TATE CYCLES AND MOTIVATED CYCLES OVER ABELIAN VARIETIES IN CHARACTERISTIC $p > 0$

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Abstract Tates conjecture for abelian varieties over finite fields has been proven by Tate himself for $H^2$ but remains open in higher degree. We prove a weaker version of the conjecture in any degree, which describes every Tate cycle ‘in terms of’ algebraic cycles. This variant also applies to abelian varieties $A$ over finitely generated fields of characteristic $p$, at least in the ‘good reduction’ case, i.e. when $A$ is the generic fibre of an abelian scheme on a normal projective variety defined over a finite field.

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0. Introduction

0.1. The theory of pure motives envisaged by Grothendieck depend on the standard conjectures on the algebraic cycles. Given the stagnant state of these conjectures, several attempts have been proposed to circumvent them. The most recent are [2] and [5]: these are in a sense two minimal modifications of the theory, one by excess and the other by default (cf. [4, Chapter 9]).

The point of view of [2] is to formally add to morphisms of the category of motives the inverses of morphisms whose Lefschetz theorem says that they induce isomorphisms in cohomology. This modification gives rise, over a field of characteristic zero, to a semi-simple Tannakian category of motives and to the battery expected of its realisations.

The morphisms of this category, baptized “motivic correspondences”, have in addition the remarkable property of being deformed by “parallel transport”. This allows us to attack certain cases of the Hodge conjecture, leave to weaken the statement by replacing the algebraic cycles by motivated cycles.

We prove also in [2] that every Hodge cycle on an abelian variety is motivated.

0.2. The objective of this article is to show an analogous result for the abelian varieties over finite fields. Recall that a Tate cycle is a $\ell$-adic cohomological class invariant under the

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action of the absolute Galois group of the base field (suppose of finite type over its prime subfield, of characteristic distinct from $\ell$).

**Theorem 0.2.1.** Every Tate cycle over an abelian variety over a finite field is a $\mathbb{Q}_\ell$-linear combination of motivated cycles.

This is a weakened variant of the Tate conjecture (motivated cycles replacing algebraic cycles). In §6, we extend this result to *abelian varieties over a field of finite type over a finite field*, at least *in the case of good reduction at every prime of height 1.*

The method of proof of Theorem 0.2.1. is strongly inspired by the work [18] of Milne, where he proves that the Tate conjecture for abelian varieties over finite fields would result from the Hodge conjecture for the abelian varieties of CM type in characteristic zero. The fact that every Hodge cycle on an abelian variety is motivated thus suggests combining [18] and [2] to obtain the theorem 0.2.1.

0.3. To avoid any misinterpretation of Theorem 0.2.1, recall that a *motivated cycle* on a projective lisse variety $X$ is a cohomology class of the form $pr_{X'}^X(\alpha \cup \ast_{L}(\beta))$, where $X'$ is an “auxilliary” projective lisse variety, $\alpha$ and $\beta$ are classes of algebraic cycles on the product $X \times X'$, and $\ast_{L}$ is the *inverse* of the Lefschetz isomorphism induced by the cup products iterated with the cohomology class of a hyperplane section of $X \times X'$ in the Segre product of projective embeddings of $X$ and of a projective embedding of $X'$.

Although we know for a long time that on an abelian variety, the inverse of the Lefschetz isomorphism is given by an algebraic correspondence (Lieberman), the theorem 0.2.1. does not imply the Tate conjecture for abelian varieties over finite fields, because the motivated cycles in play involve certain auxilliary varieties $X'$ which are not abelian varieties.

We show that we can limit ourselves to auxilliary varieties which are fibered in abelian varieties over a connected lisse projective curve (we will show in fact a much more precise result). We deduce from the theorem 0.2.1 the following corollary.

**Corollary 0.3.1 (cf. Remark 5.2.4).** If the standard conjecture of Lefschetz type is true for every variety fibered in abelian varieties over a connected lisse projective curve over $\overline{\mathbb{Q}}$ or good over $\overline{\mathbb{F}}_p$,

\[1\]  
thus the Tate conjecture is true for every abelian variety over a finite field of characteristic $p$.

0.4. The main technical point is the proof of the fact that a motivated cycle on an abelian variety $A$ of CM type over $\overline{\mathbb{Q}}_p$ specializes to a motivated cycle on the reduction of $A$. Here again, the difficulty comes from the bad reduction of auxilliary varieties (fibered over abelian varieties over connected smooth projective curves). To treat this point, we appeal to the techniques of deformation and specialization of motivated cycles of [3], as well as a detailed study of the infinite behavior of certain models of “unitary” Shimura varieties in mixed characteristics.

\[1\] It’s in this second alternative that the result is new; in the first one, it is deduced from the result of [2] cited below as Theorem 1.2.1, together with the main result of [18].
0.5. This point being established, the strategy is roughly speaking, relying on a Tannakian calculation of Milne (cf. 4.7.1), to show that the Tate cycles on abelian varieties over a finite field $k$ is obtained, by linear combination and cup product, from divisor classes on the one hand, and of specializations of motivated cycles over abelian varieties of CM type over $\overline{\mathbb{Q}}$ on the other hand.

We also show that the category of motives—cutout on abelian varieties over $k$ is \emph{abelian semi-simple} (Proposition 5.2.1). It is likewise Tannakian, but the question to know if the field of endomorphisms of the unit objects is reduced to $\mathbb{Q}$ remains open (cf. Remark 5.2.3).

0.6. To extend the theorem 0.2.1 to the case of an abelian variety $A$ over a field of finite type over a finite field in the case of good reduction (i.e. in the case where $A$ is the generic fibre of an abelian scheme over a normal projective base over a finite field), we use the study done in [3] of the variation of motivic Galois group in a family.

1. Hodge cycles and motivated cycles on abelian varieties

Let $k$ be a field.

**Notation 1.1.1.** We denote $\mathcal{V}_k$ as the smallest full sub-category of the category of lisse projective $k$-schemes which is stable under sum, product, passing to connected components, and that contains the abelian varieties over $k$ such that the total space $X$ of every abelian scheme $X \to S$, where $S$ is a geometrically connected lisse projective curve over $k$.

1.2. In [2, 6.3], we demonstrated the following theorem.

**Theorem 1.2.1.** Every Hodge cycle on a complex abelian variety is motivated, modeled over $\mathcal{V}_\mathbb{C}$.

This result implies (cf. 1.5, p.9 and Proposition 2.5.2 of [2]) that of Deligne [8] (every Hodge cycle over an abelian variety is absolute Hodge) as well as the p-adic complements by Blasius, Ogus and Wintenberger.

For the needs of the future, we will recall the principle of the proof, with some supplements. The plan of proof follows the line of [8]. We proceed in three stages.

(1) Reduction to the case of abelian varieties of CM type. Recall that an abelian variety $A$ over a field is said to be \emph{of CM type} if $\text{End} A \otimes \mathbb{Q}$ contains a semi-simple commutative sub-$\mathbb{Q}$-algebra of dimension $2 \dim A$ over $\mathbb{Q}$.

(2) Reduction to the case where the Hodge cycle is a “Weil cycle”.

(3) Reduction to the case where the abelian variety is powers of an elliptic curve of CM type.

The steps (1) and (3) are based on the particular case following from the deformation theorem of motivated cycles of [2,0.5].

**Proposition 1.2.2.** Let $S$ be a projective variety connected and lisse over $\mathbb{C}$, and $f : X \to S$ a lisse projective morphism. Let $s$ and $t$ be two points of $S$.

Let $\mathcal{V}$ be a full sub-category of the category of projective lisse varieties, stable under sum, product, passing to connected components, and containing the fibres of $f$ as well as $X$.

Let $q$ be a natural integer and $\xi$ a section $R^q f_* \mathbb{Q}(q)$ over the analytification of $S$. Then the following conditions are equivalent:

(i) There exists $s \in S(\mathbb{C})$ such that $\xi_s \in H^q(X_s, \mathbb{Q})(q)$ is a motivated cycle on $X_s$ modeled over $\mathcal{V}$,
(ii) For every $t \in S(\mathbb{C})$, $\xi_t$ is a motivated cycle on $X_t$ modeled on $V$.

(iii) $\xi$ comes from a motivated cycle on $X$ modeled over $V$.

A completely different, and more effective, proof is given in [3, 7.2.1].

1.3. First step [2, 6.3.1]

(This will not serve us later.) We start from an abelian variety $A$ and a Hodge cycle $\theta \in H^{2q}(A, \mathbb{Q})(q)$. If $q = 1$, the Lefschetz theorem shows that $\theta$ is algebraic. We will thus suppose in the following that $q > 1$.

In this step, we construct a compact abelian pencil $f : X \to S$, whose fibre $X_t$ is isogenous to $A \times A$, another $X_s$ with complex multiplication, and the inverse image of $(\theta, \theta)$ on $X_t$ is invariant under the monodromy, so is fibre at $t$ of a section $\xi$ of $R^{2q}f_*\mathbb{Q}(q)$. The application of the proposition 1.2.2 to this pencil brings us back to show that every Hodge cycle on $X_s$ is motivated modeled over $V$.

