

# The crystalline comparison of $A_{\text{inf}}$ -cohomology: the case of good reduction

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ABSTRACT

We provide a simple approach for the crystalline comparison of  $A_{\text{inf}}$ -cohomology, and reprove the comparison between crystalline and  $p$ -adic étale cohomology for formal schemes in the case of good reduction.

## 1. Introduction

The recently developed theory of  $A_{\text{inf}}$ -cohomology  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$  in [BMS18], for smooth formal schemes  $\mathfrak{X}$  over rings like  $\mathcal{O}_{C,p}$ , plays a central role in integral  $p$ -adic Hodge theory — the study of  $\mathbb{Z}_p$ -lattices in  $p$ -adic Galois representations. This paper studies the relation between the aforementioned theory and crystalline cohomology. More precisely, we provide a crystalline interpretation of  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ , when valued on certain semiperfectoid covers of  $\mathfrak{X}$ . This allows us to obtain a simplified proof of the crystalline comparison of  $A_{\text{inf}}$ -cohomology. In particular, we avoid some rather involved analysis in relative de Rham–Witt theory and the methods of “all possible coordinates” in [BMS18]. Moreover, we expect our approach to generalize relatively easily to the case of semistable reduction.

### 1.1 Background

To explain further let us fix notations. Let  $C$  be a complete algebraically closed nonarchimedean extension of  $\mathbb{Q}_p$ , with ring of integers  $\mathcal{O}_C$  and residue field  $k$ . Let  $\mathfrak{X}$  be a smooth formal scheme over  $\text{Spf } \mathcal{O}_C$ , with adic generic fiber  $X = \mathfrak{X}_C^{\text{ad}}$  over  $\text{Spa}(C, \mathcal{O}_C)$  and special fiber  $\mathfrak{X}_k$  over  $\text{Spec } k$ . We fix a compatible system of primitive  $p$ -power roots of unity in  $\mathcal{O}_C$ , which defines an element  $\epsilon := (1, \zeta_p, \zeta_{p^2}, \dots)$  in its tilt  $\mathcal{O}_C^{\flat}$ . As usual let  $\mu := [\epsilon] - 1 \in A_{\text{inf}} = W(\mathcal{O}_C^{\flat})$ , and  $\xi := \mu/\varphi^{-1}(\mu)$ . We denote by  $\theta : A_{\text{inf}} \rightarrow \mathcal{O}_C$  the natural projection whose kernel is generated by  $\xi$ , and by  $\vartheta : A_{\text{inf}} \rightarrow W(k)$  the map induced from  $\mathcal{O}_C^{\flat} \rightarrow k$  by the Witt vector functoriality. Finally, recall that  $A_{\text{cris}}$  is the  $p$ -completed PD-envelop of  $A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$ .

There are at least three constructions of the  $A_{\text{inf}}$ -cohomology  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$  of  $\mathfrak{X}$  by now: via perfectoid spaces ([BMS18]), via topological Hochschild homology ([BMS19]), and via the prismatic site ([BS19]). We are concerned with the first construction, which is defined as

$$R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) := R\Gamma(\mathfrak{X}_{\text{ét}}, L\eta_{\mu} R\nu_* \mathbb{A}_{\text{inf}, X}) \in D(A_{\text{inf}}).$$

Here  $\mathbb{A}_{\text{inf}, X}$  is (the derived  $p$ -completed) Fontaine’s period sheaf  $W(\widehat{\mathcal{O}}_{X^{\flat}}^+)$  on the pro-étale site  $X_{\text{proét}}$ , the map  $\nu : X_{\text{proét}} \rightarrow \mathfrak{X}_{\text{ét}}$  is the natural map of sites, and  $L\eta_{\mu}$  is the décalage operator (see Section 5 and 6 of [BMS18]). The perfectoid nature of this definition lies within  $X_{\text{proét}}$ , which has a basis given by affinoid perfectoid objects  $(S, S^+)$ , on which  $\mathbb{A}_{\text{inf}, X}$  takes the value

$A_{\text{inf}}(S^+) = W(S^{+,b})$ .<sup>1</sup> Note that  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$  is equipped with a Frobenius operator  $\varphi$  induced from the Frobenius on  $A_{\text{inf},X}$ , which essentially comes from the tilting equivalence. From this definition, it is not difficult to deduce the following comparison theorems of  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$ .

- étale comparison:  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}}} A_{\text{inf}}[\frac{1}{\mu}] \cong R\Gamma(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\text{inf}}[\frac{1}{\mu}]$ .<sup>2</sup>
- de Rham comparison:  $\gamma_{\text{dR}} : R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{\theta}^{\mathbb{L}} \mathcal{O}_C \xrightarrow{\sim} R\Gamma_{\text{dR}}(\mathfrak{X})$ .

Note that, since  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$  is derived  $\xi$ -complete (see the proof of Lemma 4.5), the de Rham comparison in particular ensures that  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$  takes values in perfect  $A_{\text{inf}}$ -complexes. We will take the de Rham comparison as an input in our paper, which is the easier part of [BMS18]. It can be proven either from Theorem 8.3 of *loc.cit.* or, alternatively, by the methods of [Bha17].

## 1.2 Main results

Let  $R\Gamma_{\text{crys}}((\mathfrak{X}_{\mathcal{O}_C/p})/A_{\text{cris}})$  be the crystalline cohomology of  $\mathfrak{X}_{\mathcal{O}_C/p}$  over the base  $A_{\text{cris}}$ . Let us now state our first main result.

**Theorem 1** *There exists a functorial  $\varphi$ -equivariant map*

$$h_{\text{crys}} : R\Gamma_{\text{crys}}((\mathfrak{X}_{\mathcal{O}_C/p})/A_{\text{cris}}) \longrightarrow R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}_{A_{\text{inf}}}^{\mathbb{L}} A_{\text{cris}}$$

*which is compatible with the de Rham comparison  $\gamma_{\text{dR}}^{-1}$  after base change.*

**Remark 2** *The map we construct here goes in the opposite direction of a map constructed in [BMS18] (by taking a certain limit over all choices of coordinates), which they show is an isomorphism. In fact, as we explain next, the existence of such a functorial map (in the direction that we construct) is sufficient to deduce the comparison theorems of interest.*

**Remark 3** *In [Fal02], Faltings constructs a map  $\chi : R\Gamma_{\text{crys}}((\mathfrak{X}_{\mathcal{O}_C/p})/A_{\text{cris}}) \rightarrow R\Gamma(X_{\text{proét}}, \mathbb{A}_{\text{cris}})$  (in modern language) to relate crystalline and étale cohomology (he even shows that the latter is almost isomorphic to  $R\Gamma(X_{\text{ét}}, \mathbb{Z}_p) \otimes A_{\text{cris}}$ ). Faltings's map  $\chi$  is essentially the composition of  $h_{\text{crys}}$  with the canonical map  $\delta : R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}_{A_{\text{inf}}}^{\mathbb{L}} A_{\text{cris}} \rightarrow R\Gamma(X_{\text{proét}}, \mathbb{A}_{\text{cris}})$ . It does not seem possible to reconstruct  $h_{\text{crys}}$  from  $\chi$ , and our approach for  $B_{\text{cris}}$  comparison is different from [Fal02].*

As a consequence we are able to reprove the following results of [BMS18].

**Theorem 4** *Let  $\mathfrak{X}$  be a smooth proper formal scheme over  $\mathcal{O}_C$ .*

- (1) *The cohomology  $H_{A_{\text{inf}}}^i(\mathfrak{X})$  of the perfect complex  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \in D(A_{\text{inf}})$  takes values in Breuil–Kisin–Fargues modules (see Definition 4.1).*
- (2)  *$R\Gamma_{A_{\text{inf}}}(\mathfrak{X})$  enjoys the following  $\varphi$ -equivariant comparison with the crystalline cohomology of the special fiber*

$$\gamma_{\text{crys}} : R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{\theta}^{\mathbb{L}} W(k) \cong R\Gamma_{\text{crys}}(\mathfrak{X}_k/W(k)).$$

- (3) *Now suppose that  $\mathfrak{X}_{\mathcal{O}_K}$  is a smooth proper formal scheme over  $\mathcal{O}_K$ , where  $K$  is a complete discretely valued nonarchimedean extension of  $\mathbb{Q}_p$  with residue field  $k_0$ . Let  $C = C_K$  be a  $p$ -completed algebraic closure of  $K$ . Then there exists a  $\text{Gal}_K$ ,  $\varphi$ -equivariant isomorphism*

$$H_{\text{crys}}^i(\mathfrak{X}_{k_0}/W(k_0)) \otimes_{W(k_0)} B_{\text{cris}} \xrightarrow{\sim} H_{\text{ét}}^i(\mathfrak{X}_C^{\text{ad}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}.$$

<sup>1</sup>up to  $W(\mathfrak{m}^b)$ -torsion, where  $\mathfrak{m} \subset \mathcal{O}_C$  is the maximal ideal. See Lemma 5.6 of *loc.cit.*

<sup>2</sup>For example it follows from the primitive comparison theorem of Scholze (Theorem 5.1 and the proof of Theorem 8.4 in [Sch13]). In fact, Bhatt gives another proof of the étale comparison without using the primitive comparison theorem, for which one needs the de Rham comparison (at least in the almost sense). See Remark 8.4 in [Bha17].

In particular,  $H_{\text{ét}}^i(\mathfrak{X}_C^{\text{ad}}, \mathbb{Q}_p)$  is a crystalline  $\text{Gal}_K$ -representation.

Several remarks are in order.

**Remark 5** *The étale and crystalline specialization, together with the fact that  $H_{A_{\text{inf}}}^i(\mathfrak{X})[\frac{1}{p}]$  is a Breuil–Kisin–Fargues module, implies the main result of [BMS18] on torsion relations. Namely, for each  $n \geq 1$ , we have*

$$\text{length}_{W(k)}(H_{\text{crys}}^i(\mathfrak{X}_k/W(k))_{\text{tor}/p^n}) \geq \text{length}_{\mathbb{Z}_p}(H_{\text{ét}}^i(X, \mathbb{Z}_p)_{\text{tor}/p^n}).$$

Let us briefly comment on the proof of Theorem 4 from Theorem 1.

**Remark 6** – *First we observe that part (2) (the crystalline comparison of  $A_{\text{inf}}$  cohomology) is an immediate corollary by the derived Nakayama’s lemma. This in particular recovers Theorem 1.8(i) of [BMS18].*

- *The map  $h_{\text{crys}}$  has the desired direction for applications to  $p$ -adic Hodge theory – in particular in view of the  $B_{\text{cris}}$  comparison – as it implies that “there are enough Galois invariants” after base change to  $W(k)$ .*
- *Finally let us turn to part (1), which seems to be the most subtle part, since we do not claim that  $h_{\text{crys}}$  is an isomorphism (at least we want to avoid using it). For this we perform a careful induction on the degree of cohomology following [Mor16], which reduces to show that the  $\mathbb{Q}_p$  dimension of  $H_{\text{ét}}^i(X, \mathbb{Q}_p)$  agrees with the  $W(k)$ -rank of  $H_{\text{crys}}^i(\mathfrak{X}_k/W(k))$ , which we again deduce from the existence of  $h_{\text{crys}}$ .*

**Remark 7** *One defect of the  $B_{\text{cris}}$  comparison stated as above is that we have ignored the information on filtration (after base change to  $B_{\text{dR}}$ ). To take care of the filtration compatibility of the  $B_{\text{cris}}$  comparison, we consider a certain infinitesimal cohomology of the generic fiber  $X$  over the pro-thickening  $B_{\text{dR}}^+ \rightarrow C$ , which we discuss in the end of this introduction.*

### 1.3 The construction of $h_{\text{crys}}$

Now we illustrate the strategy to prove Theorem 1. By functoriality it suffices to construct the map on affine opens  $\text{Spf } R$  in the étale site  $\mathfrak{X}_{\text{ét}}$ . The basic idea is to use quasisyntomic descent to reduce to the case of quasiregular semiperfectoid covers of  $R$ , which are certain well-behaved quotients of perfectoid rings by “quasiregular ideals”. The precise definition is recalled in Section 2, for now it suffices to think of them as certain perfectoid variants of local complete intersections. Typical examples include

$$k[[x^{1/p^\infty}, y^{1/p^\infty}]]/(x - y), \quad \mathcal{O}_C/p = \mathcal{O}_{C^b}/p^b, \quad \text{and } \mathcal{O}_C\langle X^{1/p^\infty} \rangle/(X).$$

More precisely, let  $A\Omega_S$  be the derived  $A_{\text{inf}}$ -cohomology of  $S$ , obtained as the derived  $(p, \xi)$ -completion of the left Kan extension of the functor

$$R \longmapsto A\Omega_R := R\Gamma_{A_{\text{inf}}}(\text{Spf } R, L\eta_\mu R\nu_* \mathbb{A}_{\text{inf}, X}),$$

from the category of  $p$ -completions of smooth  $\mathcal{O}_C$ -algebras to  $D(A_{\text{inf}})$ . By quasisyntomic descent it suffices to construct a functorial map from the crystalline cohomology  $R\Gamma_{\text{crys}}((R/p)/A_{\text{cris}})$  to  $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ . This in turn follows from the following lemma.

**Lemma 8** *For a quasiregular semiperfectoid  $\mathcal{O}_C$ -algebra  $S$ , its derived  $A_{\text{inf}}$ -cohomology  $A\Omega_S$  is a discrete, topologically free  $A_{\text{inf}}$ -algebra concentrated in degree 0. Moreover, there is a natural map of rings  $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \rightarrow S/p$ , which is a PD-thickening over  $A_{\text{cris}} \rightarrow \mathcal{O}_C/p$ . In particular,  $A\Omega_S \rightarrow S/p$  is an object in the crystalline site  $\text{CRIS}(R/p)_{A_{\text{cris}}}$ .*

### 1.4 The $B_{\mathrm{dR}}^+$ -cohomology of the generic fiber

As alluded to previously, to analyze the individual cohomology groups  $H_{\mathrm{Ainf}}^*(\mathfrak{X})$  further we need a variant of  $h_{\mathrm{crys}}$  over  $B_{\mathrm{dR}}^+$ . This is provided by reformulating the  $B_{\mathrm{dR}}^+$ -cohomology of [BMS18] via a certain infinitesimal site.

