7.7.4 of [Nak05] and the commutativity of the left square, which we now check by hand on the generators of $(W_n^{\mathrm{HK},i})''$. Clearly $F \circ t = t \circ F$ on $(x \otimes m_1 \wedge \cdots m_i, 0)$; for $(0, y \otimes m_2 \wedge \cdots m_i)$, we simply observe that

$$Fdy = Fd([y_0] + Vy') = [y_0]^p \mathrm{d}\log[y_0] + dy'.$$

This finishes the proof. □

Lemma 6.4 *$W^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}}$ is a saturated Dieudonné algebra.*

Proof. Retain notations from the proof of Lemma 6.3. By Corollary 4.5 (2) of [HK94], $W^{\mathrm{HK}}\omega^*$ is p-torsion free. Since $\underline{R}/\underline{k}$ is log-smooth of log-Cartier type, by the same proof of I.3.11.4 of [Ill79] in the case $n = 0$, the Cartier isomorphism (cf. Proposition A.10) implies that

$$\ker(d : W_1^{\mathrm{HK}}\omega^i \to W_1^{\mathrm{HK}}\omega^{i+1}) = \mathrm{im}\left(F : W_2^{\mathrm{HK}}\omega^i \to W_1^{\mathrm{HK}}\omega^i\right).$$

This in turn implies that $d^{-1}(p \cdot W^{\mathrm{HK}}\omega^*) = F\left(W^{\mathrm{HK}}\omega^*\right)$. To see this, suppose that $dx \in p \cdot W^{\mathrm{HK}}\omega^*$, then by the equality above we have $x = Fx_1' + Vx_2' + dVy_2' = Fx_1 + Vx_2'$ where $x_1 = x_1' + dV^2(y_2')$. This implies that $dVx_2' \in p \cdot W^{\mathrm{HK}}\omega^*$, so $dx_2' = FdVx_2' \in p \cdot W^{\mathrm{HK}}\omega^*$ and we may write $x_2' = Fx_2 + Vx_3'$. By repeating this procedure we may write $x = F(x_1 + Vx_2 + V^2x_3 + \cdots)$. This together with the injectivity of F (since $W^{\mathrm{HK}}\omega^*$ is p-torsion free) implies that $W^{\mathrm{HK}}\omega^*$ is saturated (as a Dieudonné complex).

It remains to show that $F(x) \equiv x^p \mod p$ for all $x \in W^{\mathrm{HK}}\omega^0$. Since $W^{\mathrm{HK}}\omega^*$ is saturated, in fact it suffices to check that $F(x) \equiv x^p \mod V$ for all $x \in W^{\mathrm{HK}}\omega^0$. But this follows from the commutative diagram in the proof of Lemma 6.3, since the middle vertical map is constructed to extend the Witt vector Frobenius and the map ψ preserves the V-filtration (by Proposition 7.1 and diagram 7.5.6 in [Nak05]). □

Corollary 6.5 *$W^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}}$ admits a natural structure of a log Dieudonné algebra, which is strict (in particular saturated) and p-compatible.*

Proof. For each n, the log structure $[\alpha]_n : M \xrightarrow{\alpha} R \xrightarrow{[\,]} W_n(R)$ induces a log derivation $(d, \delta_n : M \to \omega^1_{W_n(\underline{R})})$. By composing with projection we get a monoid map $\delta_n : M \to \omega^1_{W_n(\underline{R})}/I_n$. Both $[\alpha_n]$ and δ_n are compatible under the canonical projections. Now the map ψ in

$$(W_n^{\mathrm{HK},*})'' \xrightarrow{t} \omega^*_{W_n(\underline{R})}/I_n \xrightarrow{\psi} W_n^{\mathrm{HK},*}$$

(from the proof of Lemma 6.3) is also compatible with the canonical projections. Therefore we have monoid morphisms (which we still denote by the same symbols): $[\alpha]_n : M \to W_n^{\mathrm{HK}}\omega^0$ and $\delta_n : M \to W_n^{\mathrm{HK}}\omega^1$. In the inverse limit they give rise to

$$[\alpha] : M \to W^{\mathrm{HK}}\omega^0, \quad \delta : M \to W^{\mathrm{HK}}\omega^1.$$

This makes $W^{\mathrm{HK}}\omega^*$ a p-compatible log Dieudonné algebra. It is saturated by Lemma 6.4; and it is strict since for each $n \geq 1$ and any $k \geq 0$, we have the following exact sequence

$$0 \to \operatorname{im}\left(W_k^{\mathrm{HK}}\omega^* \xrightarrow{V^n + dV^n} W_{n+k}^{\mathrm{HK}}\omega^*\right) \longrightarrow W_{n+k}^{\mathrm{HK}}\omega^* \xrightarrow{\mathrm{pr}} W_n^{\mathrm{HK}}\omega^* \to 0$$

by Section 4.9 of [HK94]. □

Now we prove Theorem 6.1 with the preparations above. As discussed previously, it suffices to work locally on charts. In the affine setup, the identity map $\underline{R} \longrightarrow W_1^{\mathrm{HK}}\omega^0_{\underline{R}/\underline{k}} \cong \underline{R}$ induces a map of strict log Dieudonné algebras $\gamma : \mathcal{W}\omega^*_{\underline{R}/\underline{k}} \longrightarrow W^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}}$. Theorem 6.1 follows from

Proposition 6.6 *γ is an isomorphism of log Dieudonné algebras*

$$\gamma : \mathcal{W}\omega^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} W^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}}.$$

Proof. The induced map $\overline{\gamma} : \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}} \to W_1^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}}$ is an isomorphism by Proposition 4.22. The theorem then follows from Corollary 2.10 part (3). □

6.2 Comparison with the constructions of Matsuue

Recall that we are still under the assumption that $\underline{R}/\underline{k}$ is log-smooth of log-Cartier type. We start with a review of the definition of the category $\mathcal{C}_{FV}$ of (R-framed) log F-V-procomplexes.