The idea of the construction is to consider the family of Hodge type associated to the special Mumford-Tate group $G^1$ of $A$ (viewed as subgroup of group of symplectic similitudes), and with fixed level structure $\nu \geq 3$; on the connected Shimura variety which parametrize this family, we then trace a curve going through the point $t$ corresponding to $A$ and through a CM point, i.e. a point corresponding to an abelian variety of CM type (it exists according to [22]). To obtain a compact curve $S$, it is necessary to place oneself in the case where the boundary of the Shimura variety in its minimal compactification (Baily-Borel-Satake) is of codimension at least 2. We do it by choosing an auxiliary real quadratic field $L^+$, and by replacing the family of Hodge type associated to $G^1$ by that associated to $R_{L^+/\mathbb{Q}}(G^1 \otimes QL^+)$, where $R_{L^+/\mathbb{Q}}$ designates the Weil scalar restriction.

1.4. Second step [2, 6.3.2]

Let’s start with brief reminders on “Weil cycles” (cf. [1, 8, 21, 29]). Let $E$ be a CM field, i.e. a totally imaginary extension of a totally real number field $E^+$. A Hodge $\mathbb{Q}$-structure $V$ of even dimension $d = 2q$ and of type $(1, 0) + (0, 1)$, equipped with an action of $E$, is said of Weil if there exists an $E$-Hermitian form $\langle \cdot, \cdot \rangle$ on the $E$-space underlying $V$ admitting a totally isotropic subspace of dimension $q$, and if there exists a totally imaginary element $\epsilon$ of $E$ such that

$$(v, w) = \text{Tr}_{E/\mathbb{Q}} \epsilon(v, w)$$

polarizes $V$. According to Landherr (cf. [25, 26]), this condition determines the $E$-hermitian space underlying $V$. It implies that $V^{1,0}$ and $V^{0,1}$ are free $E \otimes \mathbb{R}$-modules.

We thus verify that the elements of $(\wedge^2_E V)(q)$ are of type $(0, 0)$, which are Hodge cycles, called Weil cycles.

Now let $A$ be an abelian variety of CM type, and let $\theta \in H^{2q}(A, \mathbb{Q})(q)$ a Hodge cycle on $A$. We show [1] that there exists a CM field $E$ Galois over $\mathbb{Q}$, of abelian varieties $A_j$ of dimension

$$g = nq, \text{ where } n = [E : \mathbb{Q}],$$

such that $\text{End} A_j \otimes \mathbb{Q}$ contains $E$, homomorphisms $f_j : A \to A_j$, and Weil cycles $\theta_j \in (\wedge^2_E H^b_{\bar{J}}(A_j, \mathbb{Q}))(q)$ such that $\theta = \sum f_j^* \theta_j$. We are thus brought back to prove the algebraicity of these Weil cycles.

Otherwise, the argument of [1] shows that we can replace $E$ by any larger CM field Galois
over \( \mathbb{Q} \). This allows us to choose ad libitum an imaginary quadratic field \( \mathbb{Q}(\sqrt{-D}) \), and to suppose that \( E \) is of the form \( E^+ \mathbb{Q}(\sqrt{-D}) \) where \( E^+ \) is totally real.

**Third step [2, 6.3.3]**

This time we start from an abelian variety \( A \) (which we could suppose to be of CM type) equipped with a Weil cycle \( \theta \). We construct a compact abelian pencil \( f : X \to S \), and points \( s, t \) of \( S \) such that

(i) \( X_s \) is isogenous to the \( g \)-th power of an elliptic curve (of CM type if we want),
(ii) \( X_t \) is isogenous to \( A \),
(iii) The inverse image of \( \theta \) over \( X_t \) extends to a global section \( \xi \) of

\[
\bigwedge^{2g} R^1 f_* \mathbb{Q}(q) \subset R^{2g} f_* \mathbb{Q}(q),
\]

(iv) The fibre of \( \xi \) at \( s \) is algebraic.

We apply the proposition 1.2.2 to this pencil, and we conclude from (iv) that \( \theta \) is algebraic. The idea is to consider, as in [8], the family of Hodge type associated to unitary group \( R_{E^+/\mathbb{Q}} SU(V, \langle \cdot, \cdot \rangle) \subset Sp(V, \langle \cdot, \cdot \rangle) \) (and with level structure \( \nu \geq 3 \)). This family contains fibres of type \( X_s, X_t \) described as above (cf. [2, proof of 6.3.3]), corresponding to points \( s, t \) of the connected Shimura variety \( Sh^0 \) which parametrize this family. On the other hand, since we suppose \( q > 1 \), the boundary of \( Sh^0 \) in its minimal compactification is of codimension at least 2. By an argument of Bertini type, we find finally a connected smooth projective curve \( S \) and a morphism

\[
S \to Sh^0
\]

(which we can suppose to be a closed immersion, no matter) whose image goes through the points \( s, t \), and such that the inverse image of the family of Hodge type over \( S \) gives us the abelian pencil desired\(^2\).

Otherwise, it is well-known that the family of Hodge type is defined over \( \overline{\mathbb{Q}} \), likewise that the minimal compactification \( Sh^{0,*} \) of \( Sh^0 \), and that the CM points of \( Sh^0 \) are defined over \( \overline{\mathbb{Q}} \). We conclude that we can choose \( f \) and \( s, t \) defined over \( \overline{\mathbb{Q}} \). This shows that we can refine the theorem 1.2.1, in the CM case, in the following manner.

**Variant 1.5.1.** Let \( A \) be an abelian variety of CM type over \( \overline{\mathbb{Q}} \). Therefore every Hodge cycle on \( A_C \) is a motivated cycle, modeled on the smallest full sub-category of the category of lisse projective \( \overline{\mathbb{Q}} \)-varieties, stable under sum, product, passing to connected components, and containing the abelian varieties of CM type as well as the total space of every abelian pencil of a lisse projective curve base.

2. **Specializations of motivated cycles on abelian varieties of CM type**

**2.1.** In [3], we gave various variants and refinements of the proposition 1.2.2 over a field of any characteristic. Here is a consequence, in the particular case of abelian pencil which interests us here.

Let \( k \) be a field, \( \overline{k} \) a separable closure of \( k \), and \( \ell \) a prime number distinct from characteristic

\(^2\)It is well-known that on an abelian variety of type \( X_s \), every Hodge cycle is algebraic (in fact linear combination of product of divisor classes (cf. [14])), whence (iv).
of $k$. As in [3] we abridge $H^j_{\text{ét}}(Y, \mathbb{Q}_\ell)$ as $H^j(Y)$.

**Proposition 2.1.1.** Let $f : X \to S$ be an abelian scheme over a lisse projective geometrically connected curve over a field $k$. Let $q$ be a natural integer and $\xi$ a global section of $R^{2q}f_{\text{ét}*}\mathbb{Q}_\ell(q)$ ($\ell \neq \text{char } k$).

If for a point $s \in S(k)$, $\xi_s \in H^2_{\ell}(X_s)(q)$ is a motivated cycle on $X_s$ modeled over $\mathcal{V}_k$, the same is true for every point.

**Proof.** See [3, 7.2.1].

2.2. The $\mathbb{Q}_\ell$-subspace $V_f$ of $H_{\ell}(X)$ introduced in [3, 6.2.1.] is none other than the part $H^\text{pair}(S, \oplus R^j f_{\text{ét}*}\mathbb{Q}_\ell)$ in the canonical splitting “in the sense of Lieberman” of the Leray filtration

$$H(X, \mathbb{Q}_\ell) = H^0(S, \oplus R^j f_{\text{ét}*}\mathbb{Q}_\ell) \oplus H^1(S, \oplus R^j f_{\text{ét}*}\mathbb{Q}_\ell) \oplus H^2(S, \oplus R^j f_{\text{ét}*}\mathbb{Q}_\ell)$$

(cf. [3,6.2.1]).

If $k = \overline{\mathbb{F}}_p$, we know that the even Künneth projector of $X$ is algebraic [12], and that the projector $\pi_{V_f}$ on $H^\text{even}(S, \oplus R^j f_{\text{ét}*}\mathbb{Q}_\ell)$ along $H^1(S, \oplus R^j f_{\text{ét}*}\mathbb{Q}_\ell)$ is given by an algebraic correspondence (cf. [3,6.2.1]).

According to [3,7.1.2], we can thus refine the proposition 2.1.1 by replacing “motivated cycles” modeled on $\mathcal{V}_k$ by “motivated cycles modeled on $\mathcal{V}_k^\ell$”, where $\mathcal{V}_k$ contains $(X_s, H_\ell(X_s))$ for every $s \in S(K)$ and the $(X, V_f)$ for every abelian scheme $X \to S$ over a lisse geometrically connected projective curve. Let’s go back to [3] for the precise definitions.

The advantage is a better control of the $\mathbb{Q}$-algebra $\text{End} 1$ in the category of motive correspondences (cf. [3,3.1]). This $\mathbb{Q}$-algebra is generated by the “numbers” $\langle \alpha, *H\alpha \rangle$, where $\alpha$ runs through the algebraic cycle classes on $X$ contained in $V_f$ (for every couple $(X, V_f)$ as above), and $*H$ is the Hodge involution on $X$ attached to any polarization of type $\eta = f^*$ (polarization of $S$)+ class of a relatively ample invertible symmetric fibre rigidified along the zero section.