**Theorem 9** *Let  $X$  be a smooth proper rigid analytic variety over  $C$ . There is a cohomology  $R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+)$  with  $B_{\mathrm{dR}}^+$  coefficients, satisfying*

$$R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \otimes^{\mathbb{L}} C \cong R\Gamma_{\mathrm{dR}}(X/C).$$

Moreover, there exists a functorial map

$$h_{\mathrm{dR}} : R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \longrightarrow R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{B}_{\mathrm{dR}}^+),$$

where  $\mathbb{B}_{\mathrm{dR}}^+$  is the period sheaf on  $X_{\mathrm{pro\acute{e}t}}$  defined in [Sch13]. Further suppose  $X = \mathfrak{X}_C^{\mathrm{ad}}$  is the generic fiber of a smooth proper formal scheme  $\mathfrak{X}$  as above, then there is a canonical isomorphism

$$R\Gamma_{\mathrm{Ainf}}(\mathfrak{X}) \otimes^{\mathbb{L}} B_{\mathrm{dR}}^+ \xrightarrow{\sim} R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+),$$

which identifies  $H_{\mathrm{inf}}^i(X/B_{\mathrm{dR}}^+)$  with the  $B_{\mathrm{dR}}^+$ -lattice  $H_{\mathrm{Ainf}}^i(\mathfrak{X}) \otimes B_{\mathrm{dR}}^+$  in  $H_{\acute{e}t}^i(X, \mathbb{Z}_p) \otimes B_{\mathrm{dR}}$ . Finally,  $h_{\mathrm{dR}}$  is compatible with  $h_{\mathrm{crys}}$  under the étale comparison in a suitable sense.

Finally, from this we prove the rest of the main theorem of [BMS18].

**Corollary 10** *Let  $\mathfrak{X}_{\mathcal{O}_K}$  be a smooth proper formal scheme over  $\mathcal{O}_K$  as in Theorem 4 (3).*

- (i) *If  $H_{\mathrm{crys}}^*(\mathfrak{X}_k/W(k))$  is torsion free for both  $* = i$  and  $i + 1$ , then the integral crystalline cohomology  $H_{\mathrm{crys}}^i(\mathfrak{X}_k)$  with its Frobenius  $\phi$  can be recovered from the  $\mathrm{Gal}_K$ -module  $H_{\acute{e}t}^i(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p)$ .*
- (ii) *The  $B_{\mathrm{cris}}$  comparison in Theorem 4 is compatible with filtrations after base change to  $B_{\mathrm{dR}}$ . In other words,  $D_{\mathrm{cris}}(H_{\acute{e}t}^i(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Q}_p))$  is given by  $(H_{\mathrm{crys}}^i(\mathfrak{X}_k/W(k))_{\mathbb{Q}}, \phi, \mathrm{Fil})$  as a filtered  $\varphi$ -module, where  $\mathrm{Fil}$  is the Hodge filtration on  $H_{\mathrm{dR}}^i(X_K/K)$ .*

### Conventions

We use the language of  $\infty$ -categories. In particular, the derived categories considered here are the natural  $\infty$ -categorical enhancements and algebra objects in derived categories are  $E_{\infty}$ -algebras. Moreover we do not distinguish derived categories of sheaves and sheaves valued in derived categories. For a category  $\mathcal{C}$ , we denote by  $\mathcal{C}^{\circ}$  the indiscrete site on  $\mathcal{C}$ , in other words, the only non-empty cover allowed in the topology is identity. In this article, a Tate algebra over a  $p$ -adic field  $K$  refers to classical Tate algebras, which are quotients of  $K\langle t_1, \dots, t_d \rangle$ . A Huber ring with a topological nilpotent unit is called a Tate Huber ring. Finally, for an  $\mathcal{O}_C$ -algebra  $S$ , we denote by  $\mathbb{L}\Omega_{S/\mathcal{O}_C}$  the  $p$ -completed derived de Rham cohomology of  $S/\mathcal{O}_C$ . We write  $\Omega_{(S/p)}$  (resp.  $\mathbb{L}\Omega_{(S/p)}$ ) to denote  $\Omega_{(S/p)/(\mathcal{O}_C/p)}$  (resp.  $\mathbb{L}\Omega_{(S/p)/(\mathcal{O}_C/p)}$ ) to ease notations.

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## 2. $A\Omega_S$ for quasisyntomic rings

### 2.1 The quasisyntomic site

Recall from [BMS19] the following definitions.

**Definition 2.1** *A ring  $A$  is quasisyntomic if it is  $p$ -complete with bounded  $p^\infty$ -torsion and the base change of the cotangent complex  $\mathbb{L}_{A/\mathbb{Z}_p} \otimes_A^{\mathbb{L}} A/p \in D(A/p)$  has Tor amplitude in  $[-1, 0]$ .*

Examples of quasisyntomic rings include  $p$ -completions of smooth algebras over perfectoid rings, and  $p$ -complete noetherian local complete intersections. In fact, by a result of Avramov, the latter essentially gives all noetherian quasisyntomic rings (see Theorem 4.13 in *loc.cit.*).

**Definition 2.2** *Let  $A, B$  be  $p$ -complete rings with bounded  $p^\infty$ -torsion. A morphism  $f : A \rightarrow B$  is quasisyntomic if*

- $B \otimes_A^{\mathbb{L}} A/p \in D(A/p)$  lives in degree 0 and is flat over  $A/p$ ;
- $\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} B/p \in D(B/p)$  has Tor amplitude  $[-1, 0]$ .

*The morphism  $f$  is proj(= projective)-quasisyntomic if*

- both  $A$  and  $B$  are  $p$ -torsion free;
- $B/p$  is projective over  $A/p$ ;
- $\mathbb{L}_{(B/p)/(A/p)} \in D(B/p)$  has projective amplitude in  $[-1, 0]$ .

We denote by  $\text{qSyn}_{\mathcal{O}_C}^{\text{op}}$  the site whose underlying category is the opposite category of quasisyntomic  $\mathcal{O}_C$ -algebras, with topology generated by quasisyntomic covers. Let  $\text{qSyn}_{\mathcal{O}_C}^{\text{proj}} \subset \text{qSyn}_{\mathcal{O}_C}$  be the full subcategory spanned by proj-quasisyntomic  $\mathcal{O}_C$ -algebras, and  $\text{qSyn}_{\mathcal{O}_C}^{\text{proj,op}}$  be the site whose topology is generated by proj-quasisyntomic covers. By Lemma 4.16 (and Variant 4.35) in *loc.cit.*,  $\text{qSyn}_{\mathcal{O}_C}^{\text{op}}$  and  $\text{qSyn}_{\mathcal{O}_C}^{\text{proj,op}}$  indeed form sites, in particular, the property of being quasisyntomic is preserved under quasisyntomic maps.

### 2.2 Quasiregular semiperfectoid rings

Now we look at a particularly useful basis for the topology of  $\text{qSyn}_{\mathcal{O}_C}^{\text{op}}$  (resp.  $\text{qSyn}_{\mathcal{O}_C}^{\text{proj,op}}$ ).

**Definition 2.3** *A ring  $S$  is quasiregular semiperfectoid if it is quasisyntomic and there exists a surjective map  $R_{\text{perfd}} \rightarrow S$  from a perfectoid ring  $R_{\text{perfd}}$ .*

The latter condition implies that  $S$  is semiperfect (i.e. the Frobenius on  $S/pS$  is surjective). Denote by  $\text{qrsPerfd}_{\mathcal{O}_C} \subset \text{qSyn}_{\mathcal{O}_C}$  the full subcategory of quasiregular semiperfectoid  $\mathcal{O}_C$ -algebras, and write  $\text{qrsPerfd}_{\mathcal{O}_C}^{\text{proj}} := \text{qSyn}_{\mathcal{O}_C}^{\text{proj}} \cap \text{qrsPerfd}_{\mathcal{O}_C}$ . The category  $\text{qrsPerfd}_{\mathcal{O}_C}^{\text{proj,op}}$  again forms a site under the topology given by proj-quasisyntomic covers. We will frequently use the fact that quasiregular semiperfectoid rings form a basis for the quasisyntomic site. More precisely,

**Lemma 2.4 (Proposition 4.30, Variant 4.35 in [BMS19])** *The natural restriction of sites of topoi  $\text{qrsPerfd}_{\mathcal{O}_C}^{\text{proj,op}} \rightarrow \text{qSyn}_{\mathcal{O}_C}^{\text{proj,op}}$  induces an equivalence*

$$\text{Shv}(\text{qSyn}_{\mathcal{O}_C}^{\text{proj,op}}) \xrightarrow{\simeq} \text{Shv}(\text{qrsPerfd}_{\mathcal{O}_C}^{\text{proj,op}}).$$

### 2.3 Flat/quasisyntomic descent

Recall from the introduction the functor from the category of  $p$ -adic completions of smooth  $\mathcal{O}_C$ -algebras to derived  $(p, \xi)$ -complete  $A_{\text{inf}}$ -algebras, sending

$$R \longmapsto A\Omega_R := R\Gamma(\text{Spf } R, L\eta_\mu R\nu_* \mathbb{A}_{\text{inf}}).$$

By left Kan extension in  $(p, \xi)$ -complete  $A_{\text{inf}}$ -complexes, we obtain a functor  $A\Omega : R \mapsto A\Omega_R$  from all  $p$ -complete (simplicial) algebras to derived  $(p, \xi)$ -complete  $A_{\text{inf}}$ -algebras, which satisfies the following de Rham comparison

$$A\Omega_R \otimes^{\mathbb{L}} A_{\text{inf}}/\xi \cong \mathbb{L}\Omega_{R/\mathcal{O}_C}.$$

To relate to the crystalline cohomology, we base change to  $A_{\text{cris}}$  to obtain a functor

$$A\Omega \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} : R \longmapsto A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}.$$

Here the completion is  $(p, \mu)$ -adic.<sup>3</sup> In order to compute  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ , we need the following lemma on flat (more precisely, quasisyntomic) descent.

**Lemma 2.5** *The functor  $R \mapsto A\Omega_R \widehat{\otimes}^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{cris}}$  forms a sheaf on  $\text{qSyn}_{\mathcal{O}_C}^{\text{proj}, \text{op}}$ .*

*Proof.* It suffices to show that  $R \mapsto A\Omega_R \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{cris}}/p$  is a sheaf. Now observe that  $A_{\text{cris}}$  is isomorphic to the  $p$ -adic completion of  $A_{\text{inf}}[\frac{\xi^{p^k}}{p^{1+p+\dots+p^{k-1}}}]_{k \geq 1}$ , therefore,

$$A_{\text{cris}}/p \cong A_{\text{inf}}[X_1, X_2, \dots]/(p, \xi^p, X_1^p, X_2^p, \dots)$$

is a countable direct sum of copies of  $A_{\text{inf}}/(p, \xi^p)$  as an  $A_{\text{inf}}$ -module. Thus it suffices to show that  $R \mapsto A\Omega_R \otimes^{\mathbb{L}} A_{\text{inf}}/(p, \xi^p)$  is a coconnective sheaf. This follows from Construction 9.5 of [BMS19], which shows that  $R \mapsto A\Omega_R$  is a coconnective sheaf on  $\text{qSyn}_{\mathcal{O}_C}^{\text{proj}, \text{op}}$  (the key input is that flat descent holds for cotangent complexes, see Theorem 3.1 of *loc.cit.*)  $\square$

### 2.4 $A\Omega_S$ for quasiregular semiperfectoid rings

Locally on quasiregular semiperfectoid rings  $S$ ,  $A\Omega_S$  is a relatively simple object.

**Lemma 2.6** ([BMS19]) *For  $S \in \text{qrsPerfd}_{\mathcal{O}_C}^{\text{proj}}$ ,  $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$  is a topologically free  $A_{\text{cris}}$ -module concentrated in degree 0. In particular it is discrete and  $p$ -torsion free.*

*Proof.* First note that  $\mathbb{L}\Omega_{(S/p)}$  is a free  $\mathcal{O}_C/p$ -module concentrated in degree 0 (by considering the graded pieces  $\wedge^i \mathbb{L}_{(S/p)}[-i]$  for the conjugate filtration on  $\mathbb{L}\Omega_{(S/p)}$ ). The  $D(A_{\text{inf}})$ -valued sheaf  $R \mapsto A\Omega_R$  on  $\text{qSyn}_{\mathcal{O}_C}^{\text{proj}, \text{op}}$  takes discrete values on quasi-regular semiperfectoid objects, hence the lifting of a basis from  $A\Omega_S \otimes^{\mathbb{L}} A_{\text{inf}}/(p, \xi) \cong \mathbb{L}\Omega_{(S/p)}$  to  $A\Omega_S$  becomes a topological basis, so  $A\Omega_S$  is topologically free. The claim thus follows.  $\square$

**Lemma 2.7** *Retain notations from above. The natural projection map  $\beta$*

$$\begin{array}{ccc} A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} & \longrightarrow & \mathbb{L}\Omega_{S/\mathcal{O}_C} \\ & \searrow \beta & \downarrow \\ & & S/p \end{array}$$

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<sup>3</sup>Note that the  $p$ -adic topology on  $A_{\text{cris}}$  agrees with the  $(p, \mu)$ -adic topology since  $\mu^{p-1}$  is divisible by  $p$ .

is a PD-thickening. Here the horizontal arrow is induced from the de Rham comparison  $A\Omega_S \otimes^{\mathbb{L}} \mathcal{O}_C \cong \mathbb{L}\Omega_{S/\mathcal{O}_C}$ , while the vertical map is the composition of  $\mathbb{L}\Omega_{S/\mathcal{O}_C} \rightarrow \mathbb{L}\Omega_{(S/p)}$  with the augmentation map  $\mathbb{L}\Omega_{(S/p)} \rightarrow \Omega_{(S/p)}^{\bullet} = S/p$ .