Definition 6.7 ([Mat17] Definition 3.4) *Let $\underline{R} = (R, M)$ be a log algebra over $\underline{k} = (k, N)$. A R-framed F-V-procomplex over $\underline{R}/\underline{k}$ is a projective system*

$$\cdots \to E^*_{m+1} \xrightarrow{R_m} E^*_m \xrightarrow{R_{m\text{-}1}} \cdots \to E^*_1 \to E^*_0 = 0$$

where each $E^*_m = (E^*_m, d_m, \delta_m)$ is a log cdga over $W_m(\underline{R})/W_m(\underline{k})$, together with

- *a collection $F : E^*_{m+1} \to E^*_m$ of graded ring homomorphisms;*
- *a collection $V : E^*_m \to E^*_{m+1}$ of graded abelian group homomorphisms.*

These data are required to satisfy the following conditions

(1) $R = \{R_m\}$ is compatible with $\delta = \{\delta_m\}$, i.e. $\delta_m = R_m \circ \delta_{m+1}$ for $m \geq 0$.

(2) R is compatible with the collection of maps F and V.

(3) The structure maps $\beta_m : W_m(R) \to E^0_m$ are compatible with F and V.

(4) The collections of F, V, d, δ satisfies relations

$$FV = p \qquad Fd_{m+1}V = d_m \qquad F\delta_{m+1} = \delta_m$$

$$(Vx)y = V(xFy) \qquad Fd_{m+1}[x] = [x]^{p-1}d_m[x].$$

Here by the structure maps, we mean the collection of ring homomorphisms $\beta_m : W_m(R) \to E^0_m$ given as part of the data of the $W_m(\underline{R})/W_m(\underline{k})$-cdga (E^*_m, d_m, δ_m) where $(d_m \circ \beta_m, \delta_m : M \to E^1_m)$ forms a log derivation of $W_m(\underline{R})/W_m(\underline{k})$ into E^1 and satisfies $d_m \circ \delta_m = 0$.

Denote the category of log F-V-procomplexes over $\underline{R}/\underline{k}$ by $\mathcal{C}_{FV,\underline{R}}$. In [Mat17], Matsuue constructs an initial object $\{W_m\Lambda^*_{\underline{R}/\underline{k}}\}$ in $\mathcal{C}_{FV,\underline{R}}$, where each $W_m\Lambda^*_{\underline{R}/\underline{k}}$ is constructed as a certain quotient of $\omega^*_{W_m(\underline{R})/W_m(\underline{k})}$. The complexes $W\Lambda^*_{\underline{R}/\underline{k}} = \lim_n W_n\Lambda^*_{\underline{R}/\underline{k}}$ is then globalized to a sheaf $W\Lambda^*_{\underline{X}/\underline{k}}$ on $X_{\text{ét}}$. In fact, the construction in [Mat17] works more generally over non-perfect bases, but we have restricted to the case of perfect base (in fact a perfect field) to compare with our version of saturated log de Rham–Witt complexes. The following lemma is clear.

Lemma 6.8 *The saturated log de Rham–Witt complexes $\mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ gives rise to an R-framed F-V-procomplex $\{W_m\omega^*_{\underline{R}/\underline{k}}\}$ over $\underline{R}/\underline{k}$.*

Therefore by the universal property of $W\Lambda^*$ we have a map $\gamma' : W\Lambda^*_{\underline{R}/\underline{k}} \longrightarrow \mathcal{W}\omega^*_{\underline{R}/\underline{k}}$ of cochain complexes which is compatible with F, V and R.

Proposition 6.9 *γ' is an isomorphism.*

Proof. The map γ' induces commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^n W_{n+1}\Lambda^*_{\underline{R}/\underline{k}} & \longrightarrow & W_{n+1}\Lambda^*_{\underline{R}/\underline{k}} & \longrightarrow & W_n\Lambda^*_{\underline{R}/\underline{k}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^n \mathcal{W}_{n+1}\omega^*_{\underline{R}/\underline{k}} & \longrightarrow & \mathcal{W}_{n+1}\omega^*_{\underline{R}/\underline{k}} & \longrightarrow & \mathcal{W}_n\omega^*_{\underline{R}/\underline{k}} & \longrightarrow & 0 \end{array}$$

of short exact sequences, where the filtrations are standard filtrations, given by $\mathrm{Fil}^n W_{n+1}\Lambda^*_{\underline{R}/\underline{k}} = V^n W_1\Lambda^*_{\underline{R}/\underline{k}} + \mathrm{d}V^n W_1\Lambda^{*-1}_{\underline{R}/\underline{k}}$ and likewise for $\mathcal{W}_{n+1}\omega^*_{\underline{R}/\underline{k}}$. By Proposition 4.22, we know that γ' induces an isomorphism $W_1\Lambda^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} \mathcal{W}_1\omega^*_{\underline{R}/\underline{k}}$ and hence $\mathrm{Fil}^n W_{n+1}\Lambda^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} \mathrm{Fil}^n \mathcal{W}_{n+1}\omega^*_{\underline{R}/\underline{k}}$ for each n. The proposition therefore follows by induction. □