We denote $\mathbb{Q}_{(\ell)}$ the field of fractions of $\text{End} 1$. This is a countable sub-field of $\mathbb{Q}_\ell$, conjecturally equal to $\mathbb{Q}$.

2.3. We fix henceforth a prime number $p$.

In [3], we applied the deformation of motivated cycles to the study of their specialisation in unequal characteristics $(0, p)$. Here is one corollary, in the particular case of abelian pencils.

Let $\mathfrak{o}$ be a complete discrete valuation ring of residue field $k = \mathbb{F}_p$, and $K$ its field of fractions (of characteristic zero).

**Proposition 2.3.1.** Let $\mathcal{G}$ be an $\mathfrak{o}$-scheme of curves , projective and flat, with generic fibre lisse geometrically connected. Let $f : \mathfrak{X} \to \mathcal{G}$ an abelian scheme. Let $s$ and $t$ be two $\mathfrak{o}$-points of $\mathcal{G}$.

Let $q$ be a natural integer and $\xi$ a global section of $R^{2q}(f_K)_{\text{ét}*}\mathbb{Q}_\ell(q)$ ($\ell \neq p$). We suppose that $\xi_{\mathfrak{p}K} \in H^2_{\ell}(\mathfrak{X}_{\mathfrak{p}K})$ is an algebraic cycle.

Thus $\xi_{\mathfrak{p}K} \in H^2_{\ell}(\mathfrak{X}_{\mathfrak{p}K})$ is a motivated cycle modeled over $\mathcal{V}_K$, and its specialisation $\xi_{\mathfrak{p}p} \in H^2_{\ell}(\mathfrak{X}_{\mathfrak{p}p})$ is a motivated cycle modeled over $\mathcal{V}_{\mathfrak{p}p}$.

**Proof.** See [3,9.2.1] (see also the remark 9.3.3 of [3]).
2.4. Recall that if an abelian variety $A$ over $K$ has good reduction, this is the generic fibre of a lisse projective $\mathfrak{a}$-scheme \( \mathfrak{A} \), which is unique up to unique isomorphism, and which admits a structure of abelian scheme [6, § 1.2]. Recall also that if $A$ is of CM type, it acquires good reduction over a finite extension (Serre-Tate). Otherwise, every abelian variety over $\mathbb{F}_p$ is of CM type (Tate).

**Theorem 2.4.1.** Let $A$ be an abelian variety of CM type having good reduction over $K$. Let $\theta \in H^{2r}(A_\mathbb{F}_p, \mathbb{Q}_\ell)(r)$ a motivated cycle on $A$. Therefore its specialisation in $H^{2r}(A_{\mathbb{F}_p}, \mathbb{Q}_\ell)(r)$ is a motivated cycle, modeled over $\mathcal{V}_{\mathbb{F}_p}$, over the abelian variety $A_{\mathbb{F}_p}$.

**Proof.** The abelian variety $A$ is defined over a certain number field $K_0 \subset K$ which embeds in $\mathbb{C}$. We can freely replace $K$ by a finite extension (and thus also $K_0$).

The second step (1.4) of §1 brings us back to the case where the motivated cycle $\theta \in H^{2r}(A_{\mathbb{F}_p}, \mathbb{Q}_\ell)(r)$ is a Weil cycle.

In the third step (1.5) of §1, we will construct an abelian pencil $f : X \to S$ (over a geometrically connected lisse projective curve defined over a finite extension of $K_0$), admitting a level structure $\nu$ (which we suppose greater or equal to 3 and not divisible by $p$), and two points $s$ and $t$ of $S$ (which we can also suppose defined over a finite extension of $K_0$), such that

(i) $X_s$ is isogenous to the $g$-th power of an elliptic curves of CM type,
(ii) $X_t$ is isogenous to $A$,
(iii) the inverse image of $\theta$ over $X_t$ extends to a global section $\xi$ of

\[
\bigwedge^2 R^1 f_* \mathbb{Q}(q) \subset R^{2g} f_* \mathbb{Q}(q),
\]

(iv) The fibre of $\xi$ at $s$ is algebraic.

Up to replacing $K_0$ by a suitable finite extension and completing $p$-adically, we can suppose that $f$ is defined over the complete field $\hat{K}$, and that $s$ and $t$ are $\hat{K}$-points. Because of complex multiplication and of the presence of level structure, $X_s$ and $X_t$ have good reduction over $\mathfrak{a}$.

To apply the proposition 2.3.1, it’s about showing that we can choose the curve $S$ as above (suppose defined over $K_0$) of such that it extends to an $\mathfrak{a}$-scheme $\mathfrak{S}$ projective and flat, and that the abelian pencil $f$ of base $S$ extends to an abelian scheme $\mathfrak{f}$ over $\mathfrak{S}$ (the most delicate case is when $p$ divides the discriminant of $E^+$, because the connected Shimura variety $Sh^0$ of unitary type which parametrizes the principally polarized abelian varieties of dimension $g$ of Weil type, quipped with level structure $\nu \geq 3$ not divisible by $p$, is lisse, but does not have good reduction relative to the ring of integers $\mathfrak{o}$ of a $p$-adic field $K$ sufficiently large). The key point will be to show that the “minimal boundary” of this reduction is of codimension at least 2 for $q > 1$.

We usually denote as $A_{g,\nu}$ the moduli scheme over $\mathbb{Z}[1/\nu]$ of principally polarized abelian varieties of dimension $g$ with (not necessarily symplectic) level structure $\nu \geq 3$, and $A^*_{g,\nu}$ the minimal compactification of $A_{g,\nu}$ constructed in [11].

We denote $Sh^0 \to A_{g,\nu} \otimes K$ the canonical morphism, and $\mathcal{M}^{0,*}$ the closure of the image of $Sh^0$ in $A^*_{g,\nu} \otimes \mathfrak{o}$. $\square$
Proposition 2.4.2. Suppose that \( E = E^+, \mathbb{Q}(\sqrt{-D}) \), and that \( p \) decomposes in \( \mathbb{Q}(\sqrt{-D}) \). Therefore the complement of \( (\mathcal{M}^{0,*} \otimes \mathbb{F}_p) \cap (A_{g,\nu} \otimes \mathbb{F}_p) \) in \( \mathcal{M}^{0,*} \otimes \mathbb{F}_p \) is of codimension at least 2.

The proof will occupy §3.

2.5. Accept the proposition 2.4.2 and continue the proof of theorem 2.4.1. We denote again \( s \) and \( t \) the \( K \)-points of \( (\mathcal{M}^{0,*} \otimes K) \cap (A_{g,\nu} \otimes K) \) defined by \( X_s \) and \( X_t \). As these varieties have good reduction, they actually extend into \( o \)-points \( s \) and \( t \) of \( \mathcal{M}^{0,*} \cap (A_{g,\nu} \otimes o) \).

Consider the blowup \( \overline{\mathcal{M}} \) of \( \mathcal{M}^{0,*} \) along the closed subscheme defined by the image of \( s \) and \( t \), and the open \( \mathcal{M} \) obtained by removing the strict transform of boundary \( \mathcal{M}^{0,*} \setminus (\mathcal{M}^{0,*} \cap (A_{g,\nu} \otimes o)) \). The proposition 2.4.2 allows us to apply to this situation the following lemma.

Lemma 2.5.1. Let \( \overline{\mathcal{M}} \) be a projective flat \( o \)-scheme of relative dimension \( d > 1 \), and \( \mathcal{M} \) and open of \( \overline{\mathcal{M}} \) dense fibre to fibre. We suppose \( \mathcal{M} \otimes K \) geometrically connected and lisse, and that \( \mathcal{M} \otimes \mathbb{F}_p \setminus \mathcal{M} \otimes \mathbb{F}_p \) is of codimension at least 2 in \( \mathcal{M} \otimes \mathbb{F}_p \).

Therefore \( \mathcal{M} \otimes \mathbb{F}_p \) is connected. In addition, every sufficiently general linear section of relative codimension \( d - 1 \) in a projective embedding \( \overline{\mathcal{M}} \hookrightarrow \mathbb{P}_s^N \) defines an \( o \)-scheme over curves \( \mathcal{G} \), projective and flat, with geometrically connected lisse generic fibre, and contained in \( \mathcal{M} \).

Proof. Let \( \mathcal{G} \) be the \( o \)-Grassmanian of linear sub-variants of codimension \( d - 1 \) in \( \mathbb{P}_s^N \). For \( \sigma_{\mathcal{G}_p} \in \mathcal{G}(\mathbb{F}_p) \) in an open dense, the section of \( \mathcal{M} \otimes \mathbb{F}_p \) corresponding to \( \sigma_{\mathcal{G}_p} \) is a curve \( \mathcal{G}_{\mathcal{F}_p} \subset \mathcal{M} \otimes \mathbb{F}_p \) which cut each connected component of \( \mathcal{M} \otimes \mathbb{F}_p \), and does not cut the boundary \( \mathcal{M} \otimes \mathbb{F}_p \setminus \mathcal{M} \otimes \mathbb{F}_p \). For \( \sigma_K \in \mathcal{G}(K) \) in an open dense, the section of \( \mathcal{M} \otimes K \) corresponds to \( \sigma_K \) is a curve \( \mathcal{G}_K \subset \mathcal{M} \otimes K \) lisse and geometrically connected since \( \mathcal{M} \otimes K \) is geometrically connected lisse of dimension at least 2. This open contains a \( \sigma_K \) that specialises to \( \sigma_{\mathcal{G}_p} \), such that \( \mathcal{G}_K \) and \( \mathcal{G}_{\mathcal{F}_p} \) are generic fibre and special fibre of a projective flat \( o \)-curve \( \mathcal{C} \). By the theorem of connectivity of Zariski, \( \mathcal{G}_{\mathcal{F}_p} \) is connected, and as \( \mathcal{G}_{\mathcal{F}_p} \) cut each connected component of \( \mathcal{M} \otimes \mathbb{F}_p \), \( \mathcal{M} \otimes \mathbb{F}_p \) is connected.