*Proof.* We first observe that the statement is equivalent to that  $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \rightarrow S$  is a PD-thickening. Moreover, since  $S$  is  $p$ -torsion free, it suffices to show that, for each  $x \in I := \ker(\theta : A\Omega_S \rightarrow S)$ ,  $x^p$  is divisible by  $p$  in  $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ .<sup>4</sup>

Pick a surjection  $\tilde{S} \rightarrow S$  from a perfectoid ring  $\tilde{S}$  with kernel  $J$ . Let  $\tilde{S}' = \tilde{S}\langle X_j^{1/p^\infty} \rangle_{j \in J}$  be the perfectoid ring obtained by adjoining all  $p$ -power roots of  $X_j$  for all  $j \in J$  (in the  $p$ -adic complete sense). Let  $S' := \tilde{S}\langle X_j^{1/p^\infty} \rangle_{j \in J}/(X_j)$ , which is a quasiregular semiperfectoid  $\mathcal{O}_C$ -algebra.

Choose a homomorphism  $\tilde{S}' \rightarrow \tilde{S}$  with  $X_j \mapsto j$  for all  $j \in J$ . This induces a surjective map  $A\Omega_{S'} \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \rightarrow A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ . In fact this map is already surjective without applying  $-\widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ . One can check the surjectivity on derived de Rham complexes  $\mathbb{L}\Omega_{(S'/p)} \rightarrow \mathbb{L}\Omega_{(S/p)}$  after modulo  $\xi$ , and observe that  $\mathbb{L}_{(S'/p)}[-1] \rightarrow \mathbb{L}_{(S/p)}[-1]$  is surjective by the construction of  $S'$ . Moreover, we claim that the map

$$\ker(A\Omega_{S'} \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \rightarrow S'/p) \longrightarrow \ker(A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \rightarrow S/p)$$

is surjective. For this we first observe that the diagram

$$\begin{array}{ccc} A\Omega_{S'} \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} & \longrightarrow & A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \\ \downarrow & & \downarrow \\ \mathbb{L}\Omega_{(S'/p)} & \longrightarrow & \mathbb{L}\Omega_{(S/p)} \end{array}$$

is a push-out diagram of commutative rings. The claim then follows from the surjectivity of

$$\ker(\mathbb{L}\Omega_{(S'/p)} \rightarrow \mathbb{L}\Omega_{(S/p)}) \longrightarrow \ker(S'/p \rightarrow S/p),$$

again by considering the filtered graded pieces. Therefore, it suffices to prove the lemma replacing  $S$  with  $S' = \tilde{S}\langle X_j^{1/p^\infty} \rangle_{j \in J}/(X_j)$ .

To this end, we consider the quasiregular semiperfectoid ring  $S_0 = \mathcal{O}_C\langle X_j^{1/p^\infty} \rangle_{j \in J}/(X_j)$ . Observe that the natural map  $A\Omega_{S_0} \widehat{\otimes}_{A_{\text{inf}}}^{\mathbb{L}} A_{\text{inf}}(\tilde{S}) \rightarrow A\Omega_{S'}$  is an isomorphism. Hence we are further reduced to proving that the kernel of  $A\Omega_{S_0} \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \rightarrow S_0$  carries a PD-structure, where  $S_0 = \mathcal{O}_C\langle X_j^{1/p^\infty} \rangle_{j \in J}/(X_j)$ , which follows from the next lemma.  $\square$

**Lemma 2.8** *Let  $S = \mathcal{O}_C\langle X^{1/p^\infty} \rangle/(X)$ , then  $\beta : A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \rightarrow S/p$  is a PD thickening.*

*Proof.* First we observe that from the Hodge–Tate comparison (Theorem 8.3 in [BMS18]) we have a functorial map of  $E_\infty$ -algebras

$$\alpha : R \rightarrow H^0(A\Omega_R/\varphi(\xi)) \rightarrow A\Omega_R/\varphi(\xi).$$

The composition  $R \rightarrow \Omega_{(R/p)}^{\bullet}$  of  $\alpha$  with the quotient map (and the de Rham comparison)

$$A\Omega_R/\varphi(\xi) \longrightarrow A\Omega_R/(\varphi(\xi), p) = A\Omega_R/(\xi^p, p) \longrightarrow A\Omega_R/(\xi, p) \cong \Omega_{(R/p)}^{\bullet}$$

<sup>4</sup>Note that the  $p$ -torsion-free condition on the target  $S$  is necessary. For example, for a discrete valuation ring  $R$  with uniformizer  $\pi$  such that  $\pi^p = p$ ,  $R \rightarrow R/\pi$  is not a PD-thickening (the requirement for the map  $R \rightarrow R/\pi$  to be a PD-thickening is precisely that the ramification index satisfies  $e \leq p - 1$ ).

can be described by  $x \mapsto \bar{x}^p$ , which lies in  $H^0(\Omega_{(R/p)}^\bullet)$ . After applying left Kan extensions from ( $p$ -completions of) smooth  $\mathcal{O}_C$ -algebras to all  $p$ -complete  $\mathcal{O}_C$ -algebras, we get a map

$$\alpha : S' \longrightarrow A\Omega_{S'}/\varphi(\xi)$$

which is a homomorphism of discrete algebras for any quasiregular semiperfectoid  $\mathcal{O}_C$ -algebra  $S'$ . Moreover, when  $S'$  is perfectoid, the map  $\bar{\alpha} : S' \longrightarrow A\Omega_{S'}/(\varphi(\xi), p) \xrightarrow{\text{can}} \mathbb{L}\Omega_{(S'/p)} \cong S'/p$  is given by  $x \mapsto \bar{x}^p$ .

Now let  $S = \mathcal{O}_C\langle X^{1/p^\infty} \rangle / (X)$  as in the lemma, and let  $\tilde{S} = \mathcal{O}_C\langle X^{1/p^\infty} \rangle$ . We have the following commutative diagram of (discrete) algebras

$$\begin{array}{ccccc} & & A\Omega_{\tilde{S}} & & \\ & & \downarrow & \dashrightarrow & \\ \tilde{S} & \xrightarrow{\alpha} & A\Omega_{\tilde{S}}/(\xi^p, p) & & A\Omega_S \\ & \dashrightarrow & \downarrow & \dashrightarrow & \downarrow \\ & & A\Omega_{\tilde{S}}/(\xi, p) = \tilde{S}/p & \xrightarrow{\alpha} & A\Omega_S/(\xi^p, p) \\ & & & & \downarrow \\ & & & & A\Omega_S/(\xi, p) = \mathbb{L}\Omega_{(S/p)} \end{array}$$

Consider  $X^{1/p} \in \tilde{S}$  and its image  $\alpha(X^{1/p}) \in A\Omega_{\tilde{S}}/(\varphi(\xi), p)$ , let  $Y \in A\Omega_{\tilde{S}}$  be a lift of  $\alpha(X^{1/p})$ , then  $Y$  maps to  $(X^{1/p})^p = X \in \tilde{S}/p$ . Now consider the images of all the elements above along the dashed arrows, which we denote by the same notation. Since  $X = 0 \in S$ , we know that the image of  $Y^p$  in  $A\Omega_S/(\xi^p, p)$  is 0, in other words,  $Y^p \in (\xi^p, p) \subset A\Omega_S$ . This in turn shows that  $Y^p$  is divisible by  $p$  in  $A\Omega_S \hat{\otimes}^{\mathbb{L}} \mathbb{A}_{\text{cris}}$ . Since  $A\Omega_S \hat{\otimes}^{\mathbb{L}} \mathbb{A}_{\text{cris}}$  is  $p$ -torsion free (by Lemma 2.6), and is equipped with a Frobenius map (which lifts the  $p$ -power Frobenius mod  $p$ ), we know that all the divided powers  $\frac{Y^n}{n!}$  lies in  $A\Omega_S \hat{\otimes}^{\mathbb{L}} \mathbb{A}_{\text{cris}}$ . Finally, by a standard computation (for example in [Bha12]),

$$\mathbb{L}\Omega_{(S/p)} \cong \mathbb{L}\Omega_{(\mathbb{F}_p[X^{1/p^\infty}]/X)/\mathbb{F}_p} \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \cong D_{\mathcal{O}_C/p[X^{1/p^\infty}]}(X)$$

is the divided power envelop of  $\mathcal{O}_C/p[X^{1/p^\infty}]$  by adjoining all divided powers of  $X$ , and  $Y \in A\Omega_S$  maps to  $X \in \mathbb{L}\Omega_{(S/p)}$ . Therefore, the kernel of  $A\Omega_S \hat{\otimes}^{\mathbb{L}} \mathbb{A}_{\text{cris}} \rightarrow S/p$  can be generated by  $\langle p, \frac{\xi^m}{m!}, \frac{Y^n}{n!} \rangle$  with  $m, n \geq 1$ .  $\square$

**Remark 2.9 (Compare with the prismatic theory in [BS19])** *By the comparison between prismatic cohomology and  $A_{\text{inf}}$ -cohomology, we know a posteriori that*

$$A\Omega_S \cong \phi_{A_{\text{inf}}}^* \Delta_S \cong A_{\text{inf}}\langle Y^{1/p^\infty} \rangle \left\{ \frac{Y^p}{\varphi(\xi)} \right\}$$

for  $S = \mathcal{O}_C\langle X^{1/p^\infty} \rangle / (X)$ , where  $\Delta_S = \Delta_S / A_{\text{inf}}$  denotes the derived prismatic cohomology defined in [BS19], and  $Y = [X^b] \in A\Omega_{\tilde{S}} \cong \phi_{A_{\text{inf}}}^* \Delta_{\tilde{S}} / A_{\text{inf}} \cong A_{\text{inf}}(\tilde{S})$ . The map  $\alpha : S \rightarrow A\Omega_S/(\varphi(\xi))$  used in the proof of Lemma 2.8 corresponds (up to Frobenius twist) to the structure map  $S \rightarrow \Delta_S/\xi = \bar{\Delta}_S$ , which sends  $X^{1/p^m} \mapsto Y^{1/p^{m-1}}$  (see Section 12 of loc.cit.).

In fact, use the fact that  $\Delta_S$  is the initial prism constructed in Section 7 of loc.cit., the map  $\alpha$  induces a map  $\Delta \rightarrow A\Omega_S$ , which one can check is a  $\phi_{A_{\text{inf}}}$ -semilinear isomorphism by reducing mod  $\xi$  on the right (hence mod  $\varphi^{-1}(\xi)$  on the left). For more general quasiregular semiperfectoid algebras  $S$ , this makes  $A\Omega_S$  an object in the prismatic site of  $R/A_{\text{inf}}$ , hence by quasisyntomic

descent inducing a functorial  $\phi_{A_{\text{inf}}}$ -semilinear map between  $\Delta_{R/A_{\text{inf}}} \rightarrow A\Omega_R$ , which provides a way to reprove the comparison between prismatic cohomology and  $A_{\text{inf}}$ -cohomology from de Rham comparisons of both sides (compare with Section 17 of [BS19]).

### 3. The crystalline nature of the $A_{\text{cris}}$ base change of $A_{\text{inf}}$ -cohomology

As in the introduction, let  $\mathfrak{X}$  be a smooth formal scheme over  $\mathcal{O}_C$ . In this section we construct a functorial map from  $R\Gamma_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}) \rightarrow R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$  which reduces to the de Rham comparison over  $\mathcal{O}_C/p$ . It is convenient to view the  $A_{\text{inf}}$ -cohomology as a  $D(A_{\text{inf}})$ -valued sheaf on  $\mathfrak{X}_{\text{ét}}$ , given by

$$\mathfrak{U} \longmapsto R\Gamma_{A_{\text{inf}}}(\mathfrak{U}, A\Omega_{\mathfrak{X}}),$$

where  $A\Omega_{\mathfrak{X}} = L\eta_{\mu} R\nu_* \mathbb{A}_{\text{inf}, X}$ . The functor  $\mathfrak{U} \mapsto R\Gamma_{A_{\text{inf}}}(\mathfrak{U}) \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$  forms a sheaf on  $\mathfrak{X}_{\text{ét}}$  by the same argument for Lemma 2.5 (here the completion is  $p$ -adic). Furthermore, as affine opens  $\mathfrak{U} = \text{Spf } R$  in  $\mathfrak{X}_{\text{ét}, \text{aff}}$  form a basis for  $\mathfrak{X}_{\text{ét}}$ , it suffices to construct a functorial map

$$R\Gamma_{\text{crys}}((R/p)/A_{\text{cris}}) \longrightarrow A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$$

for all  $\text{Spf } R = \mathfrak{U} \subset \mathfrak{X}_{\text{ét}, \text{aff}}$ .

#### 3.1 The local map $h_{\text{crys}}$

**Construction 3.1** *Let  $R$  be the  $p$ -adic completion of a smooth  $\mathcal{O}_C$ -algebra. For each  $\mathcal{O}_C$ -algebra map  $R \rightarrow S$  where  $S \in \text{qrsPerfd}_{\mathcal{O}_C}^{\text{proj}}$ , we have the following commutative diagram*

$$\begin{array}{ccc} A_{\text{cris}} & \longrightarrow & A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \\ \downarrow & & \downarrow \beta \\ \mathcal{O}_C/p & \longrightarrow & R/p \longrightarrow S/p \end{array} .$$

The map  $\beta$  is  $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \rightarrow A\Omega_S/\xi \xrightarrow{\gamma_{\text{dR}}} \mathbb{L}\Omega_{(S/p)} \rightarrow S/p$  as described in Lemma 2.7. By Lemma 2.7, the diagram (often denoted by  $A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ ) is an object in the big crystalline site of  $R/p$  over  $(A_{\text{cris}}, \mathcal{O}_C/p, \gamma)$ . Thus by restriction in the crystalline site we obtain a functorial map

$$h_S : R\Gamma_{\text{crys}}((R/p)/A_{\text{cris}}) \longrightarrow A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} .$$

We first claim that

**Lemma 3.2**  *$h_S$  is compatible with the de Rham comparison along the base change  $A_{\text{cris}} \rightarrow \mathcal{O}_C/p$ . More precisely, the following diagram commutes*

$$\begin{array}{ccc} R\Gamma_{\text{crys}}((R/p)/A_{\text{cris}}) & \xrightarrow{h_{\text{crys}, S}} & A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \\ \downarrow & & \downarrow \\ R\Gamma_{\text{dR}}(R/p) \cong \Omega_{(R/p)}^{\bullet} & \longrightarrow & \mathbb{L}\Omega_{(S/p)} \end{array}$$

where the bottom map is the functorial map  $\Omega_{(R/p)}^{\bullet} \cong \mathbb{L}\Omega_{(R/p)} \rightarrow \mathbb{L}\Omega_{(S/p)}$ , and the right vertical map is the composition of base change and the de Rham comparison  $\gamma_{\text{dR}} : A\Omega_S/(p, \xi) \xrightarrow{\sim} \mathbb{L}\Omega_{(S/p)}$ .