Corollary 6.10 *For a quasi-coherent log scheme $\underline{X}$ which is log-smooth of log-Cartier type over $\underline{k}$, the isomorphism γ' globalizes to an isomorphism of sheaves $\gamma' : W\Lambda^*_{\underline{X}/\underline{k}} \xrightarrow{\sim} \mathcal{W}\omega^*_{\underline{X}/\underline{k}}$.*

Remark 6.11 *Another way to obtain $\mathcal{W}\omega^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} W\Lambda^*_{\underline{R}/\underline{k}}$ is to use the comparison with Hyodo–Kato complexes in Subsection 6.1 and prove that the Hyodo–Kato complexes agree with $W\Lambda^*_{\underline{R}/\underline{k}}$. The latter is claimed in Proposition 3.1 of [GL17], which asserts that there is an isomorphism $W\Lambda^*_{\underline{R}/\underline{k}} \xrightarrow{\sim} W^{\mathrm{HK}}\omega^*_{\underline{R}/\underline{k}}$ identifying the log structures, F, V and R on both sides. The key assertion used in their proof is that $W^{\mathrm{HK}}_m\omega^*_{\underline{R}/\underline{k}}$ form an R-framed F-V-procomplex, which is claimed without justification. It seems to the author that proving their claim would involve proving most of the arguments in Subsection 6.1, which seems non-obvious (though probably known to experts).*

7. The monodromy operator

In this section we explain a construction of the monodromy operator on the log crystalline cohomology for log schemes of "generalized semistable type" in the following sense: let $\underline{k} = (k, \mathbf{N})$ be the standard log point, $\underline{X}/\underline{k}$ a log scheme such that étale locally, $\underline{X}$ is naively étale over a finite fiber product of log schemes that are given by log algebras of the form

$$\begin{array}{ccc} \underline{R}^{\square} = & \Big(\mathbf{N}^{r+1} \xrightarrow{\alpha} & k[X_0, ..., X_r, ..., X_d]/\prod_{0\le i\le r} X_i\Big) \\ & \uparrow{\scriptstyle 1\mapsto(1,...,1)} & \uparrow \\ \underline{k} = & \Big(\mathbf{N} \longrightarrow & k\Big) \end{array}$$

where the log structure α on $R^{\square}$ is given by $e_i \mapsto X_i$ for $0 \le i \le r$, here $e_i = (0, ..., 1, ..., 0)$ with 1 at the i^{th} position. More precisely, $\underline{X}$ can be covered by charts which are naively étale over log algebras of the form $\underline{S}^{\square} = (M, S^{\square})$, where

$$S^{\square} = k[X_{l,h}]_{\substack{1\le l\le s \\ 0\le h\le d_l}} / \Big(\prod_{1\le i\le d_1} X_{1,i}, \cdots, \prod_{1\le i\le d_s} X_{s,i}\Big)$$

and $M = \bigsqcup_{\substack{\mathbf{N} \\ 1\leq l\leq s}} \mathbf{N}^{r_l+1}$ is the pushout of the diagonal maps $\mathbf{N} \to \mathbf{N}^{r_l+1}$ for $1 \leq l \leq s$. Clearly $\underline{R}^\square$ (resp. $\underline{S}^\square$) is fine and log-smooth over $\underline{k}$ of log-Cartier type.

Lemma 7.1 *Let $\underline{R}^\square$ be as above, let $k^\circ = (k, 0)$ be the trivial log point, then $\underline{R}^\square/k^\circ$ satisfies conditions (1) and (2) in Remark 4.26. In particular, there is a canonical isomorphism*

$$\omega_{\underline{R}^\square/k^\circ} \xrightarrow{\sim} \mathcal{W}_1\omega_{\underline{R}^\square/k^\circ}.$$

The same conclusion holds for $\underline{S}^\square$.

Proof. For notational simplicity we check the claim for $\underline{R}^\square$. For (1) we may take the lifting $A^\square = W(k)[X_0, ..., X_d]/\prod_{0\leq i\leq r} X_i$ with log structure $\mathbf{N}^{r+1} \xrightarrow{e_i \mapsto X_i} A^\square$. For the lift of Frobenius we take the canonical lifting σ on $W(k)$ and $X_i \mapsto X_i^p$, while Frobenius on $\mathbf{N}^{r+1}$ is given by multiplication by p. The log differential $\omega^1_{\underline{A}^\square/\underline{k}}$ is free over $\underline{A}$ with generators $\mathrm{dlog} X_0, ..., \mathrm{dlog} X_r, \mathrm{d}X_{r+1}, ..., \mathrm{d}X_d$ (see Subsection A.2.2), hence both condition (1) and (2) in Remark 4.26 are satisfied. $\square$

The canonical map $\underline{R}^\square/k^\circ \longrightarrow \underline{R}^\square/\underline{k}$ induces a morphism of saturated log de Rham–Witt complexes

$$\Theta : \mathcal{W}\omega^*_{\underline{R}^\square/k^\circ} \longrightarrow \mathcal{W}\omega^*_{\underline{R}^\square/\underline{k}}.$$

The cochain complex $\ker\Theta$ is concentrated in (cohomological) degree ≥ 1.