We are thus in a situation to apply the proposition 2.3.1 (refined by replacing \( \mathcal{V}_{\mathcal{F}_p} \) by \( \widetilde{\mathcal{V}_{\mathcal{F}_p}} \)) to the inverse image over \( \mathcal{G} \) of universal abelian scheme over \( A_{g,\nu} \), in noting that the image of \( \mathcal{G} \) in \( A_{g,\nu} \otimes o \) contains the image of \( s \) and \( t \), and we obtain the proposition 2.3.1.

Remark 2.5.2.

(1) If we could choose \( \mathcal{G} \) lisse over \( o \), there is no need to appeal to the theorem of specialisation of [3], we will be in the easy case where the auxiliary variety intervening in the in the writing of the motivated cycle considered with also good reduction.

The obstacle to the construction of a scheme in curves \( \mathcal{G} \) lisse over \( o \), whose image in \( A_{g,\nu} \otimes o \) contains the image of \( s \) and \( t \), lies in that the reduction of \( s \) and \( t \) are not necessarily smooth points of \( \mathcal{M}^{0,*} \otimes \mathbb{F}_p \).

(2) To extend the theorem 2.4.1 of specialisation of motivated cycles to the case of any abelian variety (let’s say defined over \( \overline{\mathbb{Q}} \) for simplicity), it suffices, following the same path, to prove that we can choose, in the first step (§1.3) of the proof of theorem 1.2.1, the curve \( S \) which appears there (suppose defined over \( K_0 \)) of such sort that it extends to an \( o \)-scheme \( \mathcal{G} \) projective and flat, and that the abelian pencil \( f \) of base \( S \) extends to an abelian scheme.
To do this, we are confronted with the following general question on the Shimura varieties, which we formulate in a rather vague way.

Let's start from a connected Shimura variety parametrizing a family (of abelian varieties) of Hodge type associated to a reductive group (of special Mumford-Tate group) $G^1$ given, as in [3].

**Question 2.5.3.** is the analogue of the proposition 2.4.2 replacing $Sh^0$ by the connected Shimura variety parametrizing a family of Hodge type associated to group $R_{L+}^1(G^1 \otimes_QL^+)$, for a quadratic real field, or at least for a suitable totally real number field $L^+$?

**2.6.** Let’s translate now the theorem of specialisation 2.4.1 in terms of category of motives (defined using motivic correspondences).

Let $k$ be a field.

**Notation 2.6.1.** We denote by $W_k$ the smallest full sub-category of the category of lisse projective $k$-schemes which is stable under disjoint sum and which contains the abelian varieties of CM type over $k$.

Note that $W_k$ is stable under product and passing to connected components, and is contained in $V_k$. If $k = \mathbb{F}_p$, $W_k$ is none other than the category of disjoint sum of abelian varieties.

If $C$ is an algebraically closed field of characteristic zero, we have on the one hand a $\otimes$-category $M(W_C)_{\nu_C}$ of motives (by the motivic correspondences modelled on $V_C$) cut out on the abelian varieties of CM type over $C$. We will dimply denote $M(W_C)$ for brevity.

This is a semi-simple Tannakian category over $\mathbb{Q}$. In fact, it follows from the variant 1.5.1 that it does not depend on $C$: if we can fix the embeddings $\mathbb{Q} \hookrightarrow C$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, the natural $\otimes$-functors

$$M(W_{\overline{\mathbb{Q}}}) \to M(W_C), \quad M(W_{\overline{\mathbb{Q}}}) \to M(W_{\overline{\mathbb{Q}}_p})$$

are equivalences.

We will also need the variant $M(W_C)_{\nu_C}[Q(\ell)]$ with coefficients in the field $Q(\ell)$ (see §2.2), which is none other than the semi-simple Tannakian category $M(W_C) \otimes Q(\ell)$ deduced from $M(W_C)$ (here and further, we denote by $\mathcal{T} \otimes Q(\ell)$ the semi-simple Tannakian $Q(\ell)$-category deduced from a semi-simple Tannkian $\mathbb{Q}$-category $\mathcal{T}$ by tensoring the morphisms by $Q(\ell)$ and by passing to the pseudo-abelian envelope).

Otherwise, we have the $\otimes$-category

$$M_\ell(W_{\tilde{\mathbb{F}}_p})_{\tilde{V}_{\tilde{\mathbb{F}}_p}}[Q(\ell)]$$

of motives cutout on the disjoint sum of $\tilde{\mathbb{F}}_p$-abelian varieties, modeled on $\tilde{V}_{\tilde{\mathbb{F}}_p}$, with coefficients in $Q(\ell)$. We will simply denote $M_\ell(W_{\tilde{\mathbb{F}}_p})$. This is a rigid $Q(\ell)$-linear, pseudo-abelian $\otimes$-category, and whose morphisms form $Q(\ell)$-vector space of finite dimensions. The quotient $\overline{M}_\ell(W_{\tilde{\mathbb{F}}_p})$ of $M_\ell(W_{\tilde{\mathbb{F}}_p})$ by its largest $\otimes$-ideal is a semi-simple Tannakian category over $Q(\ell)$, the functor passing to quotient being conservative (cf. [3, §§2.4, 3.3, 4.1]).

**Corollary 2.6.2.** There exists a canonical faithful $\otimes$-functor $M(W_{\overline{\mathbb{Q}}_p}) \to M_\ell(W_{\tilde{\mathbb{F}}_p})$. A fortiori, we obtain a canonical exact faithful $\otimes$-functor of semi-simple Tannakian categories, called “specialisation functor”: $$M(W_{\overline{\mathbb{Q}}_p}) \to \overline{M}_\ell(W_{\tilde{\mathbb{F}}_p}).$$
In addition, every object of $\mathcal{M}_G(\mathcal{W}_p)$ is direct factor of the image of an object of $\mathcal{M}(\mathcal{W}_p)$.

The first two assertions can be deduced from Theorem 2.4.1 as [3, 9.3.2] of [3, 9.2.1]. The last assertion follows from the theorem of Honda-Tate that reductions of $\mathbb{Q}_p$-abelian varieties of CM type meet each isogeny class over $\mathbb{F}_p$ [28].

3. Certain unitary Shimura varieties and their local models

The subject of this paragraph is to prove the proposition 2.4.2, which will complete the proof of the results of the previous paragraph.

3.1. We fix a prime number $p$. Let $E^+$ be a totally real number field and $E$ a totally imaginary quadratic extension of $E^+$. We suppose (for simplification) $E$ is Galois over $\mathbb{Q}$, of degree $n$. We suppose that each place $v$ of $E^+$ above of $p$ decomposes in $E$. (*)

It follows from (*) that

$$E \otimes \mathbb{Q}_p \cong (E^+ \otimes \mathbb{Q}_p)^2,$$

the complex conjugation induces the exchange of two factors. We fix such an isomorphism of $E^+ \otimes \mathbb{Q}_p$-algebras.

We consider $E$ as subfield of $\overline{\mathbb{Q}}$, and we fix a complex embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and a $p$-adic embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p$. This is used to identify $\text{Hom}_{\mathbb{Q}}(K, \mathbb{C})$ and $\text{Hom}_{\mathbb{Q}}(K, \mathbb{Q}_p) = \text{Hom}_{\mathbb{Q}_p}(K \otimes \mathbb{Q}_p, \mathbb{Q}_p)$ for every subfield $K \subset \overline{\mathbb{Q}}$ (especially $K = E$ or $E^+$). In combining with the identification $E \otimes \mathbb{Q}_p = (E^+ \otimes \mathbb{Q}_p)^2$ followed by the first projection, we obtain thus a section

$$\varphi \mapsto \tilde{\varphi}$$

of the restriction map $\text{Hom}_{\mathbb{Q}}(E, \mathbb{C}) \to \text{Hom}_{\mathbb{Q}}(E^+, \mathbb{C})$.

We denote by $F$ the completion of $E^+$ at the place induced by $\overline{\mathbb{Q}} \hookrightarrow \mathbb{Q}_p$, and $m$ (respectively, $e$) its degree (respectively, its ramification index) over $\mathbb{Q}_p$. $F_0$ the subfield of $F$ maximal unramified over $\mathbb{Q}_p$, $k$ its residue field. To simplify later the notations, we fix an isomorphism between $F$ and the completion of $E^+$ at each $p$-adic place of $E^+$.