*Proof.* By functoriality of the crystalline site, we may replace the top horizontal map by  $\bar{h}_{\text{crys}, S} : R\Gamma_{\text{crys}}((R/p)/(\mathcal{O}_C/p)) \rightarrow \mathbb{L}\Omega_{(S/p)}$ . As the crystalline cohomology of  $R/p$  over the trivial PD-ring

$(\mathcal{O}_C/p, 0)$  computes the de Rham cohomology, the problem is now the following: we have two maps  $\bar{h}_{\text{crys}}$  and  $\bar{h}_{\text{can}}$  from  $R\Gamma_{\text{dR}}(R/p)$  to  $\mathbb{L}\Omega_{(S/p)}$ , one constructed by viewing  $\mathbb{L}\Omega_{(S/p)}$  as an object in the crystalline site, the other induced by functoriality of derived de Rham cohomology, and we need to show that the two maps agree (up to homotopy). For this we first reduce to the case where  $R/p$  is a polynomial ring over  $\mathcal{O}_C/p$ , by considering a surjective map  $\Sigma \cong \mathcal{O}_C/p[X_i] \rightarrow R/p$  (and view  $\mathbb{L}\Omega_{(S/p)}$  as an object in the crystalline site over  $\Sigma$  via the map  $\Sigma \rightarrow R/p$ ).

In the rest of the proof we assume that  $R/p = \Sigma$  is a polynomial ring over  $\mathcal{O}_C/p$ . Now let  $\tilde{S} \rightarrow S$  be a perfectoid  $\mathcal{O}_C$ -algebra which surjects onto  $S$  (as  $S$  is quasiregular semiperfectoid). Choose a homomorphism  $\Sigma \rightarrow \tilde{S}/p$  such that its composition with the surjection  $\tilde{S}/p \rightarrow S/p$  gives the map  $\Sigma \rightarrow S/p$ . By functoriality of derived de Rham cohomology we obtain a map  $\mathbb{L}\Omega_{(\tilde{S}/p)} \rightarrow \mathbb{L}\Omega_{(S/p)}$ , which is a morphism in the crystalline site over  $\Sigma$ . Thus the composition

$$R\Gamma_{\text{crys}}(\Sigma/(\mathcal{O}_C/p)) \xrightarrow{\bar{h}_{\text{crys}, \tilde{S}}} \mathbb{L}\Omega_{(\tilde{S}/p)} \longrightarrow \mathbb{L}\Omega_{(S/p)}$$

agrees with  $\bar{h}_{\text{crys}, S}$ , and we are further reduced to the case where  $S$  is perfectoid, which we now assume. In this case  $\mathbb{L}\Omega_{(S/p)} = S/p$ , and by construction the map  $\bar{h}_{\text{crys}}$  is given by  $\Sigma(\bullet) \rightarrow S/p$  of cosimplicial algebras, where  $\Sigma(n)$  is the  $(n+1)$ -folded product of the polynomial ring  $\Sigma = R/p$  in the crystalline site (equivalently, the PD-envelop of  $\Sigma^{\otimes(n+1)} \rightarrow \Sigma$ ), and  $S/p$  denotes the constant cosimplicial algebra. By the proof of the crystalline and de Rham comparison in [BdJ11] (or similarly, by the proof of Proposition 5.5), we see that the map  $\bar{h}_{\text{crys}}$  is given by

$$\Sigma(\bullet) \cong \Omega_{\Sigma/(\mathcal{O}_C/p)}^\bullet \longrightarrow S/p,$$

where the second map is  $\Sigma \rightarrow S/p$  in degree 0, and 0 in other degrees. This agrees with the description of  $\bar{h}_{\text{can}}$ .  $\square$

### 3.2 The global map $h_{\text{crys}}$

**Construction 3.3** *Since the functor  $R \mapsto A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$  forms a sheaf on the quasisyntomic site (Lemma 2.5), by taking the homotopy limit over maps  $R \rightarrow S$  in the quasisyntomic site with  $S \in \text{qrsPerfd}_{\mathcal{O}_C}^{\text{proj}}$ , we arrive at the desired homomorphism*

$$R\Gamma_{\text{crys}}((R/p)/A_{\text{cris}}) \xrightarrow{h_{\text{crys}, R}} \text{Rlim } A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \cong A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}.$$

As discussed in the beginning of the section, we then take the homotopy limit over formal affine opens  $\text{Spf } R$  in the étale site  $\mathfrak{X}_{\text{ét}}$ , which allows us to write  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \cong \text{Rlim } A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$ . Consequently, we obtain the construction of  $h_{\text{crys}}$  as the limit of  $h_{\text{crys}, R}$ .

**Remark 3.4** *An equivalent construction of the map  $h_S$  can be given as follows. As in [BMS19], let  $\mathbb{A}_{\text{cris}}(S/p)$  be the  $p$ -completed PD-envelop of  $A_{\text{inf}}(S) \rightarrow S/p$ , and let  $\mathbb{L}W\Omega_{S/p}$  be the derived de Rham–Witt complex of  $S/p$  over  $\mathbb{F}_p$ , which is defined via left Kan extension from smooth  $\mathbb{F}_p$ -algebras. In particular  $\mathbb{L}W\Omega_{S/p}$  is equivalent to the derived crystalline cohomology  $\mathbb{L}R\Gamma_{\text{crys}}(S/p)$  (over the PD-ring  $(\mathbb{Z}_p, p)$ ). By Proposition 8.12 of [BMS19], there is a canonical isomorphism  $\mathbb{A}_{\text{cris}}(S/p) \xrightarrow{\sim} \mathbb{L}R\Gamma_{\text{crys}}(S/p)$ . Thus there is a natural map  $\psi_S : \mathbb{A}_{\text{cris}}(S/p) \rightarrow A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}$  by Lemma 2.7. This in turn induces a map*

$$R\Gamma_{\text{crys}}((R/p)/A_{\text{cris}}) \longrightarrow \mathbb{L}R\Gamma_{\text{crys}}(S/p) \xrightarrow{\sim} \mathbb{A}_{\text{cris}}(S/p) \longrightarrow A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}}.$$

It might be possible to show that  $\psi_S$  (hence  $h_{\text{crys}}$ ) is an isomorphism, which we do not pursue as the existence of  $h_{\text{crys}}$  already suffices to deduce the main results of [BMS18] (essentially all except

for  $A_{\text{cris}}$ -specialization). Moreover, we prefer Construction 3.3 for the following reasons. First, this avoids using Proposition 8.12 in [BMS19].<sup>5</sup> More importantly, Construction 3.3 (via the crystalline site) compares well with the  $B_{\text{dR}}^+$ -cohomology by functoriality. Finally, our construction generalizes to the logarithmic setting, while  $\psi_S$  does not take log structures into account.

We summarize this discussion in the following

**Theorem 3.5** *Let  $\mathfrak{X}$  be a smooth formal scheme over  $\mathcal{O}_C$ . There exists an  $A_{\text{cris}}$ -linear map*

$$h_{\text{crys}} = h_{\text{crys}, \mathfrak{X}} : R\Gamma_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_C/p} / A_{\text{cris}}) \longrightarrow R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}_{A_{\text{inf}}}^{\mathbb{L}} A_{\text{cris}}$$

which is  $\varphi$ -equivariant and functorial on  $\mathfrak{X}$ . Moreover, it is compatible with the de Rham comparison. In other words, the following diagram commutes:

$$\begin{array}{ccc} R\Gamma_{\text{crys}}((R/p) / A_{\text{cris}}) & \longrightarrow & R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}_{A_{\text{inf}}}^{\mathbb{L}} A_{\text{cris}} \\ \downarrow & & \downarrow \\ R\Gamma_{\text{dR}}(\mathfrak{X}_{\mathcal{O}_C/p}) & \xrightarrow{\gamma_{\text{dR}}^{-1}} & R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}, \theta}}^{\mathbb{L}} \mathcal{O}_C/p \end{array}$$

### 3.3 The crystalline specialization of $A_{\text{inf}}$ -cohomology

As a direct corollary of the construction above (Theorem 3.5), we obtain the crystalline comparison of  $A_{\text{inf}}$ -cohomology.

**Corollary 3.6** *Let  $\mathfrak{X}$  be a smooth formal scheme over  $\mathcal{O}_C$ . There is a canonical  $\varphi$ -compatible quasi-isomorphism*

$$h_W : R\Gamma_{\text{crys}}(\mathfrak{X}_k / W(k)) \xrightarrow{\sim} R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \widehat{\otimes}_{A_{\text{inf}, \vartheta}}^{\mathbb{L}} W(k)$$

relating  $A_{\text{inf}}$ -cohomology of  $\mathfrak{X}$  to the crystalline cohomology of  $\mathfrak{X}_k$ .

*Proof.* We base change  $h_{\text{crys}}$  along  $\vartheta : A_{\text{cris}} \rightarrow W(k)$  to obtain the desired morphism. Now we further base change from  $W(k) \rightarrow k$  and consider the following obviously commutative diagram

$$\begin{array}{ccc} A_{\text{cris}} & \xrightarrow{\vartheta} & W(k) \\ \downarrow \theta & & \downarrow \\ \mathcal{O}_C/p & \longrightarrow & k \end{array}$$

of maps of rings. Therefore, the base change of  $h_W$  to  $k$  is  $\gamma_{\text{dR}}^{-1} \otimes^{\mathbb{L}} k$ , which is a quasi-isomorphism, then apply derived Nakayama's lemma.  $\square$

**Remark 3.7** *We do not know whether this directly implies that  $h_{\text{crys}}$  is an quasi-isomorphism. The issue is that the ideal  $I = \ker(A_{\text{cris}} \rightarrow \mathcal{O}_C/p)$  is not finitely generated, so we do not know whether derived  $I$ -completion is a well-behaved notion (in particular we do not know whether the derived Nakayama's lemma still holds in this context).*

## 4. The $B_{\text{cris}}$ comparison theorem

In this section we analyze the cohomology groups  $H_{A_{\text{inf}}}^*(\mathfrak{X})$ , and prove the remaining part of Theorem 4 from the introduction. Throughout this section we further assume that  $\mathfrak{X}$  is proper.

<sup>5</sup>which is elementary enough but still involves analyzing the conjugate filtration on a certain PD envelop, see Proposition 8.11 thereof.

### 4.1 The individual cohomology groups

**Definition 4.1** A Breuil–Kisin–Fargues module is a pair  $(M, \varphi_M)$  where  $M$  is a finitely presented  $A_{\text{inf}}$ -module such that  $M[\frac{1}{p}]$  is free over  $A_{\text{inf}}[\frac{1}{p}]$ , and  $\varphi_M$  is a  $\varphi$ -linear isomorphism  $\varphi_M : M[\frac{1}{\xi}] \xrightarrow{\sim} M[\frac{1}{\varphi(\xi)}]$ .

We will need the following classification result of Fargues for the proof of Theorem 5.9.<sup>6</sup>

**Theorem 4.2 (Fargues)** There is an equivalence of categories between finite free Breuil–Kisin–Fargues modules and the category of pairs  $(T, \Xi)$ , where  $T$  is a finite free  $\mathbb{Z}_p$ -module and  $\Xi$  is a  $B_{\text{dR}}^+$ -lattice in  $T \otimes_{\mathbb{Z}_p} B_{\text{dR}}$ , given by the functor

$$(M, \varphi_M) \mapsto \left( (M \otimes W(C^b))^{\varphi_M=1}, M \otimes B_{\text{dR}}^+ \right).$$

**Theorem 4.3** Let  $\mathfrak{X}$  be a smooth proper formal scheme over  $\text{Spf } \mathcal{O}_C$ . The cohomology group  $H_{A_{\text{inf}}}^i(\mathfrak{X})$  takes value in Breuil–Kisin–Fargues modules, and vanishes for  $i > 2 \dim_{\mathcal{O}_C} \mathfrak{X}$ .

**Remark 4.4** The claim on vanishing of cohomology is immediate from the de Rham comparison. For the (implicit) claim on Frobenius, observe that the Frobenius on  $R\nu_* \mathbb{A}_{\text{inf}, \mathfrak{X}}$  induces the desired map on the level of sheaves

$$L\eta_\mu R\nu_* \mathbb{A}_{\text{inf}, \mathfrak{X}} \xrightarrow{\sim} L\eta_{\varphi(\xi)} L\eta_\mu R\nu_* \mathbb{A}_{\text{inf}, \mathfrak{X}} \longrightarrow L\eta_\mu R\nu_* \mathbb{A}_{\text{inf}, \mathfrak{X}}$$

on the pro-étale site, which becomes an isomorphism after inverting  $\varphi(\xi)$ . The first isomorphism essentially follows from definition of  $L\eta$  (note that  $\mu$  is regular in  $A_{\text{inf}}$ ). The existence of the second arrow is not entirely formal: it uses in particular that  $\mathcal{H}^0(L\eta_\mu R\nu_* \mathbb{A}_{\text{inf}, \mathfrak{X}})$  has no  $\varphi(\xi)$ -torsion and properties of the  $L\eta$  operator (see Lemma 5.8 of [Mor16]).

It remains to show that  $H_{A_{\text{inf}}}^i(\mathfrak{X})$  is finitely presented and becomes free after inverting  $p$ . Our approach follows the method outlined in [Mor16], which is different from the proof given in [BMS18], where they need to identify  $R\Gamma_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_C/p} / A_{\text{cris}})$  with  $R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}} A_{\text{cris}}$  (at least after inverting  $p$ ). Our proof uses a careful descending induction on the degree  $i$  of cohomology, where the most difficult part is to show that  $H_{A_{\text{inf}}}^i(\mathfrak{X})[\frac{1}{p}]$  is finite free. We will complete the proof in the next subsection.

### 4.2 Valuation in Breuil–Kisin–Fargues modules

We continue to assume that  $\mathfrak{X}$  is smooth proper over  $\mathcal{O}_C$ .