Proposition 7.2 *The cochain complex $(\ker\Theta)[1]$ admits a structure of a Dieudonné complex, which is saturated and strict. Moreover, there is a short exact sequence*

$$0 \to \mathcal{W}\omega^*_{\underline{R}^\square/\underline{k}}[-1] \longrightarrow \mathcal{W}\omega^*_{\underline{R}^\square/k^\circ} \xrightarrow{\Theta} \mathcal{W}\omega^*_{\underline{R}^\square/\underline{k}} \to 0.$$

The same conclusion holds when $\underline{R}^\square$ is replaced by $\underline{S}^\square$. In particular, $(\ker\Theta)[1]$ carries a (log) Dieudonné algebra structure.

Proof. Again we check the proposition for $\underline{R}^\square$ since the only complication for $\underline{S}^\square$ is notational. Write $K^* = \ker\Theta$ which is a p-torsion free sub-Dieudonné complex of $\mathcal{W}\omega^*_{\underline{R}^\square/k^\circ}$ (here we ignore the algebra structure). It is straightforward to check that

$$\phi_F : K^* \longrightarrow \eta_p(K^*), \quad x \in K^i \mapsto p^i F(x)$$

is an isomorphism, therefore $K^*[1]$ is a saturated Dieudonné complex as multiplication by p gives an isomorphism $\eta_p(K^*[1]) \xrightarrow{\sim} \eta_p(K^*)[1]$. By Lemma 7.3, we have a short exact sequence

$$0 \to W_m(K^*) \longrightarrow \mathcal{W}_m\omega^*_{\underline{R}^\square/k^\circ} \longrightarrow \mathcal{W}_m\omega^*_{\underline{R}^\square/\underline{k}} \to 0$$

for each m. Taking the inverse limit we see that $K^*[1]$ is strict.

Next we construct a map of cochain complexes $\omega^*_{\underline{A}^\square/\underline{k}} \to \omega^*_{\underline{A}^\square/k^\circ}[1]$ preserving Frobenius on both sides. For this we define an element $\theta \in \omega^1_{\underline{A}^\square/k^\circ}$ from the log differential $\mathrm{dlog} : \mathbf{N}^{r+1} \to \omega^1_{\underline{A}^\square/k^\circ}$ by

$$\theta := \mathrm{dlog}(e_0) + \cdot + \mathrm{dlog}(e_r).$$

(More generally, for the product $\underline{S}^\square = (S^\square, M)$, we have a map $\mathbf{N} \to M$ coming from each diagonal map $\mathbf{N} \to \mathbf{N}^{r_l+1}$. Let $e \in M$ be the image of $1 \in \mathbf{N}$. Then θ is the element $\mathrm{dlog}(e)$). We abuse notation and use the same symbol to denote the image of θ in $\mathcal{W}\omega^*_{\underline{R}^\square/k^\circ}$, then clearly

$\theta \in K^1$. Since $\omega^*_{\underline{A}^\square/\underline{k}}$ is generated over $A^\square$ by $(\wedge_i \mathrm{dlog} X_i) \wedge (\wedge_j \mathrm{d}X_j)$, we may a map of complexes $\Psi : \omega^*_{\underline{A}^\square/\underline{k}} \to \omega^*_{\underline{A}^\square/k^\circ}[1]$ by

$$(\wedge_i \mathrm{dlog} X_i) \wedge (\wedge_j \mathrm{d}X_j) \mapsto (\wedge_i \mathrm{dlog} X_i) \wedge (\wedge_j \mathrm{d}X_j) \wedge \theta.$$

The map is well defined, since

$$\sum_{0\le i\le r} \mathrm{dlog} X_i \mapsto (\sum_{0\le i\le r} \mathrm{dlog} X_i) \wedge \theta = \theta \wedge \theta = 0.$$

Moreover, since $d(\theta) = 0$ (by $d \circ \mathrm{dlog} = 0$) and $F(\theta) = \theta$, the map Ψ preserves differential and Frobenius structures on both sides.

By passing to the (p-adic) completion and strictification (complete saturation), we get a map $\Psi : \mathcal{W}\omega^*_{\underline{R}^\square/\underline{k}} \to \mathcal{W}\omega^*_{\underline{R}^\square/k^\circ}[1]$. The image of Ψ lie in $K^*[1]$, so we have now constructed a map of strict Dieudonné complexes

$$\Psi : \mathcal{W}\omega^*_{\underline{R}^\square/\underline{k}} \longrightarrow K^*[1].$$

To finish the proof of the proposition, it remains to prove that this map is an isomorphism. By Corollary 2.7.4 in [BLM18] [17], it suffices to show that

$$W_1(\Psi) : \mathcal{W}_1\omega^*_{\underline{R}^\square/\underline{k}} \longrightarrow W_1(K^*[1])$$

is an isomorphism. By Lemma 7.3 and Lemma 7.1, we need to check that

$$\omega^*_{\underline{R}^\square/\underline{k}} \xrightarrow{\wedge\theta_1} \ker(\omega^*_{\underline{R}^\square/k^\circ} \to \omega^*_{\underline{R}^\square/\underline{k}})[1]$$

is an isomorphism of cochain complexes, which follows from their explicit descriptions. □