We otherwise fix a purely imaginary element $\epsilon \in \mathcal{O}_E$ of norm prime to $p$ (it exists: (*) assures that the kernel of the trace $\mathcal{O}_E \to \mathcal{O}_{E^+}$ contains a unit of $\mathcal{O}_E/p$, and lifts to the kernel of the trace $\mathcal{O}_E \to \mathcal{O}_{E^+}$). We suppose (for simplicity) that

$$\text{for every } \varphi \in \text{Hom}_{\mathbb{Q}}(E^+, \mathbb{C}), \text{ Im } \tilde{\varphi}(\epsilon) > 0. \quad (**)$$

Remark 3.1.1. The conditions (*) and (**) are satisfied if $E = E^+.\sqrt{-D}$ with $p$ decomposes in $\mathbb{Q}(\sqrt{-D})$ and $\epsilon = \sqrt{-D}$.

3.2. We give ourselves a $\mathbb{Q}$-vector space $V$, equipped with a non-degenerate alternating form $(\cdot, \cdot)$. We suppose that $V$ is equipped with a structure of $E$-vector space such that

$$(\alpha v, w) = (v, \overline{\alpha} w), \quad v, w \in V, \alpha \in E. \quad (***)$$

We denote by $d$ the dimension of $V$ as $E$-vector space, and we put

$$g = \frac{1}{2}nd.$$
The data of $(\cdot, \cdot)$ equivalent to that of an $E$-hermitian form $(\cdot, \cdot)$, via the formula:

$$(v, w) = \text{Tr}_{E/Q} \epsilon(v, w).$$

When tensoring with $\mathbb{Q}_p$, the data of $(\cdot, \cdot)$ amounts to that of a decomposition

$$V \otimes \mathbb{Q}_p = W \oplus W^\vee$$

in a free $E^+ \otimes \mathbb{Q}_p$-module $W = \bigoplus_{v|p} W_v$ of rank $d$ and its dual (note that due to (*), $V \otimes \mathbb{Q}_p$ is a free $(E^+ \otimes \mathbb{Q}_p)^2$-module and use (***)).

Let $G \subset (V, (\cdot, \cdot))$ be the closed $\mathbb{Q}$-subgroup formed of $E$-linear symplectic similitudes. We thus have an exact sequence

$$\{1\} \to R_{E/+/Q}U((\cdot, \cdot)) \to G \to \mathbb{G}_m \to \{1\} \quad (3.1)$$

and it follows from the above that

$$G_{\mathbb{Q}_p} \cong (R_{F/\mathbb{Q}_p} \text{GL}_d)^{n/m} \times \mathbb{G}_m_{\mathbb{Q}_p}.$$

3.3. Let $h : R_{C/\mathbb{R}} \mathbb{G}_m_{\mathbb{C}} \to G_\mathbb{R}$ be a homomorphism defining on $V$ a rational Hodge structure of type $(1, 0) + (0, 1)$, and such that $(v, h(i)w)$ is a positive definite form on $V_\mathbb{R}$ (Riemann relations). Up to changing $h$ by a conjugate, we can suppose, and we will suppose that the decomposition

$$V^{1,0} \oplus V^{0,1}$$

is defined over $\overline{\mathbb{Q}} \subset \mathbb{C}$ ($V^{1,0}$ being the part on which $h(i)$ acts by $i$).

The structure of $E \otimes \overline{\mathbb{Q}}$-module of $V^{1,0}$ is described by the $n$-tuple $r = (\cdots, r_\varphi, \cdots)$ of natural integers $r_\varphi \leq d$ defined by

$$r_\varphi = \dim_{\overline{\mathbb{Q}}} V^{1,0} \otimes_{E, \varphi} \overline{\mathbb{Q}}.$$

That $v \otimes w \mapsto (v, h(i)w)$ is a positive definite form is translated, considering (**), to:

the signature of the Hermitian form $(\cdot, \cdot)$ on $V \otimes_{E^+} \mathbb{R} \cong \mathbb{C}^d$ is $(r_\varphi, d - r_\varphi)$.

Otherwise, considering the definition of $\varphi$ and of $W$, we also have

$$W \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p = \bigoplus_{\varphi} V \otimes_{E, \varphi} \overline{\mathbb{Q}}_p$$

and

$$r_\varphi = \dim_{\overline{\mathbb{Q}_p}} (V^{1,0} \otimes_{E, \varphi} \overline{\mathbb{Q}}_p) \cap (W \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p) \quad (3.2)$$

(in considering $\varphi$ as an embedding into $\overline{\mathbb{Q}_p}$). For every embedding $\psi : F_v \hookrightarrow \overline{\mathbb{Q}}_p$ and every $p$-adic place $v$ of $E^+$, we denote by $r_{v, \psi}$ the $e$-tuple of $r_\varphi$ where $\varphi$, viewed as embedding $E^+ \hookrightarrow \overline{\mathbb{Q}}_p$, satisfies two conditions:

- it induces the $p$-adic place $v$,
- the homomorphism $F \cong E^+_v \hookrightarrow \overline{\mathbb{Q}}_p$ which extends $\varphi$ is compatible with $\psi$,

which is indicated by the abbreviation $\varphi \to (v, \psi)$.

3.4. The couple $(h, G)$ gives birth to a Shimura (pro-)variety “of unitary type” $\text{Sh}_{r, g}$, defined over $E$.

\footnote{In fact over the “reflex” field, which is a subfield since $E$ is supposedly Galois over $\mathbb{Q}$.} The Shimura variety appearing at §1.5 is a quotient, under the hypothesis
of the proposition 2.4.2, for the data \( g = nq, d = 2q \), \( r \) being the \( n \)-tuple \( (q, \cdots, q) \).

We give ourselves an \( \mathcal{O}_F \)-lattice \( \Lambda_v \) of \( W_v \), hence an \( \mathcal{O}_E \otimes \mathbb{Q}_p \)-lattice \( \Lambda \) of \( V \otimes \mathbb{Q}_p \):

\[
\Lambda = \bigoplus \Lambda_v \oplus \bigoplus \Lambda_v^\vee
\]

where \( \Lambda_v^\vee \) denotes the \( \mathbb{Z}_p \)-dual of \( \Lambda_v \). This is a self-dual lattice with regard to \((\cdot, \cdot)\) since \( \epsilon \) is a \( p \)-adic unit. Put \( C_p := \text{End}_{\mathbb{Z}_p} \Lambda \cap G(\mathbb{Q}_p) \).

Otherwise, we choose a (sufficiently small) compact open subgroup \( C_p \) of \( G(\mathbb{A}_p) \), where \( \mathbb{A}_p \) denotes as usual the ring of adeles of \( \mathbb{Q} \) deprived of their components at infinity and at \( p \). We suppose \( C_p \) is contained in the principal congruence subgroup of sufficiently large level \( \nu \) prime to \( p \).

The compact group \( C = C_p.C_p \) acts on \( \text{Sh}_{r,g} \) and we put \( \text{Sh}_{r,g,C} := \text{Sh}_{r,g}/C \).

3.5. Consider the following moduli problem (functor): to every \( \mathcal{O}_F \)-scheme \( S \), we associate the set of isomorphism classes of the following data:

(I) An abelian \( \mathcal{O}_E \)-scheme \( A \) over \( S \), of relative dimension \( nq, d = 2q \), up to isogeny of prime-to-\( p \) order;

(II) a \( \mathbb{Q} \)-homogenous principal polarisation of \( A \);

(III) A level structure

\[
H^1(A, \mathbb{A}_f^p) \cong V \otimes \mathbb{A}_f^p \mod C_p
\]

which respects the bilinear form up to a constant \( \in (\mathbb{A}_f^p)^* \).

It is required that

\[
\text{det}(\alpha | \text{Lie}_S^\vee(A)) = \text{det}(\alpha | V^{1,0})
\]

as polynomial function of \( \alpha \in \mathcal{O}_E \), that is to say

\[
\text{det}(\alpha | \text{Lie}_S^\vee(A)) = \prod_{\varphi} \tilde{\varphi}(\alpha)^{r_{\varphi}}\varphi(\alpha)^{d-r_{\varphi}} \tag{3.3}
\]

(\( \text{Lie}_S^\vee(A) \) denotes the \( \mathcal{O}_S \)-dual of \( \text{Lie}_S(A) \)).

We refer to [24, Chapter 6] for the (sufficiently long) precise definition of these notions. This type of moduli problem on a trait to first been considered by Kottwitz [13, §5] in the case where \( F \) is non-ramified over \( \mathbb{Q}_p \).

**Proposition 3.5.1.**

(1) The moduli problem as above is representable by a quasi-projective \( \mathcal{O}_F \)-scheme \( \mathcal{M}_{r,g} \). It is lisse if \( F \) is non-ramified over \( \mathbb{Q}_p \), but not smooth or even flat in general.

(2) \( (\mathcal{M}_{r,g})_F \) is identified to \( \text{Sh}_{r,g,C} \otimes \mathcal{O}_F \), where \( \text{Sh}_{r,g,C} \) is the Shimura variety associated to the situation.

(3) Forgetting the action of \( \mathcal{O}_E \) (given I) induces a projective morphism

\[
f : \mathcal{M}_{r,g} \to \mathcal{A}_{g,\nu} \otimes \mathcal{O}_F,
\]

and \( f_F \) is finite étale.

---

\(^4\)That is to say, if we prefer, the \( \mathcal{O}_F \)-dual tensored with the codifference of \( \mathcal{O}_F \).