**Lemma 4.5** Via base change along  $A_{\text{cris}} \rightarrow B_{\text{dR}}^+$ , we have a functorial isomorphism  $h_{\text{crys}} \otimes B_{\text{dR}}^+ :$

$$R\Gamma_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_C/p} / A_{\text{cris}}) \otimes_{A_{\text{cris}}}^{\mathbb{L}} B_{\text{dR}}^+ \xrightarrow{\sim} R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}}}^{\mathbb{L}} B_{\text{dR}}^+.$$

*Proof.* As mentioned in the introduction, the  $A_{\text{inf}}$ -cohomology  $R\Gamma_{A_{\text{inf}}}$  is derived  $\xi$ -adically complete thus it is a perfect complex in  $D(A_{\text{inf}})$  by the de Rham comparison. On the other hand, the crystalline cohomology  $R\Gamma_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_C/p} / A_{\text{cris}})$  is a perfect complex in  $D(A_{\text{cris}})$  (see, for example, Tag 07MY in [dJea]). In particular, both sides of the map given above are derived  $\xi$ -adically complete (to see this, note that  $M \in D(B_{\text{dR}}^+)$  is derived complete if and only if each  $H^i(M)$  is derived  $\xi$ -adically complete, and then observe that finitely presented modules over  $B_{\text{dR}}^+$  are derived  $\xi$ -complete). Therefore, to prove the lemma it suffices to show that  $h_{\text{crys}} \otimes B_{\text{dR}}^+$  becomes an isomorphism after reducing mod  $\xi$ , this in turn follows from Theorem 3.5 and the de Rham comparison.  $\square$

<sup>6</sup>As remarked in [BMS18], only the easy direction of the equivalence (namely fully faithfulness) is needed.

We also need the following variant of the Berthelot–Ogus comparison isomorphism, as already observed in [BMS18]. For this lemma we fix a section  $k \rightarrow \mathcal{O}_C/p$ .

**Lemma 4.6 (BMS)** *There is a natural  $\varphi$ -equivariant isomorphism*

$$H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}})[\frac{1}{p}] \xrightarrow{\sim} H_{\text{crys}}^i(\mathfrak{X}_k/W(k)) \otimes_{W(k)} B_{\text{cris}}^+.$$

*In particular,  $H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}})[\frac{1}{p}]$  is a finite free  $B_{\text{cris}}^+$ -module.*

*Proof.* The point is that the Frobenius  $\varphi$  on the crystalline cohomology  $H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}})$  becomes an isomorphism after inverting  $p$  (since on affine opens  $U$ , there is an isomorphism  $U \cong U_k \times_{\text{Spec } k} \text{Spec } \mathcal{O}_C/p$  by the smoothness assumption, then apply base change). Moreover, there exists a large enough  $n$  such that

$$\iota : \mathfrak{X}_{\mathcal{O}_C/p^{1/p^n}} \cong \mathfrak{X}_k \times_{\text{Spec } k} \text{Spec } \mathcal{O}_C/p^{1/p^n},$$

and any two such isomorphisms agree once we enlarge  $n$ . Now by repeatedly applying Frobenius we obtain the following diagram

$$\begin{array}{ccc} H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p^{1/p^n}}/A_{\text{cris}}^{(\varphi^n)}) \otimes_{\varphi^n} B_{\text{cris}}^+ & \xrightarrow[\varphi^n]{\sim} & H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}})[\frac{1}{p}] \\ \uparrow & & \uparrow \text{---} \\ H_{\text{crys}}^i(\mathfrak{X}_k/W(k)) \otimes_{\varphi^n} B_{\text{cris}}^+ & \xrightarrow[\sim]{\varphi^n \otimes \text{id}} & H_{\text{crys}}^i(\mathfrak{X}_k/W(k)) \otimes B_{\text{cris}}^+ \end{array}$$

where  $A_{\text{cris}}^{(\varphi^n)}$  denotes the  $p$ -completed PD-envelop of  $A_{\text{inf}} \rightarrow \mathcal{O}_C/p^{1/p^n}$ . The top horizontal isomorphism is induced by the absolute Frobenius  $\varphi^n$  on  $\mathfrak{X}_{\mathcal{O}_C/p}$ , which factors as  $\mathfrak{X}_{\mathcal{O}_C/p} \xrightarrow{\text{pr}} \mathfrak{X}_{\mathcal{O}_C/p^{1/p^n}} \xrightarrow{\varphi^n} \mathfrak{X}_{\mathcal{O}_C/p}$ , and the left vertical isomorphism comes from the identification  $\iota$  noted above.  $\square$

**Corollary 4.7** *Continue to assume that  $\mathfrak{X}$  is smooth proper over  $\mathcal{O}_C$ , then*

$$\text{rk}_{W(k)} H_{\text{crys}}^i(\mathfrak{X}_k/W(k)) = \text{rk}_{\mathbb{Z}_p} H_{\text{ét}}^i(X, \mathbb{Z}_p).$$

*Proof.* By Lemma 4.5 and the étale comparison for  $A_{\text{inf}}$ -cohomology, we have an isomorphism  $R\Gamma_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}) \otimes_{A_{\text{cris}}}^{\mathbb{L}} B_{\text{dR}} \cong R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes B_{\text{dR}}$ . Denote their common value by  $D$ , and for ease of notation write  $H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p})$  for the crystalline cohomology of  $\mathfrak{X}_{\mathcal{O}_C/p}$  over  $A_{\text{cris}}$ , then

$$H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p}) \otimes_{A_{\text{cris}}} B_{\text{dR}} \cong H^i(D)$$

by the spectral sequence  $\text{Tor}_a(H_{\text{crys}}^b(\mathfrak{X}_{\mathcal{O}_C/p}), B_{\text{dR}}) \Rightarrow H^{b-a}(D)$ . Note that all higher ( $a \geq 1$ ) Tor terms  $\text{Tor}_a^{A_{\text{cris}}}(H_{\text{crys}}^b(\mathfrak{X}_{\mathcal{O}_C/p}), B_{\text{dR}})$  vanishes, since  $p$  is already inverted, and  $H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}})[\frac{1}{p}]$  is finite free by Lemma 4.6. Similarly, we have the following isomorphism

$$H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p}) \otimes_{A_{\text{cris}}} B_{\text{dR}} \cong H_{\text{ét}}^i(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{dR}}$$

of free  $B_{\text{dR}}$ -modules. The lemma then follows from Lemma 4.6 once again.  $\square$

**Lemma 4.8** *Assume that  $H_{A_{\text{inf}}}^k(\mathfrak{X})[\frac{1}{p}]$  is finite free  $A_{\text{inf}}[\frac{1}{p}]$ -modules for  $k \geq i+1$ , then the cohomology groups fit in the following short exact sequences*

$$0 \rightarrow H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes W(k) \rightarrow H_{\text{crys}}^i(\mathfrak{X}_k/W(k)) \rightarrow \text{Tor}^1(H_{A_{\text{inf}}}^{i+1}(\mathfrak{X}), W(k)) \rightarrow 0.$$

*In particular, if  $H_{\text{crys}}^i(\mathfrak{X}_k/W(k))$  is torsion free, then so is  $H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes W(k)$ .*

**Remark 4.9** *This lemma is used in the proof of Theorem 4.3. Of course, once that theorem is proven, the assumption here is no longer necessary.*

*Proof.* By the crystalline comparison (the derived version, see Corollary 3.6), the spectral sequence for Tor becomes  $E_2^{-a,b} = \mathrm{Tor}_a^{\mathrm{A}_{\mathrm{inf}}}(H_{\mathrm{A}_{\mathrm{inf}}}^b(\mathfrak{X}), W(k)) \Rightarrow H_{\mathrm{crys}}^{b-a}(\mathfrak{X}_k/W(k))$ . Now apply Lemma A.5 in [Mor16], which implies that  $\mathrm{Tor}_a^{\mathrm{A}_{\mathrm{inf}}}(H_{\mathrm{A}_{\mathrm{inf}}}^b(\mathfrak{X}), W(k)) = 0$  for all  $b > i, a > 1$ .  $\square$

The final ingredient we need is the following lemma of Morrow (which replaces the role of Lemma 4.19 of [BMS18]).

**Lemma 4.10 (Morrow)** *Let  $M$  be a finitely presented  $\mathrm{A}_{\mathrm{inf}}$ -module equipped with a  $\varphi$ -semilinear endomorphism, which becomes an isomorphism upon inverting  $\xi$ . Assume that  $M[\frac{1}{p\mu}]$  is a finite free  $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p\mu}]$ -module of the same rank of  $M \otimes_{\mathrm{A}_{\mathrm{inf}}} W(k)$ . Then  $M[\frac{1}{p}]$  is finite free over  $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]$ .*

*Proof.* This is Lemma A.4 of [Mor16].  $\square$

Now we are ready to prove Theorem 4.3 (using the method of [Mor16]).

*Proof of Theorem 4.3.* As in Remark 4.4 it remains to show that  $H_{\mathrm{A}_{\mathrm{inf}}}^i(\mathfrak{X})$  is finitely presented and becomes free after inverting  $p$ , which we prove by a descending induction on  $i$ . The claim is vacuously true for  $i > 2 \dim \mathfrak{X}$ . By inductive hypothesis, each  $H^k(\tau^{>i} R\Gamma_{\mathrm{A}_{\mathrm{inf}}}(\mathfrak{X}))$  is finitely presented over  $\mathrm{A}_{\mathrm{inf}}$  and becomes free after inverting  $p$ , so as a complex it is perfect over  $\mathrm{A}_{\mathrm{inf}}$ , so the bounded complex  $\tau^{>i} R\Gamma_{\mathrm{A}_{\mathrm{inf}}}(\mathfrak{X})$  is perfect, and hence  $\tau^{\leq i} R\Gamma_{\mathrm{A}_{\mathrm{inf}}}(\mathfrak{X})$  is perfect. From this it follows that the cohomology  $H_{\mathfrak{X}}^i(\mathfrak{X})$  is finitely presented as it is the cokernel of a map between two finite projective  $\mathrm{A}_{\mathrm{inf}}$ -modules. To show that  $H_{\mathrm{A}_{\mathrm{inf}}}^i(\mathfrak{X})[\frac{1}{p}]$  is finite free over  $\mathrm{A}_{\mathrm{inf}}[\frac{1}{p}]$ , we apply Lemma 4.10. Note that from the étale comparison (and note that  $\mathrm{A}_{\mathrm{inf}}[\frac{1}{\mu}]$  is  $p$ -torsion free) we have  $H_{\mathrm{A}_{\mathrm{inf}}}^i(\mathfrak{X})[\frac{1}{\mu}] = H_{\mathrm{ét}}^i(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{A}_{\mathrm{inf}}[\frac{1}{\mu}]$ , thus it suffices to show that  $\mathbb{Z}_p$ -rank of  $H_{\mathrm{ét}}^i(X, \mathbb{Z}_p)$  is the same the  $W(k)$ -rank of  $H_{\mathrm{crys}}^i(\mathfrak{X}_k/W(k))$ . This is precisely the statement of Corollary 4.7.  $\square$

**Remark 4.11** *As mentioned in the introduction, Theorem 4.3 together with the comparison theorems of  $R\Gamma_{\mathrm{A}_{\mathrm{inf}}}(\mathfrak{X})$  is enough to deduce the following results of [BMS18] on torsion. First of all, for each  $n \geq 1$ , we have*

$$\mathrm{length}_{W(k)}(H_{\mathrm{crys}}^i(\mathfrak{X}_k/W(k))_{\mathrm{tor}/p^n}) \geq \mathrm{length}_{\mathbb{Z}_p}(H_{\mathrm{ét}}^i(X, \mathbb{Z}_p)_{\mathrm{tor}/p^n}).$$

*Moreover,  $H_{\mathrm{crys}}^i(\mathfrak{X}_k/W(k))$  is  $p$ -torsion free if and only if  $H_{\mathrm{dR}}^i(\mathfrak{X}/\mathcal{O}_C)$  is  $p$ -torsion free, in which case,  $H_{\mathrm{A}_{\mathrm{inf}}}^i(\mathfrak{X})$  is a finite free  $\mathrm{A}_{\mathrm{inf}}$ -module. These claims follow from some basic properties of finitely presented  $\mathrm{A}_{\mathrm{inf}}$ -modules, notably Lemma 4.15, 4.17 and 4.18 of loc.cit, for which it is important to know that  $H_{\mathrm{A}_{\mathrm{inf}}}^i(\mathfrak{X})[\frac{1}{p}]$  is free.*

**Remark 4.12** *Note that after inverting  $p$ , the map  $h_{\mathrm{crys}}$  induces the following  $\varphi$ -equivariant map on cohomology groups (which we denote again by  $h_{\mathrm{crys}}$ )*

$$h_{\mathrm{crys}} : H_{\mathrm{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes_{A_{\mathrm{cris}}} B_{\mathrm{cris}}^+ \rightarrow H_{\mathrm{A}_{\mathrm{inf}}}^i(\mathfrak{X}) \otimes B_{\mathrm{cris}}^+,$$

*which is functorial on  $\mathfrak{X}$ . This directly follows from the freeness of  $H_{\mathrm{A}_{\mathrm{inf}}}^i(\mathfrak{X})[\frac{1}{p}]$ .*

### 4.3 The $B_{\mathrm{cris}}$ comparison

In this subsection we prove the  $B_{\mathrm{cris}}$  comparison theorem (except for the part on filtration, which we leave to the last section of this paper). We adopt the following setup: let  $K$  be a finite

extension of  $\mathbb{Q}_p$ ,<sup>7</sup> with residue field  $k_0$ , and maximal unramified subfield  $K_0 = W(k_0)[\frac{1}{p}]$ . Let  $C = C_K$  be a  $p$ -completed algebraic closure of  $K$ , with residue field  $k \cong \overline{\mathbb{F}}_p$ . Let  $\mathfrak{X}_{\mathcal{O}_K}$  be a smooth proper formal scheme over  $\mathcal{O}_K$ , with adic generic fiber  $X_K$  and special fiber  $\mathfrak{X}_0$ . Let  $\mathfrak{X}$  be the base change of  $\mathfrak{X}_{\mathcal{O}_K}$  to  $\mathcal{O}_C$ , so  $X = X_K \times \text{Spa}(C, \mathcal{O}_C)$  and  $\mathfrak{X}_k = \mathfrak{X}_0 \times_{\text{Spec } k_0} \text{Spec } k$ . Note that there is a unique section from  $k = \overline{\mathbb{F}}_p \rightarrow \mathcal{O}_C/p$  in this setup. By combining Lemma 4.6 and Remark 4.12, we get a  $\text{Gal}_K$ -equivariant (by functoriality) and  $\varphi$ -compatible map

$$\alpha_{\text{crys}} : H_{\text{crys}}^i(\mathfrak{X}_0/W(k_0)) \otimes B_{\text{cris}}^+ \longrightarrow H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes B_{\text{cris}}^+.$$

More precisely  $\alpha_{\text{crys}}$  is the following composition

$$\begin{array}{ccc} H_{\text{crys}}^i(\mathfrak{X}_0/W(k_0)) \otimes_{W(k_0)} B_{\text{cris}}^+ & \xrightarrow{\alpha_{\text{crys}}} & H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes B_{\text{cris}}^+ \\ \downarrow \wr & & \uparrow h_{\text{crys}} \\ H_{\text{crys}}^i(\mathfrak{X}_k/W(k)) \otimes_{W(k)} B_{\text{cris}}^+ & \xrightarrow[\text{by 4.6}]{\sim} & H_{\text{crys}}^i(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\text{cris}}) \otimes B_{\text{cris}}^+ \end{array}$$

By construction, the base change of  $\alpha_{\text{crys}}$  along  $B_{\text{cris}}^+ \rightarrow K_0$  becomes a  $\varphi$ -equivariant isomorphism  $H_{\text{crys}}^i(\mathfrak{X}_0/W(k_0))[\frac{1}{p}] \xrightarrow{\sim} H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes K_0$ , which is precisely the map on cohomology groups obtained from the quasi-isomorphism  $h_W$  in Corollary 3.6.