The following lemma is used in the proof above

Lemma 7.3 *Let $\Theta : A^* \longrightarrow B^*$ be a map between two saturated Dieudonné complexes. Let $K^* = \ker\Theta$ be the sub Dieudonné complex (which is saturated). Define the standard filtration on A^* (resp. on B^* and K^*) as in Subsection 3.4 by $\mathrm{Fil}^i(A^*) = V^i(A^*) + \mathrm{d}V^i(A^*)$. Then*

$$\mathrm{Fil}^i(K^*) = \ker(\mathrm{Fil}^i(A^*) \xrightarrow{\Phi} \mathrm{Fil}^i(B^*)).$$

Proof. We need to show that $\ker(\mathrm{Fil}^i(A^*) \xrightarrow{\Phi} \mathrm{Fil}^i(B^*)) \subset \mathrm{Fil}^i(K^*)$. In other words, if $x = V^i y + dV^i z$, and $\Phi(x) = 0$, we claim that $x \in \mathrm{Fil}^i(K^*)$. For notational simplicity we write $\overline{x} = \Phi(x)$ for the image of $x \in A^*$ in B^*. By assumption we know that

$$d\overline{z} = F^i dV^i \overline{z} = -F^i(V^i \overline{y}) = -p^i \overline{y} \in B^*.$$

Since B^* is saturated, we know that $\overline{z} = F^i \overline{w}$ for some $\overline{w} \in B^*$, so there exists $w \in A^*$ and $\alpha \in K^*$ such that

$$x = V^i y + dV^i(F^i w + \alpha) = V^i(y + F^i dw) + dV^i \alpha.$$

Therefore, it suffices to prove the claim for x of the form $x = V^i y$, but this is clear since $V^i \overline{y} = 0$ implies that $\overline{y} = 0$ as B^* is p-torsion free. □

We now prove Theorem 11 in the introduction.

[17] see also Corollary 2.10. Note that the proof we cite from [BLM18] is stated for Dieudonné complexes

Corollary 7.4 *Let $\underline{X}/\underline{k}$ be a log scheme of "generalized semistable type", then there is a short exact sequence of cochain complexes*

$$0 \to \mathcal{W}\omega^*_{\underline{X}/\underline{k}}[-1] \longrightarrow \mathcal{W}\omega^*_{\underline{X}/k^\circ} \longrightarrow \mathcal{W}\omega^*_{\underline{X}/\underline{k}} \to 0$$

that preserves the Frobenius structures. As a consequence, this gives rise to a connecting homomorphism on the log crystalline cohomology

$$N : H^*_{\text{log-cris}}(X/W(\underline{k})) \longrightarrow H^*_{\text{log-cris}}(X/W(\underline{k})).$$

This operator satisfies $N\varphi = p\varphi N$ and agrees with the monodromy operator constructed in [HK94] when $\underline{X}$ is of semistable type.

Proof. Let R be a k-algebra with an étale coordinate, namely an étale morphism $S^\square \to R$ where $S^\square$ has the form described as in the beginning of this section. Equip R with the log structure from $M \to S^\square$. Then for each m we have

$$0 \to \mathcal{W}_m\omega^*_{\underline{R}/\underline{k}}[-1] \longrightarrow \mathcal{W}_m\omega^*_{\underline{R}/k^\circ} \longrightarrow \mathcal{W}_m\omega^*_{\underline{R}/\underline{k}} \to 0$$

from the étale base change of the corresponding sequence for $\mathcal{W}_m\omega^*_{\underline{S}^\square/\underline{k}}$. To globalize from these étale coordinates, we observe that the element $\theta \in \mathcal{W}\omega^1_{\underline{R}/k^\circ}$ is independent of the choice of coordinates, since it can be described as the image of $\operatorname{dlog}(e)$ in $\omega^1_{W(\underline{R})/W(k^\circ)} \longrightarrow W(\omega^*_{W(\underline{R})/W(k^\circ)})^1_{\mathfrak{sat}} =: \mathcal{W}\omega^1_{\underline{R}/k^\circ}$, so the sequences above glue to an exact sequence of sheaves

$$0 \to \mathcal{W}_m\omega^*_{\underline{X}/\underline{k}}[-1] \longrightarrow \mathcal{W}_m\omega^*_{\underline{X}/k^\circ} \longrightarrow \mathcal{W}_m\omega^*_{\underline{X}/\underline{k}} \to 0.$$

Since there is no higher Rlim terms, taking the inverse limit we reach the conclusion. The relation $N\varphi = p\varphi N$ follows from $dF = pFd$ since φ is induced by $\phi_F : x \in \mathcal{W}\omega^i_{\underline{X}/\underline{k}} \mapsto p^i F(x) \in \mathcal{W}\omega^i_{\underline{X}/\underline{k}}$.

Finally, the comparison with Hyodo–Kato's construction follows from Theorem 6.1 and the description of $(W_n\widetilde{\omega}^q_Y)'$ in 4.20 in [HK94]. □

Appendix A. Log geometry

In this section we fix notations we use in log geometry, following [Kat89], and is slightly different from notations in literature.[18] Most of the material (if not everything) presented in this section is considered standard, but we make some remarks that might only be obvious for experts.