\(^5\)More precisely, an “\( \mathcal{L} \)-set” of such abelian schemes, where \( \mathcal{L} \) is the set (multi-chain polarized) of lattices \( \Lambda' \) obtained as \( \Lambda \) but by changing each \( \Lambda_v \) and \( \Lambda_v^\vee \) by a homothety of arbitrary rapport in \( F^* \). In the formalism of [24], that translates the choice of the maximal parahoric structure.

(2) According to the Landherr classification of $E$-hermitian forms (cf. [25,26]), the Hasse principle is true for these latest, and we deduce as in [24, pp. 296, 300, 301] that $\mathcal{M}_F = \text{Sh}_{r,g,C} \otimes_E F$.

(3) Forgetting the action of $\mathcal{O}_E$ (and also the $\mathbb{Z}_p$-lattice $\Lambda'$ of $L$ which are not in the form $p^{\ell}A$) induces a morphism between the functor represented by $\mathcal{M}_{r,g}$ and the analogous functor where $\mathcal{O}_E$ is replaced by $\mathbb{Z}$ and $C^\nu$ by its principal congruence subgroup of level $\nu$ in $(\mathbb{A}_F^\nu)$. It is easy to see that the latter functor is represented by $\mathcal{A}_{g,\nu} \otimes \mathcal{O}_F$, via the choice of a self-dual $\mathbb{Z}[1/p]$-lattice of $V$.

A direct application of the valuative criterion of properness shows that the morphism $f$ thus obtained between quasi-projective $\mathcal{O}_F$-schemes is proper, thus projective. In addition, it is well known that the natural morphism between Shimura varieties $\text{Sh}_{r,g,C} \to \mathcal{A}_{g,\nu} \otimes \mathcal{O}_F$, via the choice of a self-dual $\mathbb{Z}[1/p]$-lattice of $V$.

3.6. The local structure of $\mathcal{M}_{r,g}$, for the étale topology, is given by the “local models” studied in greater generality in [23]. This construction is based on the Grothendieck-Messing theorem, that deformations of $A$ are “controlled” by the local direct factor $\text{Lie}^\nu A$ of $H_{\text{dR}}^1(A)$. Locally over the $\mathcal{O}_F$-scheme $S$, we can identify $H_{\text{dR}}^1(A)$ to $$\Lambda \otimes_{\mathbb{Z}_p} \mathcal{O}_S \cong \bigoplus_v (\Lambda_v \otimes_{\mathbb{Z}_p} \mathcal{O}_S) \oplus \bigoplus_v (\Lambda_v^\nu \otimes_{\mathbb{Z}_p} \mathcal{O}_S)$$ and we have a corresponding decomposition

$$\text{Lie}^\nu A = \bigoplus_v \mathcal{F}_v \oplus \bigoplus_v ((\Lambda_v \otimes_{\mathbb{Z}_p} \mathcal{O}_S)/\mathcal{F}_v)^\nu.$$ 

Otherwise, we have the “following $\psi$” decomposition:

$$(\Lambda_v \otimes_{\mathbb{Z}_p} \mathcal{O}_S) = \bigoplus_{\psi:F_0 \to F \subset \mathbb{Q}_p} \Lambda_{v,\psi},$$

where $\Lambda_{v,\psi} := \Lambda \otimes_{\mathcal{O}_{F_0,\psi}} \mathcal{O}_S$ (locally free $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0,\psi}} \mathcal{O}_S$-module of rank $d$), and we have a corresponding decomposition

$$\mathcal{F}_v = \bigoplus_{\psi:F_0 \to F \subset \mathbb{Q}_p} \mathcal{F}_{v,\psi}.$$

where, by virtue of (3.2) and (3.3),

$$\det_{\mathcal{O}_S}(\alpha|_{\mathcal{F}_{v,\psi}}) = \prod_{\varphi \to (v,\psi)} \varphi(\alpha)^{r_{\psi}}.$$ 

Note that due to the eventual ramification of $F$, $\mathcal{F}_{v,\psi}$ is not necessarily locally direct factor of $\Lambda_{v,\psi}$ as $\mathcal{O}_F \otimes_{\mathcal{O}_{F_0,\psi}} \mathcal{O}_S$-module (but only as $\mathcal{O}_S$-module). It follows from above that a local model for $\mathcal{M}_{r,g}$ is given by the product $\mathcal{O}_F$-scheme

$$\prod_{(v,\psi)} M(\Lambda_{v,\psi}, r_{v,\psi})$$
where \( M(\Lambda_{v,\psi}, r_{v,\psi}) \) denotes the \( \mathcal{O}_F \)-scheme which represents the functor

\[
S \mapsto \left\{ F \subset \Lambda \otimes_{\mathcal{O}_F} \mathcal{O}_S, \text{locally direct factor sub-} \mathcal{O}_F \otimes_{\mathcal{O}_F} \mathcal{O}_S \text{-module}, \right. \\
\left. \text{as } \mathcal{O}_S \text{-module, with } \det(\alpha|_F) = \prod_{\varphi \mapsto (v,\psi)} (\varphi(\alpha))^{r_\varphi} \right\}.
\]

This is a projective \( \mathcal{O}_F \)-scheme studied in detail in [23]. Its generic fibre is a product of Grassmannians, thus lisse. Its special fibre is singular; it admits a partition into “affine Schubert cells”.

**Proposition 3.6.1.** \( \dim_F \mathcal{M}_{r,g} \otimes F = \dim_E \text{Sh}_{r,g,C} = n \sum r_\varphi (d - r_\varphi) \) and

\[
\dim_{\overline{F}_p} \mathcal{M}_{r,g} \otimes \overline{F}_p \leq n. \left(\frac{1}{2}d\right)^2 = \frac{1}{2} gd.
\]

**Proof.** Indeed, [23, 3.1] says that \( M(\Lambda_{v,\psi}, r_{v,\psi}) \otimes \overline{F}_p \) is irreducible of dimension

\[
d|_{r_{v,\psi}} - e\left[ \frac{|r_{v,\psi}|}{e} \right]^2 - \left( 2\left[ \frac{|r_{v,\psi}|}{e} \right] + 1 \right) \left( |r_{v,\psi}| - e\left[ \frac{|r_{v,\psi}|}{e} \right] \right)
\]

where \([\cdot]\) denotes the integer part and \( |r_{v,\psi}| = \sum_{\varphi \mapsto (v,\psi)} r_\varphi \).

This dimension is

\[
\leq |r_{v,\psi}| \left( d - \frac{|r_{v,\psi}|}{e} \right) \leq e^2 \left( \frac{1}{2} d \right)^2,
\]

from which we deduce the terminal of (1) because there are \( n/e \) couples \((v,\psi)\). \( \square \)

**3.7.** Let’s go back to the proof of the proposition 2.4.2. The connected Shimura variety \( \text{Sh}^0 \) considered in the proposition 2.4.2 is a connected component of a Shimura variety of type \( \text{Sh}_{r,g,C} \otimes F \), in the case where \( d = 2q \) is even and where every \( r_\varphi \) is equal to \( q \). The dimension of fibres of model \( \mathcal{M}_{r,g} \) is thus constant, equal to \( nq^2 = \frac{1}{2} gd \), which is by the way the maximal value according to proposition 3.6.1.

To prove the proposition 2.4.2, it suffices thus to prove that the boundary of the closure of the image of

\[
\mathcal{M}_{r,g} \otimes \overline{F}_p \to \mathcal{A}_{g,\nu}^* \otimes \overline{F}_p
\]

is of codimension at least 2 since \( d = 2q \) is even and every \( r_\varphi \) is equal to \( q > 1 \).

Otherwise, Chai and Faltings [11, Chapters V, VI] have constructed lisse toroidal compactifications \( \overline{\mathcal{A}}_{g,\nu} \otimes \overline{F}_p \) of \( \mathcal{A}_{g,\nu} \otimes \overline{F}_p \) and a semi-abelian scheme \( G \) over \( \overline{\mathcal{A}}_{g,\nu} \otimes \overline{F}_p \) extending the universal abelian scheme over \( \mathcal{A}_{g,\nu} \otimes \overline{F}_p \), as well as a surjective projective morphism \( \pi : \overline{\mathcal{A}}_{g,\nu} \otimes \overline{F}_p \to \mathcal{A}_{g,\nu}^* \otimes \overline{F}_p \). The boundary of \( \mathcal{A}_{g,\nu}^* \otimes \overline{F}_p \) is stratified by the stratum isomorphic to \( \mathcal{A}_{g',\nu} \otimes \overline{F}_p \) for \( g' < g \), whose geometric points \( \overline{\pi}(\overline{x}) \) parametrizing the abelian part of semi-abelian varieties \( G_\pi \) (this abelian part does not depend on \( \pi \) in the fibre of \( \pi \)) equipped with a level structure \( \nu \) described in [11, VI. 2.3.5].