**Theorem 4.13** *Retain the setup from above. Then after composing with the étale comparison,  $\alpha_{\text{crys}}$  induces a functorial  $(\text{Gal}_K, \varphi)$ -equivariant isomorphism*

$$\beta_{\text{crys}} : H_{\text{crys}}^i(\mathfrak{X}_0/W(k_0)) \otimes B_{\text{cris}} \xrightarrow{\sim} H_{\text{ét}}^i(X, \mathbb{Z}_p) \otimes B_{\text{cris}}.$$

In particular,  $H_{\text{ét}}^i(X, \mathbb{Q}_p)$  is crystalline.

*Proof.* We consider the map  $\alpha_{\text{crys}}$  constructed above and take the  $\text{Gal}_K$ -invariant subspaces, which leads to the following diagram of  $K_0$ -vector spaces<sup>8</sup>

$$\begin{array}{ccc} H_{\text{crys}}^i(\mathfrak{X}_0/W(k_0))[\frac{1}{p}] & & \\ \downarrow & & \\ \left( H_{\text{crys}}^i(\mathfrak{X}_0/W(k_0)) \otimes B_{\text{cris}}^+ \right)^{\text{Gal}_K} & \longrightarrow & \left( H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes B_{\text{cris}}^+ \right)^{\text{Gal}_K} \\ \downarrow & & \downarrow \\ H_{\text{crys}}^i(\mathfrak{X}_0/W(k_0))[\frac{1}{p}] & \xrightarrow[\sim]{\alpha_{\text{crys}} \otimes K_0} & H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes K_0 \end{array}$$

Here the commutative square is obtained from applying  $\text{Gal}_K$ -invariance to the morphism  $\alpha_{\text{crys}}$  and its base change  $\alpha_{\text{crys}} \otimes K_0$  (the Galois group acts trivially on the latter). The composition of the left vertical map (induced by  $W(k_0) \rightarrow A_{\text{cris}} \rightarrow W(k_0)$ ) is an isomorphism, therefore the following composition

$$H_{\text{crys}}^i(\mathfrak{X}_0/W(k_0))[\frac{1}{p}] \longrightarrow \left( H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes B_{\text{cris}}^+ \right)^{\text{Gal}_K} \longrightarrow H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes K_0$$

is an isomorphism. In particular, we have

$$\dim_{K_0} \left( H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes B_{\text{cris}}^+ \right)^{\text{Gal}_K} \geq \dim_{K_0} H_{\text{crys}}^i(\mathfrak{X}_0/W(k_0))[\frac{1}{p}].$$

<sup>7</sup>or more generally a discretely valued nonarchimedean extension

<sup>8</sup>Note that all terms in this commutative square are  $K_0$ -vector spaces, as  $K_0 \subset (B_{\text{cris}}^+)^{\text{Gal}_K} \subset (B_{\text{cris}})^{\text{Gal}_K} = K_0$ .

Now let  $V = H_{\text{ét}}^i(X, \mathbb{Q}_p)$ , then  $D_{\text{cris}}(V) \cong (H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes B_{\text{cris}})^{\text{Gal}_K}$  by the étale comparison, therefore we conclude that

$$\dim_{K_0} D_{\text{cris}}(V) \geq \dim_{K_0} (H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes B_{\text{cris}}^+)^{\text{Gal}_K} \geq \dim_{\mathbb{Q}_p} V$$

where the last inequality follows from the previous inequality and Corollary 4.7. The theorem thus follows.  $\square$

**Remark 4.14** *Let us rephrase the proof in words. To show that  $V$  is crystalline we want to show that  $D_{\text{cris}}(V)$  is large enough (namely there are enough Galois invariants). We know that the crystalline cohomology  $H_{\text{crys}}^i(\mathfrak{X}_k/W(k))$  has large enough rank by Corollary 4.7, so we need to relate the crystalline cohomology to  $D_{\text{cris}}(V)$  (which we know a posteriori are the same after inverting  $p$ ). This is precisely provided by  $h_{\text{crys}}$  (and  $\alpha_{\text{crys}}$  on the cohomology groups). It does not seem possible to deduce the  $B_{\text{cris}}$  comparison directly from Corollary 4.7.*

## 5. The infinitesimal nature of $B_{\text{dR}}^+$ -cohomology

In this section we give a reformulation of the  $B_{\text{dR}}^+$ -cohomology of [BMS18]. We then use it to study the filtration compatibility of the  $B_{\text{cris}}$  comparison, and to recover  $H_{A_{\text{inf}}}^i(\mathfrak{X})$  from the generic fiber in certain restricted cases.

### 5.1 An infinitesimal site

In this subsection let  $X$  be a smooth proper rigid analytic variety over  $\text{Spa}(C, \mathcal{O}_C)$ .<sup>9</sup> Let  $B_{\text{dR},m}^+$  be the complete Tate Huber ring  $B_{\text{dR},m}^+ := B_{\text{dR}}^+/\xi^m = A_{\text{inf}}[\frac{1}{p}]/\xi^m$  over  $\mathbb{Q}_p$ .

**Definition 5.1** *Let  $A$  be a Tate algebra over  $C$ . The indiscrete infinitesimal site  $\text{Inf}(A/B_{\text{dR},m}^+)^{\text{op},\circ}$  of  $A$  relative to  $B_{\text{dR},m}^+ \rightarrow C$  is defined as follows. The objects of  $\text{Inf}(A/B_{\text{dR},m}^+)^{\text{op}}$  are of the form*

$$(B, J) = \begin{array}{ccc} B_{\text{dR},m}^+ & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & A \longrightarrow B/J \end{array}$$

where  $B \rightarrow B/J$  is pro-nilpotent thickening. Here  $B$  is a complete Huber ring, whose topology is compatible with  $B_{\text{dR},m}^+$  (so in particular it is Tate with  $p$  being a pseudo-uniformizer). The topology of the site  $\text{Inf}(A/B_{\text{dR},m}^+)^{\text{op},\circ}$  is the indiscrete topology.

Similarly we define the site  $\text{Inf}(A/B_{\text{dR}}^+)^{\text{op},\circ}$ , with the requirement that  $B = \varprojlim B/\xi^m$  (as a topological ring), and  $(B/\xi^m, J) \in \text{Ob}(\text{Inf}(A/B_{\text{dR},m}^+))$ .  $\text{Inf}(A/B_{\text{dR}}^+)^{\text{op},\circ}$  (resp.  $\text{Inf}(A/B_{\text{dR},m}^+)^{\text{op},\circ}$ ) is a ringed site with the structure sheaf  $\mathcal{O}_{\text{Inf}}$ , which sends  $(B, J) \mapsto B$ .

**Definition 5.2** *Let  $K$  be any complete nonarchimedean extension of  $\mathbb{Q}_p$ . A smooth Tate algebra  $A$  over  $K$  is petit if there exists an étale map  $K\langle t_i \rangle \rightarrow A$  from a Tate polynomial with finitely many variables.*

---

<sup>9</sup>We view  $X$  as an adic space. In fact, it is possible to work the slightly more general class of analytic adic spaces that are locally noetherian – the former condition in particular implies that  $X$  can be covered by the adic spectrum of Tate Huber pairs.

For a petit Tate algebra  $A$  over  $C$ , the infinitesimal cohomology  $R\Gamma_{\text{inf}}(A/B_{\text{dR}}^+)$  of  $A$  with  $B_{\text{dR}}^+$  coefficients is defined as the cohomology of the structure sheaf  $\mathcal{O}_{\text{Inf}}$ .<sup>10</sup> To define the global infinitesimal cohomology  $R\Gamma_{\text{inf}}(X/B_{\text{dR}}^+)$  of  $X$ , we let  $X_{\text{min},\text{ét}}$  be the site generated by petit affinoid objects in the étale site  $X_{\text{ét}}$ . Note that the petit affinoids form a basis for  $X_{\text{ét}}$ , by Corollary 1.6.10 of [Hub96]. We then define

$$R\Gamma_{\text{inf}}(X/B_{\text{dR}}^+) := \lim_{\text{Spa}(A, A^\circ)} R\Gamma_{\text{inf}}(A/B_{\text{dR}}^+)$$

where the derived inverse limit is taken over all affinoids  $\text{sp}A = \text{Spa}(A, A^\circ)$  in  $X_{\text{min},\text{ét}}$ . Equivalently, we may regard  $R\Gamma_{\text{inf}}(A/B_{\text{dR}}^+)$  as a functor from  $X_{\text{min},\text{ét}}$  to  $D(B_{\text{dR}}^+)$ , which takes values in derived  $\xi$ -complete objects.<sup>11</sup> This turns out to be a sheaf on  $X_{\text{ét}}$  (by the comparison with de Rham cohomology below), and thus induces a sheaf on  $X_{\text{ét}}$ , whose derived global section gives the global infinitesimal cohomology.

**Remark 5.3** *It should be possible to define a global version of the infinitesimal site (with the étale topology) and then define  $R\Gamma_{\text{inf}}(X/B_{\text{dR}}^+)$  directly using this site. However, for simplicity we prefer to work locally on affinoids, for which it often suffices to consider the indiscrete topology by the vanishing of higher cohomology for coherent sheaves on affinoids (in our simplified setup this is already ensured by Tate's acyclicity theorem).*

## 5.2 Comparison with de Rham cohomology

**Lemma 5.4** *There is a quasi-isomorphism (obtained by base change)*

$$R\Gamma_{\text{inf}}(A/B_{\text{dR}}^+) \otimes_{B_{\text{dR}}^+}^{\mathbb{L}} C \xrightarrow{\sim} R\Gamma_{\text{inf}}(A/C).$$

Note that  $\text{Inf}(A/B_{\text{dR},1}^+)^{\text{op},\circ} = \text{Inf}(A/C)^{\text{op},\circ}$ .

*Proof.* Choose a surjection  $C\langle X_i \rangle \twoheadrightarrow A$  from a Tate polynomial ring to  $A$ . Let  $B_{\text{dR}}^+\langle X_i \rangle = \varprojlim B_{\text{dR},m}^+\langle X_i \rangle$ , where  $B_{\text{dR},m}^+\langle X_i \rangle = (A_{\text{inf}}/\xi^m\langle X_i \rangle)_{[1/p]}$ , and let  $\Sigma_{\text{dR}}$  be the classical completion of  $B_{\text{dR}}^+\langle X_i \rangle$  with respect to the kernel

$$J = J(0) := \ker(B_{\text{dR}}^+\langle X_i \rangle \twoheadrightarrow A).$$

$\Sigma_{\text{dR}}$  is a weakly terminal object in  $\text{Inf}(A/B_{\text{dR}}^+)^{\text{op},\circ}$ . Its  $(n+1)$ -folded product  $\Sigma_{\text{dR}}(n)$  is given by the completion of  $\varprojlim B_{\text{dR},m}^+\langle X_i \rangle^{\widehat{\otimes}(n+1)}$  with respect to  $J(n)$  (defined similarly as  $J(0)$ ). The cosimplicial complex  $\Sigma_{\text{dR}}(\bullet)$  then computes the cohomology  $R\Gamma_{\text{inf}}(A/B_{\text{dR}}^+)$ .

Note that  $B_{\text{dR}}^+\langle X_i \rangle$  is  $\xi$ -torsion free, and  $J$  is finitely generated since  $C\langle X_i \rangle$  is noetherian, hence  $\Sigma_{\text{dR}}$  (resp.  $\Sigma_{\text{dR}}(n)$ ) is flat over  $B_{\text{dR}}^+$ .<sup>12</sup> One then checks that the derived quotient  $\Sigma_{\text{dR}}(\bullet)/\xi$  is isomorphic to  $\Sigma_C(\bullet)$ , where  $\Sigma_C(n)$  is the  $J(n)$ -adic completion of  $C\langle X_i \rangle^{\widehat{\otimes}(n+1)}$ . This agrees with the completion with respect to  $\ker(C\langle X_i \rangle^{\widehat{\otimes}(n+1)} \twoheadrightarrow A)$ , so the complex  $\Sigma_C(\bullet)$  computes  $R\Gamma_{\text{inf}}(A/C)$ . The lemma hence follows.  $\square$

<sup>10</sup>Equivalently, we may take the derived pushforward of  $\mathcal{O}_{\text{Inf}}$  along the map of topoi

$$\text{Shv}(\text{Inf}(A/B_{\text{dR}}^+)^{\text{op},\circ}) \rightarrow \text{Shv}(\text{Aff}(B_{\text{dR}}^+)^{\text{op},\circ})$$

and then take the (derived) global sections there. Here the map is induced by the obvious cocontinuous map from  $\text{Inf}(A/B_{\text{dR}}^+)$  to the big affine ‘‘Zariski’’ site  $\text{Aff}(B_{\text{dR}}^+)^{\text{op},\circ}$  with indiscrete topology, where objects there are simply ring homomorphisms  $B_{\text{dR}}^+ \rightarrow S$ .