A.1 Log schemes (after Fontaine–Illusie–Kato)

A pre-log structure on a scheme X is a morphism $\alpha : M \to \mathcal{O}_X$ of sheaves of monoids on $X_{\text{ét}}$. The pre-log structure α is a log structure if $\alpha : \alpha^{-1}(\mathcal{O}_X^\times) \xrightarrow{\sim} \mathcal{O}_X^\times$ is an isomorphism. We will suppress notation and write (X, M) or $\underline{X}$ for (pre-) log schemes. For a pre-log structure M on X, we often denote by M^a (or $\mathcal{M}^a$) its associated log structure, which is the push-out of $\alpha^{-1}(\mathcal{O}_X^\times) \to M$ and $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ in the category of sheaves of monoids on $X_{\text{ét}}$. We make the same definitions for formal schemes. A log structure is *fine* if it is coherent and integral. In this article we work with fine log schemes, but in the application to $\mathrm{A}_{\mathrm{inf}}$-cohomology in [Yao19] we relax the condition of coherence to quasi-coherence. A chart for a log scheme $\underline{X} = (X, M)$ is a morphism $P_X \to M$ from a constant pre-log structure P_X to M which induces an isomorphism $P_X^a \xrightarrow{\sim} M$. Charts exist étale locally for fine log schemes. A chart for a morphism $f : (X, M) \to (Y, N)$ between log

[18] For example, we always use X to denote a scheme and $\underline{X}$ a (pre-) log scheme, which is opposite to the convention adopted by some algebraic geometers.

schemes is a triple $(P_X \xrightarrow{\psi_X} M, Q_Y \xrightarrow{\psi_Y} N, Q \xrightarrow{\beta} P)$ where ψ_X, ψ_Y are charts for $(X, M), (Y, N)$ respectively, and β a morphism of monoids making the obvious diagram commute.

Lemma A.1 *Let X be a scheme, $L_X \xrightarrow{\alpha} \mathcal{O}_X$ be a constant pre-log structure with underlying monoid L. Let $\mathcal{L} = (L_X)^a$ denote the log structure associated to L_X, and $L^{\mathrm{sh}} := \Gamma(X, \mathcal{L})$ be the global sections of $\mathcal{L}$, then its associated log structure $\mathcal{L}^{\mathrm{sh}} := (L_X^{\mathrm{sh}})^a$ is identified with $\mathcal{L}$.*

Proof. We view L^{sh} as a constant pre-log structure $\beta : L_X^{\mathrm{sh}} \to \mathcal{O}_X$ by composing the natural map $L_X^{\mathrm{sh}} \to \mathcal{L}$ with the induced log structure $\alpha^a : \mathcal{L} \to \mathcal{O}_X$. We have the following diagram

$$\begin{array}{ccccc} & & L_X^{\mathrm{sh}} & \xrightarrow{\iota_\beta} & \mathcal{L}^{\mathrm{sh}} \\ & {\scriptstyle h}\nearrow & & \searrow{\scriptstyle i} & \\ L_X & & \xrightarrow{\iota_\alpha} & & \mathcal{L} \end{array}$$

of pre-log structures on $X_{\text{ét}}$, where h is given by ι_α valued on X. By the universal property, $\iota_\beta \circ h : L_X \to \mathcal{L}^{\mathrm{sh}}$ induces a morphism $f : \mathcal{L} \to \mathcal{L}^{\mathrm{sh}}$ of log structures, while i induces $g : \mathcal{L}^{\mathrm{sh}} \to \mathcal{L}$. The maps f and g are inverses to each other. To see that $f \circ g = \mathrm{id}$, note that under the isomorphism $\mathrm{Hom}_{\log}(\mathcal{L}^{\mathrm{sh}}, \mathcal{L}) \cong \mathrm{Hom}_{\text{pre-log}}(L_X^{\mathrm{sh}}, \mathcal{L})$, g corresponds to i, thus after composing by $f : \mathcal{L} \to \mathcal{L}^{\mathrm{sh}}$, we know that $f \circ g$ corresponds to $f \circ i$ under the isomorphism $\mathrm{Hom}_{\log}(\mathcal{L}^{\mathrm{sh}}, \mathcal{L}^{\mathrm{sh}}) \cong \mathrm{Hom}_{\text{pre-log}}(L_X^{\mathrm{sh}}, \mathcal{L}^{\mathrm{sh}})$. But $f \circ i = \iota_\beta$ by construction, which corresponds to $\mathrm{id} : \mathcal{L}^{\mathrm{sh}} \to \mathcal{L}^{\mathrm{sh}}$, hence $f \circ g = \mathrm{id}$. The other verification is similar. □

Lemma A.2 *Let $(\operatorname{Spec} R, \mathcal{M}^a)$ be the log scheme associated to a log algebra $\underline{R} = (R, \alpha : M \to R)$, and $R \to S$ be an* étale *morphism, then $(\operatorname{Spec} S, \mathcal{M}^a|_{\operatorname{Spec} S})$ is the log scheme associated to $(S, L \to S)$. In other words, the composition of monoid maps $L \to R \to S$ gives rise to a chart for $\mathcal{M}^a|_{\operatorname{Spec} R}$.*