If \( \mathcal{O}_F \) acts by endomorphisms on an abelian scheme \( A \), this action extends to every semi-abelian scheme extending \( A \) over a normal base (cf. [11, I.2.7]). Otherwise, if \( \mathcal{O}_E \) acts on a semi-abelian variety, it also induces an action on the abelian part. We deduce that the components of the boundary of the closure of the image of \( \mathcal{M}_{r,g} \otimes \overline{F}_p \to \mathcal{A}_{g,\nu}^* \otimes \overline{F}_p \) each admits a dense open contained in the image of canonical morphism \( \mathcal{M}_{r,g'} \otimes \overline{F}_p \to \mathcal{A}_{g',\nu} \otimes \overline{F}_p \) for \( g' < g \) and suitable \( r \). According to the previous proposition, this boundary is thus of dimension \( \leq \frac{1}{2} gd = g'q \), which is \( \leq \frac{1}{2} gd - 2 \) since \( q > 1 \).
4. The pro-torus of Serre, Weil and Milne

4.1. The pro-torus of Serre and Weil, over $\mathbb{Q}$, are defined as follows by their groups of characters equipped with the natural action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

- $X(T_{\text{Serre}})$ is the group of CM types. Recall their definition.
  Let $\text{Gal}(\mathbb{Q}^{cm}/\mathbb{Q})$ be the largest quotient of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in which the complex conjugation $c$ is central; a CM type is a locally constant function $f : \text{Gal}(\mathbb{Q}^{cm}/\mathbb{Q}) \to \mathbb{Z}$ such that $f(\sigma) + f(c \circ \sigma)$ is independent of $\sigma$.

- $X(T_{\text{Weil}})$ is the group of Weil $p$-numbers modulo torsion. Recall that a Weil $p$-number is an algebraic number whose archimedean absolute values are all equal to the same integer power of $\sqrt{p}$, and all of whose non-archimedean non-$p$-adic absolute values.

We refer to [16] for a detailed study of this pro-torus.

4.2. It is well-known that $T_{\text{Serre}}$ is the Tannakian group attached to the category $\text{Hod}_{CM}$ of Hodge structures “of CM type” (cf. [18, p. 57]). Here is one motivic interpretation, resuming the notations of §2.6.

**Theorem 4.2.1.** $T_{\text{Serre}}$ is the Tannakian group of the category $\mathcal{M}(\mathcal{W}_{\mathbb{Q}_p}^\vee)$.

*Proof.* Indeed, the obvious $\otimes$-functor $\mathcal{M}(\mathcal{W}_{\mathbb{Q}}^\vee) \to \text{Hod}_{CM}$ is an equivalence: this will follow from the fact that the group of CM types is generated by those taking values 0 and 1 (CM types of abelian varieties), and from this that every Hodge cycle over an abelian variety is motivated, modeled on $\mathcal{V}_{\mathbb{Q}}$ (Variant 1.5.1). In addition, the $\otimes$-functor $\mathcal{M}(\mathcal{W}_{\mathbb{Q}}^\vee) \to \mathcal{M}(\mathcal{W}_{\mathbb{Q}_p}^\vee)$ is an equivalence. $\square$

4.3. We will demonstrate that $\mathcal{M}_t(\mathcal{W}_{\mathbb{F}_p}^\vee)$ is semi-simple Tannakian of group $T_{\text{Weil}} \otimes \mathbb{Q}(\ell)$ (Theorem 5.1.1 above). Failing to already know that $\mathcal{M}_t(\mathcal{W}_{\mathbb{F}_p}^\vee)$ is semi-simple, we will work with the quotient $\overline{\mathcal{M}}_t(\mathcal{W}_{\mathbb{F}_p}^\vee)$ of the category $\mathcal{M}_t(\mathcal{W}_{\mathbb{F}_p}^\vee)$ by its largest $\otimes$-ideal, which is a semi-simple Tannakian category over $\mathbb{Q}(\ell)$, the functor passing to quotient being conservative. Denote by $T(\ell)$ its Tannakian group in the sense of [9]: a priori, this is a group scheme internal to the category $\text{Ind}_{\ell}(\overline{\mathcal{M}}_t(\mathcal{W}_{\mathbb{F}_p}^\vee))$, but because the objects of $\mathcal{W}_{\mathbb{F}_p}^\vee$ have “a lot of endomorphisms”, this group scheme is abelian (see [18, p. 53]), which allows us to consider it as a usual affine group scheme over $\mathbb{Q}(\ell)$.

The $\otimes$-functor of specialisation of corollary 2.6.2

$$\mathcal{M}(\mathcal{W}_{\mathbb{K}}^\vee) \otimes \mathbb{Q}(\ell) \to \overline{\mathcal{M}}_t(\mathcal{W}_{\mathbb{F}_p}^\vee)$$

between semi-simple Tannakian $\mathbb{Q}(\ell)$-categories is conservative, thus automatically faithful exact, and every object of the target is factor of the image of an object of the source. This functor corresponds thus to a monomorphism

$$T(\ell) \hookrightarrow T_{\text{Serre}} \otimes \mathbb{Q}(\ell).$$

4.4. We have a canonical homomorphism

$$T_{\text{Weil}} \otimes \mathbb{Q}(\ell) \hookrightarrow T(\ell)$$

\[6\]Those are polarizable Hodge structures whose Mumford-Tate group is abelian.
which is built in level of characters, like this. For every \( X \in \mathcal{W}_{\mathbb{Q}} \), choose a field \( \mathbb{F}_p^m \) of definition, and denote by \( Fr_X \) the Frobenius of a model \( X \) over \( \mathbb{F}_p^m \). Denote by \( T(\ell)_X \) be the quotient of \( T(\ell) \) attached to the Tannakian sub-category generated by \( \mathfrak{h}(X) \). Thus it remains to replace \( m \) by a multiple, \( Fr_X^* \) defines an element of \( T(\ell)_X(\mathbb{Q}(\ell)) \), and for every character \( \chi \) of \( T(\ell)_X \), \( \chi(Fr_X^*) \) is a Weil \( p \)-number, hence the homomorphism that we’re looking for.

4.5. In [17,18], Milne constructed, by analogy with the category of motives numerically cutout on \( \mathcal{W}_{\mathbb{Q}} \) and \( \mathcal{W}_{\mathbb{F}_p} \) respectively, a category \(^7\mathcal{L}(\mathcal{W}_{\mathbb{Q}}) \) and a category \( \mathcal{L}(\mathcal{W}_{\mathbb{F}_p}) \) where algebraic correspondances are replaced by “Lefschetz correspondances”, defined as linear combinations of intersections of divisor classes on product varieties. It shows that these are actually semi-simple Tannakian categories over \( \mathbb{Q} \). We will denote \( T_{\mathbb{Q}}^{Milne} \) and \( T_{\mathbb{F}_p}^{Milne} \) the respective Tannakian groups. These are pro-tori, whose characters are explicitly described in [18].

4.6. We have a \( \otimes \)-functor of specialisation \( \mathcal{L}(\mathcal{W}_{\mathbb{Q}}) \rightarrow \mathcal{L}(\mathcal{W}_{\mathbb{F}_p}) \), which gives rise to a monomorphism

\[
T_{\mathbb{F}_p}^{Milne} \hookrightarrow T_{\mathbb{Q}}^{Milne},
\]

and a natural \( \otimes \)-functor \( \mathcal{L}(\mathcal{W}_{\mathbb{Q}}) \rightarrow Hod_{CM} \), which gives rise to a monomorphism

\[
T_{Serre} \hookrightarrow T_{\mathbb{Q}}^{Milne}.
\]

4.7. Otherwise, an analogous construction to that of \( \S 4.4 \) gives rise to a canonical monomorphism

\[
T_{Weil} \hookrightarrow T_{\mathbb{F}_p}^{Milne},
\]

and relying on the theory of Taniyama-Shimura, Milne showed that \( T_{Weil} \) is also contained in \( T_{Serre} \) [18, \( \S 5 \)]. The following result is crucial for the following.

4.7.1. ([18, Theorem 6.1]). \( T_{Weil} = T_{Serre} \cap T_{\mathbb{F}_p}^{Milne} \) in \( T_{\mathbb{Q}}^{Milne} \).

5. Tate cycles and motivated cycles on abelian varieties over a finite field

5.1. We have a natural commutative square of \( \otimes \)-functors

\[
\begin{array}{ccc}
\overline{\mathcal{M}}(\mathcal{W}_{\mathbb{F}_p}) & \xleftarrow{\sim} & \mathcal{L}(\mathcal{W}_{\mathbb{F}_p}) \otimes \mathbb{Q}(\ell) \\
\uparrow & & \uparrow \\
\mathcal{M}(\mathcal{W}_{\mathbb{Q}}) \otimes \mathbb{Q}(\ell) & \xleftarrow{\sim} & \mathcal{L}(\mathcal{W}_{\mathbb{Q}}) \otimes \mathbb{Q}(\ell)
\end{array}
\]

which induces a natural commutative square of monomorphisms of pro-tori

\[
\begin{array}{ccc}
T(\ell) & \longrightarrow & T_{\mathbb{F}_p}^{Milne} \otimes \mathbb{Q}(\ell) \\
\downarrow & & \downarrow \\
T_{Serre} \otimes \mathbb{Q}(\ell) & \longrightarrow & T_{\mathbb{Q}}^{Milne} \otimes \mathbb{Q}(\ell)
\end{array}
\]

\(^7\)Denoted as \( LCM(\mathbb{Q}) \) in [18].
By the previous theorem, we deduce that $T(ℓ) ⊂ T_{Weil} ⊗ \mathbb{Q}(ℓ)$. Otherwise, the parallelism of constructions of §4.4 and 4.7 shows that the canonical monomorphism

$$T_{Weil} ⊗ \mathbb{Q}(ℓ) \hookrightarrow T_{Milne}^p ⊗ \mathbb{Q}(ℓ)$$

is factorized by $T(ℓ)$; in other words, we have the inclusion in the other direction

$$T_{Weil} ⊗ \mathbb{Q}(ℓ) \hookrightarrow T(ℓ),$$

hence finally:

**Theorem 5.1.1.** $T(ℓ) = T_{Weil} ⊗ \mathbb{Q}(ℓ)$.

### 5.2.

We deduce, firstly, that the category $\mathcal{M}_ℓ(W_{\overline{F}_p})$ is semi-simple Tannakian.