<sup>11</sup>This is justified in the proof of the lemma below.

<sup>12</sup>In particular  $\Sigma_{\text{dR}}$  is  $\xi$ -complete and  $\xi$ -torsion free. This also shows that  $R\Gamma_{\text{inf}}(A/B_{\text{dR}}^+)$  is derived  $\xi$ -adically complete.

The follow Proposition compares the infinitesimal cohomology with de Rham cohomology. The proof is similar to the proof of [BdJ11].

**Proposition 5.5** *Let  $A$  be a petit Tate algebra over  $C$ . There is a natural quasi-isomorphism between*

$$R\Gamma_{\text{inf}}(A/C) \cong \Omega_{A/C}^{\bullet}.$$

*In particular,  $R\Gamma_{\text{inf}}(X/C) \cong R\Gamma_{\text{dR}}(X/C)$ , and therefore we have*

$$R\Gamma_{\text{inf}}(X/B_{\text{dR}}^+) \otimes_{B_{\text{dR}}^+}^{\mathbb{L}} C \xrightarrow{\sim} R\Gamma_{\text{dR}}(X/C).$$

*Proof.* As in the proof of the previous lemma, choose  $C\langle X_i \rangle \rightarrow A$  and consider the complex  $(\Sigma \rightarrow \Sigma(1) \rightarrow \Sigma(2) \rightarrow \dots)$  which computes  $R\Gamma_{\text{inf}}(A/C)$ , here we use  $\Sigma$  for  $\Sigma_C$  to simplify notation. We will make a more refined choice of such a surjection later. Now consider the double complex

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \Omega_{\Sigma}^2 & \longrightarrow & \Omega_{\Sigma(1)}^2 & \longrightarrow & \Omega_{\Sigma(2)}^2 & \longrightarrow & \dots \\ d\uparrow & & d\uparrow & & d\uparrow & & \\ \Omega_{\Sigma}^1 & \longrightarrow & \Omega_{\Sigma(1)}^1 & \longrightarrow & \Omega_{\Sigma(2)}^1 & \longrightarrow & \dots \\ d\uparrow & & d\uparrow & & d\uparrow & & \\ \Sigma & \longrightarrow & \Sigma(1) & \longrightarrow & \Sigma(2) & \longrightarrow & \dots \end{array}$$

where  $\Omega_{\Sigma(n)}^i$  denotes the  $J(n)$ -adic completed differentials. For  $i \geq 1$ , each row  $\Omega_{\Sigma}^i \rightarrow \Omega_{\Sigma(1)}^i \rightarrow \Omega_{\Sigma(2)}^i \rightarrow \dots$  is homotopic to 0 (by the [dJea] 07L9, and note that being homotopic to 0 is preserved under term-wise  $p$ -completions and  $J(\bullet)$ -completions). Moreover, for each of the maps  $\Sigma \rightarrow \Sigma(n)$ , the induced morphism  $\Omega_{\Sigma}^{\bullet} \rightarrow \Omega_{\Sigma(n)}^{\bullet}$  is a quasi-isomorphism. This follows from the same proof of Lemma 2.13 of [BdJ11], using the formal version of the Poincaré lemma (instead of the PD Poincaré lemma), and using the observation that each  $\Sigma \rightarrow \Sigma(n)$  induces an isomorphism between  $\Sigma(n)$  and a formal power series  $\Sigma[[t_i]]$  over  $\Sigma$ . Now by comparing the two filtration spectral sequences associated to the double complex above, we have a natural quasi-isomorphism  $R\Gamma_{\text{inf}}(A/C) \cong \Omega_{\Sigma/C}^{\bullet}$ .

It remains to compare this with  $\Omega_{A/C}^{\bullet}$ . For this we make the choice of the map  $C\langle X_i \rangle \rightarrow A$  as follows. By definition (of petit Tate algebras) there exists an étale map  $f : C\langle T_i \rangle \rightarrow A$ , we enlarge the source by adding (possibly infinitely many) variables  $Y_j$  to obtain a surjection  $g$  as below.

$$\begin{array}{ccc} C\langle T_i \rangle & \longrightarrow & C\langle T_i, Y_j \rangle \\ f\downarrow & \swarrow g & \\ A & & \end{array}$$

Write  $\beta_j = g(Y_j) \in A$ . Let  $\Sigma$  be the  $\ker(g)$ -completion of  $C\langle T_i, Y_j \rangle$ . Since  $\Sigma$  is a pro-nilpotent thickening of  $A$ , and  $C\langle T_i \rangle$  is étale (in the sense of [Hub96] 1.5.1), the map  $C\langle T_i \rangle \rightarrow \Sigma$  canonically lifts to a section  $A \rightarrow \Sigma$ . This induces a map  $A[Y_j] \rightarrow \Sigma$ , and thus induces a map  $g : A[[Y_j - \beta_j]] \rightarrow \Sigma$  by passing to completions. Now these maps fit into the following commutative diagram (with

only solid arrows)

$$\begin{array}{ccc}
 C\langle X_i, Y_j \rangle & & \\
 \downarrow & \searrow \tilde{f} & \\
 \Sigma & \dashrightarrow & A[[Y_j - \beta_j]] \\
 & \searrow & \downarrow \\
 & & A
 \end{array}$$

where  $\tilde{f}$  is induced by  $f$ . Therefore by the universal property there exists a map  $g' : \Sigma \rightarrow A[[Y_j]]$  (the dashed arrow) such that  $g \circ g'$  is the identity on  $\Sigma$ . Note that  $g' \circ g$  is also identity by considering the image of  $Y_j$ , so we have  $\Sigma \cong A[[Y_j - \beta_j]]$ , hence  $\Omega_{\Sigma/C}^\bullet \xrightarrow{\sim} \Omega_{A/C}^\bullet$ .  $\square$

### 5.3 Comparison with $A_{\text{inf}}$ - and étale cohomology

Next we return to the setup of the introduction, and compare the infinitesimal cohomology of  $X$  with the crystalline cohomology over  $A_{\text{cris}}$  (and hence the  $A_{\text{inf}}$ -cohomology). We use this to obtain a  $B_{\text{dR}}^+$ -lattice in  $H_{\text{ét}}^i(X, \mathbb{Z}_p) \otimes B_{\text{dR}}$ .

**Lemma 5.6** *Let  $\mathfrak{X}$  be a smooth formal scheme over  $\text{Spf } \mathcal{O}_C$ . There is a functorial isomorphism*

$$b : R\Gamma_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_C/p} / A_{\text{cris}}) \widehat{\otimes}^{\mathbb{L}} B_{\text{dR}}^+ \xrightarrow{\sim} R\Gamma_{\text{inf}}(X / B_{\text{dR}}^+).$$

*Proof.* Note that  $R\Gamma_{\text{crys}}(\mathfrak{X} / A_{\text{cris}}) \cong R\Gamma_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_C/p} / A_{\text{cris}})$ . It suffices to restrict to affine opens  $\text{Spf } R$  in  $\mathfrak{X}_{\text{ét}}$ , and construct a map  $R\Gamma_{\text{crys}}(R / A_{\text{cris}}) \rightarrow R\Gamma_{\text{inf}}(R[\frac{1}{p}] / B_{\text{dR}}^+)$ . For this we consider the following continuous functor

$$\mu : \text{CRIS}(R / A_{\text{cris}})^{\text{op}, \circ} \rightarrow \text{Inf}(R[\frac{1}{p}] / B_{\text{dR}}^+)^{\text{op}, \circ}$$

from the (big) indiscrete crystalline site to the indiscrete infinitesimal site, which sends  $(B \rightarrow B/J) \mapsto (B \rightarrow (B/J)[\frac{1}{p}])$ . Here  $\tilde{B}$  is the completion of  $B' := \varprojlim (B \otimes_{A_{\text{cris}}} B_{\text{dR}, m}^+)$  with respect to  $\ker(B' \rightarrow B/J[\frac{1}{p}])$ . The functor  $\mu$  preserves fibre products and equalizers (both exist in the big crystalline site), hence the pullback functor  $\mu_s$  is exact<sup>13</sup>. Thus  $\mu$  induces a map  $\text{Shv}(\text{CRIS}(R / A_{\text{cris}})^{\text{op}, \circ}) \rightarrow \text{Shv}(\text{Inf}(R[\frac{1}{p}] / B_{\text{dR}}^+)^{\text{op}, \circ})$  on topoi. The desired map  $b$  follows from base change. To show that  $b : R\Gamma_{\text{crys}}(\mathfrak{X} / A_{\text{cris}}) \widehat{\otimes}^{\mathbb{L}} B_{\text{dR}}^+ \xrightarrow{\sim} R\Gamma_{\text{inf}}(X / B_{\text{dR}}^+)$  is an isomorphism, it suffices to do so after reducing mod  $\xi$  as both sides are derived  $\xi$ -complete. After mod  $\xi$ , the map constructed above becomes  $R\Gamma_{\text{dR}}(\mathfrak{X})[\frac{1}{p}] \xrightarrow{\sim} R\Gamma_{\text{dR}}(X)$  by the de Rham comparison.  $\square$

In particular, in this setup we obtain a comparison between  $A_{\text{inf}}$ -cohomology and the infinitesimal cohomology:

**Corollary 5.7** *Now assume that  $\mathfrak{X}$  is smooth proper over  $\text{Spf } \mathcal{O}_C$ . Then we have a functorial isomorphism*

$$R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}} B_{\text{dR}}^+ \xrightarrow{\sim} R\Gamma_{\text{inf}}(X / B_{\text{dR}}^+).$$

*In particular, on cohomology groups we have  $H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes B_{\text{dR}}^+ \cong H_{\text{inf}}^i(X / B_{\text{dR}}^+)$ .*

*Proof.* We combine Lemma 4.5 and Lemma 5.6 to obtain the composition  $b \circ (h_{\text{crys}} \otimes B_{\text{dR}}^+)^{-1}$ . The rest follows from the fact that  $H_{A_{\text{inf}}}^i(\mathfrak{X})$  is a Breuil–Kisin–Fargues module.  $\square$

<sup>13</sup>see Tag 00WX, 00X4 and 00X5 of [dJea].

**Corollary 5.8** *When  $X = \mathfrak{X}_C^{\text{ad}}$  is the generic fiber of  $\mathfrak{X}$ , the isomorphism above plus the étale comparison gives us a canonical isomorphism*

$$H_{\text{inf}}^i(X/B_{\text{dR}}^+) \otimes B_{\text{dR}} \cong H_{\text{ét}}^i(X, \mathbb{Z}_p) \otimes B_{\text{dR}}.$$

As a consequence we can reprove the following theorem of [BMS18].

**Theorem 5.9** *Suppose that  $H_{\text{crys}}^i(\mathfrak{X}_k/W(k))$  is torsion free, then we can recover  $H_{A_{\text{inf}}}^i(\mathfrak{X})$  with its  $\varphi$ -action from the generic fiber  $X$ , more precisely from  $H_{\text{ét}}^i(X, \mathbb{Z}_p)$  and  $H_{\text{inf}}^i(X/B_{\text{dR}}^+)$ . If moreover  $H_{\text{crys}}^{i+1}(\mathfrak{X}_k/W(k))$  is torsion free, then we can further recover the integral crystalline cohomology  $H_{\text{crys}}^i(\mathfrak{X}_k/W(k))$  with its  $\varphi$ -action.*

*Proof.* If  $H_{\text{crys}}^i(\mathfrak{X}_k/W(k))$  is torsion free, then by Remark 4.11  $H_{A_{\text{inf}}}^i(\mathfrak{X})$  is a free Breuil–Kisin–Fargues module, thus by Theorem 4.2 it is determined by the pair  $(T, \Xi)$ , where  $T = (H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes W(C^b))^{\varphi=1} \cong H_{\text{ét}}^i(X, \mathbb{Z}_p)$ , and  $\Xi = H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes B_{\text{dR}}^+ \cong H_{\text{inf}}^i(X/B_{\text{dR}}^+)$  by Corollary 5.7. The first claim hence follows. Now suppose in addition that  $H_{\text{crys}}^{i+1}(\mathfrak{X}_k/W(k))$  is torsion free, then by Lemma 4.8 we have a  $\varphi$ -equivariant isomorphism  $H_{\text{crys}}^i(\mathfrak{X}_k/W(k)) \cong H_{A_{\text{inf}}}^i(\mathfrak{X}) \otimes W(k)$ , hence the second claim follows from the first.  $\square$

**Remark 5.10** *As explained in [BMS18], by Kisin’s construction (see Proposition 4.34 in loc.cit.) the theorem above implies the version with Galois actions (when  $\mathfrak{X}$  is defined over  $\mathcal{O}_K$  for some discretely valued extension  $K/\mathbb{Q}_p$ ), namely part (2) of Corollary 10.*

#### 5.4 Compatibility of $h_{\text{crys}}$ with filtration

We first construct a map

$$h_{\text{dR}} : R\Gamma_{\text{Inf}}(X/B_{\text{dR}}^+) \longrightarrow R\Gamma_{\text{proét}}(X, \mathbb{B}_{\text{dR}}^+).$$

The construction is analogous to that of  $h_{\text{crys}}$ , this time making use of pro-étale descent instead of flat/quasisyntomic descent.