Proof. Let $U = \operatorname{Spec} R$ and $V = \operatorname{Spec} S$. By definition $\mathcal{M}^a$ is the sheafification of the presheaf pushout of $\alpha^{-1}(\mathcal{O}_U^\times) \to L$ and $\alpha^{-1}(\mathcal{O}_U^\times) \to \mathcal{O}_U^\times$, so its restriction $\mathcal{M}^a|_V$ can be computed by first restricting the presheaf pushout to V, and then taking sheafification as presheaves on $V_{\text{ét}}$. Unwinding definitions, the latter is the log structure associated to $L \to \mathcal{O}_V$ on $V_{\text{ét}}$. □

Definition A.3 *A morphism $i = (i, \psi) : \underline{X} \to \underline{Y}$ between fine log schemes $\underline{X} = (X, \alpha : M \to \mathcal{O}_X)$ and $\underline{Y} = (Y, \beta : N \to \mathcal{O}_Y)$ is a closed immersion if*

(i) $i : X \to Y$ is a closed immersion; and

(ii) $\psi^a : i^*N \to M$ is surjective (here ψ^a is induced from $\psi : i^{-1}(N) \to M$).

A closed morphism $i : \underline{X} \to \underline{Y}$ is exact if moreover in (2) above ψ^a is an isomorphism. More generally, a morphism $f : (X, M) \to (Y, N)$ is exact if the following diagram is Cartesian:

$$\begin{array}{ccc} f^{-1}(N) & \longrightarrow & M \\ \downarrow & & \downarrow \\ f^{-1}(N)^{\mathrm{gp}} & \longrightarrow & M^{gp} \end{array}$$

A log thickening (of order n) of a fine log scheme (T, L) is an exact closed morphism

$$\iota : (T, L) \to (T', L')$$

such that T is defined in T' by an ideal I where $I^{n+1} = 0$.

Example A.4 *The standard log point over a field k is* $\operatorname{Spec} k$ *with pre-log structure* $\mathbf{N} \to k$ *by sending* $0 \mapsto 1$ *and everything else to* 0. *Let* (X, M) *be a log scheme on which* p *is nilpotent. For each integer* $r \geq 1$, *then* r*-th Witt log scheme* $W_r(X, M)$ *consists of the underlying scheme* $W_r(X)$ *and the pre-log structure* $W_r(\alpha) : M \to W_r(\mathcal{O}_X)$ *given by* $m \mapsto [\alpha(m)]$, *here* $[\alpha(m)]$ *is the* r*-th Teichmuller lift of* $\alpha(m)$. *Another frequently encountered example is the divisorial log structure, defined by* $M_D(U) = \{f \in \mathcal{O}_X(U) : f|_{U \backslash D} \in \mathcal{O}_X^\times(U \backslash D)\}$ *where* $D \subset X$ *is a divisor.*

A.2 Log differentials

Let $f = (f, \psi) : \underline{X} \to \underline{Y}$ be a morphism between fine log schemes $\underline{X} = (X, \alpha : M \to \mathcal{O}_X)$ and $\underline{Y} = (Y, \beta : N \to \mathcal{O}_Y)$.

A.2.1 *Log derivations* A log derivation of $\underline{X} / \underline{Y}$ into a sheaf of modules D is a pair (d, δ) where $d : \mathcal{O}_X \to D$ is a derivation of X/Y into D, and $\delta : M \to D$ is a morphism of sheaf of monoids, such that $d(\alpha(m)) = \alpha(m)\delta(m)$ for any local section m of M and $\delta(\psi(n)) = 0$ for any local section n of N.

Remark A.5 *Let* $f : (X, \alpha : M \to \mathcal{O}_X) \to (Y, \beta : N \to \mathcal{O}_Y)$ *be a morphism of pre-log schemes. We define a pre-log derivation in the same way as above. Then a pre-log derivation* (d, δ) *of* $(X, M)/(Y, N)$ *in to* D *extends uniquely to a log derivation* $(d, \widetilde{\delta})$ *of* $(X, M^a)/(Y, N^a)$.

A.2.2 *Log differentials* The relative log differential $\omega^1_{\underline{X} / \underline{Y}}$ equipped with $(\mathrm{d}, \mathrm{dlog})$ is the universal log derivation of $\underline{X} / \underline{Y}$ representing the functor $D \mapsto \mathrm{Der}_D^{\log}$, where $\mathrm{Der}_D^{\log}$ is the set of log derivations of $\underline{X} / \underline{Y}$ into the sheaf of $\mathcal{O}_X$ module D. More concretely, $\omega^1_{\underline{X} / \underline{Y}}$ is the quotient of

$$\Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbf{Z}} M^{\mathrm{gp}})$$

by (the $\mathcal{O}_X$ module generated by) local sections $(d\alpha(m), 0) - (0, \alpha(m) \otimes m)$ and $(0, 1 \otimes \psi(n))$. In the construction $\mathrm{dlog}(m) := (0, 1 \otimes m)$.