**Proposition 5.2.1.** $\mathcal{M}_ℓ(W_{\overline{F}_p}) = \overline{\mathcal{M}_ℓ(W_{\overline{F}_p})}$. In particular, $\mathcal{M}_ℓ(W_{\overline{F}_p})$ is abelian semi-simple.

**Proof.** The proof relies on an argument of Tate [27] included in [18, p. 75]. For every $X ∈ W_{\overline{F}_p}$, choose a model $\overline{X}$ of $X$ over a finite field $F_{p^m}$. Put $M = h(\overline{X}) ∈ \mathcal{M}_ℓ(W_{\overline{F}_p})$ and denote by $\overline{M}$ its image in $\overline{\mathcal{M}_ℓ(W_{\overline{F}_p})}$. It’s about showing that $\text{End} \ M = \text{End} \ \overline{M}$, or, equivalently, that the inequality $\dim_{\mathbb{Q}(ℓ)} \text{End} \ M ≥ \dim_{\mathbb{Q}(ℓ)} \text{End} \ \overline{M}$ is an equality. Consider $Fr_X$ as an endomorphism of $M$, and denote by $\overline{Fr}_X$ the endomorphism corresponding to $\overline{M}$. Since $X$ is an abelian variety, we say that $Fr_X$ acts in a semi-simple way on $H_ℓ(X)$. Moreover, for every fibre functor $\overline{H}$ on $\overline{\mathcal{M}_ℓ(W_{\overline{F}_p})}$, the characteristic polynomial of $Fr_X$ on $H_ℓ(X)$ coincides with the characteristic polynomial of $Fr_X$ on $H_ℓ(X)$. Even replace $m$ with a multiple, we have on the one hand

$$\dim_{\mathbb{Q}(ℓ)} \text{End} \ \overline{M} ≤ \dim_{\mathbb{Q}(ℓ)} \text{End}_{Fr_X} H_ℓ(X),$$

and on the other hand (according to Theorem 5.1.1)

$$\dim_{\mathbb{Q}(ℓ)} \text{End} \ M = \dim \text{End}_{Fr_X} \overline{H}(\overline{M}).$$

or the dimension of $\text{End}_{Fr_X} \overline{H}(\overline{M})$ is also that of $\text{End}_{Fr_X} H_ℓ(X)$ by virtue of a lemma of Tate (cf. [18, p.75]): the dimension of commutant in question depends only on characteristic polynomial. □

**Corollary 5.2.2.** Every $ℓ$-adic Tate cycle on an abelian variety over a finite field $F_{p^m}$ ($ℓ ≠ p$) is a $\mathbb{Q}_ℓ$-linear combination of motivated cycles modeled on $\overline{V}_{\overline{F}_p}$.

**Proof.** By the previous results, $\mathcal{M}_ℓ(W_{\overline{F}_p})$ is a Tannakian category over $\mathbb{Q}_ℓ$ of Tannakian group $T_{Weil} ⊗ \mathbb{Q}_ℓ$. As an $ℓ$-adic cohomology class is fixed by $T_{Weil} ⊗ \mathbb{Q}_ℓ$ if and only if it is fixed by Frobenius, the theorem follows immediately. □

**Remark 5.2.3.**

(1) If we could prove that $\mathbb{Q}_ℓ = \mathbb{Q}$, it would follow that $T(ℓ) = T_{Weil}$, and in fact $\mathcal{M}_ℓ(W_{\overline{F}_p})$ is equivalent to the Tannakian category constructed “abstractly” by Langlands-Rapoport [15] (cf. [16, §3.31] and [20]) (the equivalence in question being unique up to isomorphism). A fortiori, $\mathcal{M}_ℓ(W_{\overline{F}_p})$ would be independent of $ℓ$. 
(2) The equality \( Q(\ell) = \mathbb{Q} \) attached to a positive response to the question 2.5.3 on the reduction of Shimura varieties would answer positively to a question raised by Deligne [10, §6]: two abelian varieties defined over \( \overline{\mathbb{Q}} \) having same reduction \( A \) over \( \mathbb{F}_p \) being given, as well as Hodge cycles over each of them, we can interpret the cycles as \( \ell \)-adic cohomology classes over the common reduction \( A \); their intersection number is rational?

(3) To know that \( Q(\ell) \) is formally real would already be very interesting (see also [20, 7.7]): \( \mathcal{M}_\ell(W_{\mathbb{F}_p}) \) would have a canonical polarization (cf. [16, §2]), and by adapting the arguments of [19], we would get that the standard Weil forms are positive for the canonical polarisation. A fortiori, the standard conjecture of Hodge type would be true for the abelian varieties over finite fields.

**Remark 5.2.4.** It follows from [2] that the standard conjecture of Lefschetz type in characteristic 0 for certain varieties fibred in abelian varieties over a lisse projective curve implies the Hodge conjecture for complex abelian varieties, thus the Tate conjecture for abelian varieties over finite fields via [18]. Likewise, it follows from the results above that the standard conjecture of Lefschetz type over \( \mathbb{F}_p \) for certain varieties fibred in abelian varieties over a lisse projective curve implies the Tate conjecture for abelian varieties over finite fields of characteristic \( p \), and according to [19], every standard conjecture for these abelian varieties (now, we only have the partial result [7]).

6. **Tate cycles on abelian varieties over a field of finite type over a finite field**

6.1. Let \( k \) be a field of finite type over \( \mathbb{F}_p \), and \( \overline{k} \supset \mathbb{F}_p \) a separable closure fixed from \( k \). Let \( A \) be an abelian variety over \( k \). We suppose that \( A \) is generic fibre of an abelian scheme

\[
\overline{X} \stackrel{f}{\rightarrow} \overline{S}
\]

over geometrically connected normal projective variety \( \overline{S} \) over a finite field \( \mathbb{F}_{p^m} \).

**Theorem 6.1.1.** Every \( \ell \)-adic Tate cycle over \( A \) is a \( \mathbb{Q}_\ell \)-linear combination of motivated cycles modeled on \( \overline{Y}_K \). In addition the rigid sub-\( \otimes \)-category of \( \mathcal{M}_\ell(W_k) \) generated by \( A \) is semi-simple Tannakian.

**Proof.** We can find a lisse open \( S \subset \overline{S} \) whose complement is of codimension at least 2. We denote \( X \xrightarrow{f} S \) the restriction of \( f \). Let \( s \) be a \( \overline{k} \)-point of \( S \) of image the generic point, and \( s_0 \) a \( \overline{k} \)-point of image a closed point of \( S \). We are in the situation to apply the theorem 8.4.1 of [3] (in the case (\( a' \)), and with the concordance of notations: \( K = \overline{k}, K_0 = \mathbb{F}_p \)) and its corollary 8.4.3. This corollary implies the second assertion, considering the proposition 5.2.1. Denote by \( G_{X_s} \) and \( G_{X_{s_0}} \) the motivic Galois groups of \( X_s = A_{\overline{k}} \) and \( X_{s_0} \) respectively. According to the theorem 5.1.1, \( G_{X_{s_0}} \) is a torus, and the arithmetic monodromy representation

\[
\text{Gal}(\mathbb{F}_p/\mathbb{F}_{p^n}) \rightarrow G_{X_{s_0}}
\]

(for \( n \geq m \) sufficiently large) is of Zariski-dense image. Denote by \( H_{X_s} \) and \( H_{X_{s_0}} \) the connected geometric monodromy group of \( f_{\overline{k}} \) pointed in \( s \) and in \( s_0 \) respectively. Up to replacing
S by a finite etale cover, we can suppose, and we will suppose, that the monodromy group is connected; $H_{X_s}$ is thus another Zariski closure of the geometric monodromy representation
\[ \text{Gal}(\bar{k}/kF_p) \rightarrow G_{X_s}. \]
According to [3, 8.4.1.3], we have an exact sequence
\[ \{1\} \rightarrow H_{X_s} \rightarrow G_{X_s} \rightarrow T \rightarrow \{1\} \]
where $T$ is the quotient torus $G_{X_{s_0}}/(G_{X_{s_0}} \cap H_{X_{s_0}})$. It is easy to deduce that the arithmetic monodromy representation
\[ \text{Gal}(\bar{k}/kF_{p^n}) \rightarrow G_{X_s} \]
is of Zariski-dense image in $G_{X_s}$, which establishes the theorem. □

6.2. To get rid of the hypothesis of good reduction, it should apply [3, 8.4.1] and [3, 8.4.3] in the case (b’) of [3], which brings us back to question 6.3.1 of [3] on the K"unnemann compactifications.

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