**Construction 5.11** *Let  $X$  be a smooth rigid analytic variety over  $C$ . It suffices to construct  $h_{\text{dR}} : R\Gamma_{\text{Inf}}(A/B_{\text{dR}}^+) \rightarrow R\Gamma_{\text{proét}}(\text{Spa}(A, A^\circ), \mathbb{B}_{\text{dR}}^+)$  for petit affinoid Tate algebras  $A$ . Now as affinoid perfectoids form a basis for  $\text{Spa}(A, A^\circ)_{\text{proét}}$ , it suffices to construct a functorial map from  $R\Gamma_{\text{Inf}}(A/B_{\text{dR}}^+)$  to  $\mathbb{B}_{\text{dR}}^+(S, S^+)$ , for all  $U \rightarrow \text{Spa}(A, A^\circ)_{\text{proét}}$  affinoid perfectoid with  $\hat{U} = \text{Spa}(S, S^+)$  (following notation in [Sch13], see also Theorem 6.5 in loc.cit.). For this we observe that  $\mathbb{B}_{\text{dR}}^+(S, S^+) \twoheadrightarrow S$  is an object in  $\text{Inf}(A/B_{\text{dR}}^+)$ , and we get a functorial map  $\Gamma_{\text{Inf}}(A/B_{\text{dR}}^+) \rightarrow \mathbb{B}_{\text{dR}}^+(S, S^+)$  as in the construction of  $h_{\text{crys}}$ .*

Our final goal is to prove the filtration compatibility of  $h_{\text{crys}} \otimes B_{\text{dR}}$ . In order to do this, we first relate  $h_{\text{crys}} \otimes B_{\text{dR}}$  to the map  $h_{\text{dR}}$  constructed above. Then, in the case when  $\mathfrak{X}$  is base changed from  $\mathfrak{X}_{\mathcal{O}_K}$ , we relate the latter map ( $h_{\text{dR}}$ ) to a map from  $R\Gamma_{\text{proét}}(X_{\overline{K}}, \Omega_{X_{\overline{K}}}^\bullet) \otimes^{\mathbb{L}} B_{\text{dR}}^+$  to  $R\Gamma_{\text{proét}}(X, \mathbb{B}_{\text{dR}}^+)$  constructed in [Sch13], which is known to be a filtered isomorphism there.

**Lemma 5.12** *Let  $\mathfrak{X}$  be a smooth proper formal scheme over  $\mathcal{O}_C$  with generic fiber  $X$ , then we have the following commutative diagram*

$$\begin{array}{ccccc} R\Gamma_{\text{crys}}(\mathfrak{X}/A_{\text{cris}}) \otimes^{\mathbb{L}} B_{\text{dR}}^+ & \longrightarrow & R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}} B_{\text{dR}}^+ & \longrightarrow & R\Gamma_{A_{\text{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}} B_{\text{dR}} \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_{\text{Inf}}(X/B_{\text{dR}}^+) & \longrightarrow & R\Gamma_{\text{proét}}(X, \mathbb{B}_{\text{dR}}^+) & \longrightarrow & R\Gamma_{\text{proét}}(X, \mathbb{B}_{\text{dR}}) \end{array}$$

where the left vertical map comes from the base change isomorphism.

*Proof.* It suffices to replace  $\mathfrak{X}$  by  $\text{Spf } R$  where  $R$  is the  $p$ -completion of a smooth  $\mathcal{O}_C$ -algebra. Moreover, by functoriality we further reduce to show the following: for a map  $R \rightarrow S$  from  $R$  to a perfectoid  $\mathcal{O}_C$ -algebra  $S$  (the  $S$  we have in mind is pro-étale but not necessarily quasisyntomic over  $R$ ), the following diagram commutes.

$$\begin{array}{ccc} R\Gamma_{\text{crys}}(R/A_{\text{cris}}) & \xrightarrow{h_{\text{crys}}} & A\Omega_R \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} \\ & \searrow & \downarrow \\ & & A\Omega_S \widehat{\otimes}^{\mathbb{L}} A_{\text{cris}} = A_{\text{cris}}(S) \end{array}$$

The vertical map comes from the natural map  $A\Omega_R \rightarrow A\Omega_S$ , while the map  $R\Gamma_{\text{crys}}(R/A_{\text{cris}}) \rightarrow A_{\text{cris}}(S)$  is induced from the crystalline site, by viewing the PD-thickening  $A_{\text{cris}}(S) \rightarrow S$  as an object in the crystalline site  $\text{CRIS}(R/A_{\text{cris}})^{\text{op}}$ . The commutativity of the triangle is clear.  $\square$

Now we return the setup in Subsection 4.3, and let  $\mathfrak{X}$  be the base change of  $\mathfrak{X}_{\mathcal{O}_K}$  to  $\mathcal{O}_C$ . For a petit Tate algebra  $A_K$  over  $K$ , we define  $\text{Inf}(A_K/K)$  similarly as  $\text{Inf}(A/C)$ , and by the same proof of Proposition 5.5 (replacing  $C$  by  $K$ ) we have a functorial isomorphism  $R\Gamma_{\text{inf}}(X_K/K) \cong R\Gamma_{\text{dR}}(X_K)$ . The base change morphism for the indiscrete infinitesimal sites gives rise to a map

$$R\Gamma_{\text{inf}}(X_K/K) \otimes_K B_{\text{dR}}^+ \longrightarrow R\Gamma_{\text{inf}}(X_C/B_{\text{dR}}^+),$$

which is an isomorphism (again by considering its derived quotient mod  $\xi$ ). Now we compose this isomorphism with  $h_{\text{dR}}$ , and observe that from Lemma 5.12, in order to prove part (1) of Corollary 10 (the filtration compatibility of the  $B_{\text{cris}}$  comparison), it suffices to show that the following composition

$$R\Gamma_{\text{dR}}(X_K) \otimes_K B_{\text{dR}} \xrightarrow{\sim} R\Gamma_{\text{inf}}(X_C/B_{\text{dR}}^+) \otimes B_{\text{dR}} \xrightarrow{h_{\text{dR}}} R\Gamma_{\text{proét}}(X_C, \mathbb{B}_{\text{dR}})$$

is a filtered isomorphism. This follows from the following lemma (making use of Theorem 7.11 of [Sch13]).

**Lemma 5.13** *Retain the setup from above and adopt the notation from [Sch13]. Then the following diagram commutes*

$$\begin{array}{ccccc} R\Gamma_{\text{inf}}(X_K/K) & \longrightarrow & R\Gamma_{\text{inf}}(X_C/B_{\text{dR}}^+) & \xrightarrow{h_{\text{dR}}} & R\Gamma(X_C, \mathbb{B}_{\text{dR}}^+) \\ \downarrow \wr & & & & \parallel \\ R\Gamma(X_K, \Omega_{X_K}^\bullet) & \longrightarrow & R\Gamma(X_C, \mathcal{O}_{\mathbb{B}_{\text{dR}}^+} \otimes_{\mathcal{O}_{X_C}} \Omega_{X_C}^\bullet) & \xrightarrow{\sim} & R\Gamma(X_C, \mathbb{B}_{\text{dR}}^+) \end{array}$$

where  $\Omega_{X_K}^\bullet$  is the pullback of  $\Omega_{X_K, \text{ét}}^\bullet$  to the pro-étale site (left vertical isomorphism reflects the fact that the de Rham cohomology can be computed in the pro-étale site). The bottom arrows are constructed in [Sch13].

*Proof.* It suffices to assume that  $X_K = \text{Spa}(R_K, R_K^\circ)$  for a very small Tate algebra  $R_K$  over  $K$ . Recall that  $R_K$  is very small if there are enough units  $u_i, \beta_j \in (R_K^\circ)^\times$  that induces an étale map  $K\langle T_i^\pm \rangle \rightarrow R_K$  and a surjective map  $K\langle T_i^\pm, Y_j^\pm \rangle \twoheadrightarrow R_K$ , with  $T_i \mapsto u_i, Y_j \mapsto \beta_j$  (compare with petit Tate algebras). The pro-étale cover  $K\langle T_i^\pm \rangle \rightarrow K\langle T_i^{\pm 1/p^\infty} \rangle$  induces a pro-étale cover  $R_K \rightarrow R_{K, \infty}$  (resp. a pro-étale cover  $R_C \rightarrow R_\infty$  via base change). It suffices to check the

commutativity of the diagram after passing to the pro-étale cover for the horizontal arrows on the bottom of the diagram. In other words, we want to show that the following diagram commutes

$$\begin{array}{ccccc}
 R\Gamma_{\text{inf}}(R_K/K) & \longrightarrow & R\Gamma_{\text{inf}}(X_C/\mathbb{B}_{\text{dR}}^+) & \xrightarrow{h_{\text{dR},\infty}} & \mathbb{B}_{\text{dR}}^+(R_\infty) \\
 \parallel & & & & \parallel \\
 \Omega_{R_K}^\bullet & \longrightarrow & \Omega_{R_\infty}^\bullet \longrightarrow \mathcal{O}\mathbb{B}_{\text{dR}}^+(R_\infty) \otimes_{R_\infty} \Omega_{R_\infty}^\bullet & \xrightarrow{\sim} & \mathbb{B}_{\text{dR}}^+(R_\infty)
 \end{array}$$

where  $h_{\text{dR},\infty}$  is constructed similarly as  $h_{\text{dR}}$ , by viewing  $\mathbb{B}_{\text{dR}}^+(R_\infty) \rightarrow R_\infty$  as a pro-nilpotent thickening, and the last isomorphism in the bottom arrow comes from  $\mathbb{B}_{\text{dR}}^+$ -Poincaré Lemma of Scholze (Corollary 6.13 in *loc.cit.*).

As in the proof of Proposition 5.5 (replacing  $K\langle T_i, Y_j \rangle$  by  $K\langle T_i^\pm, Y_j^\pm \rangle$ ), let  $\Sigma_K$  be the completion of  $K\langle T_i^\pm, Y_j^\pm \rangle$  along the kernel of its projection to  $R_K$ . Again we have an isomorphism  $\Sigma_K \xrightarrow{\sim} R_K[[Y_j - \beta_j]]$  (note that there is indeed a map from  $K\langle T_i^\pm, Y_j^\pm \rangle \rightarrow R_K[[Y_j - \beta_j]]$  because  $Y_j$  is invertible in the latter power series ring, as each  $\beta_j$  is a unit). This describes the map  $\Omega_{R_K}^\bullet \xrightarrow{\sim} \Omega_{R_K[[Y_j - \beta_j]]}^\bullet \cong R\Gamma_{\text{inf}}(R_K/K)$  on the left. The top horizontal arrow is given by lifting the map  $K\langle T_i^\pm, Y_j^\pm \rangle \rightarrow R_K \rightarrow R_\infty$  along  $\mathbb{B}_{\text{dR}}^+(R_\infty) \rightarrow R_\infty$ , sending  $T_i$  to  $[T_i^\flat] \in \mathbb{A}_{\text{inf}}(R_\infty)$ , where  $T_i^\flat = (T_i, T_i^{1/p}, T_i^{1/p^2}, \dots) \in R_\infty^\flat$ . This in particular determines a unique lift  $R_K \rightarrow \mathbb{B}_{\text{dR}}^+(R_\infty)$ . Now unwinding definitions, the top horizontal arrows composed with the left vertical isomorphism can be described by  $\Omega_{R_K}^\bullet \rightarrow R_K \rightarrow \mathbb{B}_{\text{dR}}^+(R_\infty)$ . The bottom horizontal map factors through

$$R_K \longrightarrow \mathcal{O}\mathbb{B}_{\text{dR}}^+(R_\infty) \xrightarrow{\sim} \mathbb{B}_{\text{dR}}^+(R_\infty)[[X_i]] \xrightarrow{\text{proj}} \mathbb{B}_{\text{dR}}^+(R_\infty),$$

where the isomorphism in the middle is described by Proposition 6.10 in *loc.cit.*, which sends  $T_i \otimes 1 - 1 \otimes [T_i^\flat] \mapsto X_i$ , and the projection map induces a quasi-isomorphism on the de Rham complexes. Now the commutativity of the previous diagram follows from the fact that both maps agree on  $T_i$ : as the bottom morphism sends  $T_i \mapsto (X_i + 1 \otimes [T_i^\flat]) \mapsto [T_i^\flat]$ . This concludes the lemma.  $\square$

Finally, this finishes the proof of the  $B_{\text{cris}}$  comparison, which we restate.

**Theorem 5.14** *Suppose that  $\mathfrak{X}_{\mathcal{O}_K}$  is a smooth proper formal scheme over  $\mathcal{O}_K$ , with  $K$  a discrete  $p$ -adic field as before, with residue field  $k_0$ . Let  $C = C_K$ . Then  $h_{\text{cris}}$  induces an isomorphism*

$$H_{\text{cris}}^i(\mathfrak{X}_{k_0}/W(k_0)) \otimes_{W(k_0)} B_{\text{cris}} \xrightarrow{\sim} H_{\text{ét}}^i(\mathfrak{X}_C^{\text{ad}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}$$

*compatible with the  $\text{Gal}_K$ -action, Frobenius, and filtration.*

## REFERENCES

- BdJ11 B. Bhatt and A. J. de Jong. Crystalline cohomology and de Rham cohomology. *Preprint.*, 2011.
- Bha12 B. Bhatt.  $p$ -adic derived de Rham cohomology. *Preprint.*, 2012.
- Bha17 B. Bhatt. Specializing varieties and their cohomology from characteristic 0 to characteristic  $p$ . *Preprint.*, 2017.
- BMS18 B. Bhatt, M. Morrow, and P. Scholze. Integral  $p$ -adic Hodge Theory. *Publications Mathématiques de l’IHÉS*, 128, 2018.
- BMS19 B. Bhatt, M. Morrow, and P. Scholze. Topological Hochschild homology and integral  $p$ -adic Hodge theory. *Publications Mathématiques de l’IHÉS*, 129, 2019.

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- BS19 B. Bhatt and P. Scholze. Prisms and prismatic cohomology. *Preprint.*, 2019.
- dJea A. J. de Jong et al. Stacks project.
- Fal02 G. Faltings. Almost etale extensions. *Asterisque*, 2002.
- Hub96 R. Huber. *Etale cohomology of rigid analytic varieties and adic spaces*. Springer, 1996.
- Mor16 M. Morrow. Notes on the  $\mathbb{A}_{\text{inf}}$ -cohomology of integral  $p$ -adic Hodge theory. *Preprint.*, 2016.
- Sch13 P. Scholze.  $p$ -adic hodge theory for rigid analytic varieties. *Forum Math. Pi*, 2013.

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