Remark A.6 *If* $f : \underline{X} \to \underline{Y}$ *is a morphism of pre-log schemes, then we can make the same definitions. Moreover, we have*

$$\omega^1_{(X,M)/(Y,N)} \xrightarrow{\sim} \omega^1_{(X,M^a)/(Y,N)} \xrightarrow{\sim} \omega^1_{(X,M^a)/(Y,N^a)}.$$

A.2.3 *Log smoothness* A morphism $f = (f, \psi) : \underline{X} \to \underline{Y}$ of fine log schemes is log-smooth (resp. log étale) if $X \xrightarrow{f} Y$ is locally of finite presentation and for any commutative diagram

$$\begin{array}{ccc} (T, L) & \xrightarrow{s} & (X, M) \\ \downarrow{\scriptstyle \iota} & \overset{g}{\nearrow} & \downarrow{\scriptstyle f} \\ (T', L') & \xrightarrow{t} & (Y, N) \end{array}$$

where ι is a first order log thickening, étale locally there exists (resp. there exists a unique) $g : (T', L') \to (X, M)$ making all diagrams commute.

As in the classical case, if $f : \underline{X} \to \underline{Y}$ is log-smooth, then the log differential $\omega^1_{\underline{X} / \underline{Y}}$ is locally free of finite type.

Proposition A.7 ([Kat89] 3.5) *Let* $f : \underline{X} \to \underline{Y}$ *be a morphism between fine log schemes as above, and* $Q \to N$ *a chart for* $\underline{Y}$. *Then* f *is log-smooth if and only if* étale *locally on* X *there exists a chart* $(P_X \to M, Q_Y \to N, \beta : Q \to P)$ *of* f *extending the chart for* Y *such that* *(i)* $\ker(\beta^{gp})$ *and* $\operatorname{coker}(\beta^{gp})_{tor}$ *are finite groups of order invertible on* X*;*

(ii) *the induced morphism* $X \to Y \times_{\operatorname{Spec}(\mathbf{Z}[Q])} \operatorname{Spec}\mathbf{Z}[P]$ *is* étale.

Similarly, f *is log* étale *if and only if a similar condition holds with the torsion part of the kernel* $\operatorname{coker}(\beta)_{tor}$ *replaced by* $\operatorname{coker}(\beta)$ *in* (1) *above.*

A.3 Log-Cartier type

A.3.1 *Integral morphisms* A log-smooth morphism might fail to be flat. For example, consider $\underline{X} = (\operatorname{Spec}\mathbf{Z}[x,y], \mathbf{N}^2)$ with log structure $(1,0) \mapsto x; (0,1) \mapsto y$. The morphism $\underline{X} \to \underline{X}$ given by $x \mapsto x, y \mapsto xy$ is log-smooth (even log-étale) but not flat. This leads to the following definition:

Definition A.8 *A morphism* $f : \underline{X} \to \underline{Y}$ *is integral if for any* $\underline{Y'} \to \underline{Y}$ *where* $\underline{Y'}$ *is a fine log scheme, the base change* $\underline{X} \times_{\underline{Y}} \underline{Y'}$ *is a fine log scheme.*

This is equivalent to requiring that étale locally on X and Y, f has charts given by $\beta : Q \to P$ such that the induced morphism $\mathbf{Z}[Q] \to \mathbf{Z}[P]$ is flat. If $f : \underline{X} \to \underline{Y}$ is log-smooth and integral, then the underlying morphism $f : X \to Y$ is flat.

A.3.2 *log-Cartier type* Let $\underline{X} = (X, M)$ be a log scheme in characteristic p, the absolute (p-power) Frobenius $F_{\underline{X}}$ is given by the usual absolute Frobenius on X and $M \xrightarrow{\times p} M$. Note that we have implicitly identified $F_{\underline{X}}^{-1}(M) \cong M$ on $X_{\text{ét}}$.

Definition A.9 *A morphism* $f : \underline{X} \to \underline{Y}$ *over* $\mathbf{F}_p$ *is of log-Cartier type if* f *is integral and the relative Frobenius* $F_{\underline{X}/\underline{Y}}$ *in the diagram below is exact.*

$$\begin{array}{ccccc} & & F_{\underline{X}} & & \\ \underline{X} & \overset{F_{\underline{X}/\underline{Y}}}{\dashrightarrow} & \underline{X}^{(p)} & \xrightarrow{h} & \underline{X} \\ & & \downarrow & \square & \downarrow f \\ & & \underline{Y} & \xrightarrow{F_{\underline{Y}}} & \underline{Y} \end{array}$$

The most important feature for a log-smooth morphism of log-Cartier type is that the Cartier isomorphism holds. This will be a key step to relate our log de Rham–Witt complex with the de Rham complex of a log Frobenius lift.

Proposition A.10 ([HK94] 2.12) *Let* f *be a log-smooth morphism of log-Cartier type, then there exists a (Cartier) isomorphism*

$$F = C^{-1} : \omega^k_{\underline{X}^{(p)}/\underline{Y}} \cong \mathcal{H}^k(\omega^*_{\underline{X}/\underline{Y}}),$$

which, on local sections $a \in \mathcal{O}_X, m_1, ..., m_k \in M$*, is given by*

$$x \operatorname{dlog}(h^* m_1) \wedge \cdots \wedge \operatorname{dlog}(h^* m_k) \longmapsto F_{\underline{X}/\underline{Y}}(x) \operatorname{dlog}(m_1) \wedge \cdots \wedge \operatorname{dlog}(m_k).$$

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