Abstract A conjecture due to Grothendieck predicts that the parallel transport of any monodromy-invariant algebraic cycle class remains an algebraic cycle class. We prove that this follows from the standard conjectures.

We have shown in a previous work a variant of this deformation conjecture in characteristic zero, in which algebraic cycles are replaced by motivated cycles (formal adjunction of inverses of Lefschetz isomorphisms attached to polarizations). We extend this result in any characteristic under various assumptions, for instance if the base is ‘almost complete’.

We give an unconditional construction of motivic Galois group in this context, and study their variation in a family of projective smooth varieties.

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0. INTRODUCTION

0.1. Most of the problems of the theory of algebraic cycles are problems of existence. Their extreme difficulty comes from the rarity, even the absence, of general techniques of construction of algebraic cycles, especially by deformation. The theory of the obstruction to the infinitesimal deformation of sub-schemes is not very effective in the context of cycle classes modulo homological equivalence, where we do not just consider the effective cycles. It only applies, for example, under hypotheses that are not verifiable in practice (cf. [10]) to the deformation by parallel transport predicted by the following conjecture.

Conjecture 0.1.1 (of deformation by Grothendieck\textsuperscript{1}). Let $S$ be a connected scheme of finite type over a field $K$, and let $f : X \to S$ a projective lisse morphism. Let $q$ be a natural integer and $\xi$ a global section of $R^{2q}f_{\text{ét}}^*Q_\ell(q) (\ell \neq \text{char } K)$.

\textsuperscript{1}See note 13 in [18] (variant in de Rham cohomology) and [19, 3.2] (case of abelian schemes)
If for a point \( s \in S(K) \), the image \( \xi_s \) of \( \xi \) in \( H^{2q}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \) is the class of an algebraic cycle, it is the same for every \( K \)-point.

(\( \overline{K} \) designates a fixed separable closure of \( K \).) If \( K = \mathbb{C} \), and in de Rham cohomology rather than \( \ell \)-adic cohomology, this conjecture follows from the Hodge conjecture and is sometimes called “variational Hodge conjecture” (cf. [33]).

0.2. In the article [3], we demonstrated, in characteristic zero, that this conjecture of deformation follows from the standard conjectures of Grothendieck on algebraic cycles [20]. Recall briefly these last.

**Conjecture 0.2.1 (standard of Grothendieck)**. For every projective lisse \( K \)-variety of dimension \( d \),

(I) The Kunneth components \( \pi^i_X \) of the diagonal of \( X \) are classes of algebraic correspondences;

(II) The inverse of the Lefschetz isomorphism

\[
L^ {d-s}_\eta : H^{d}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\cup \eta^{d-s}} H^{2d-s}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)(d-s)
\]

attached to any polarization \( \eta \in H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)(1) \) is induced by an algebraic correspondence;

(III) for every algebraic cycle class \( \alpha \) nonzero on a power of \( X \), \( \langle \alpha, *H\alpha \rangle \) is a strictly positive rational number;\(^3\)

(IV) The homological equivalence of algebraic cycles on every power of \( X \) coincide with the numerical equivalence.

These conjectures are connected by these implications (cf. [25])

\[
(\text{II}) \Rightarrow (\text{I}), \quad (\text{II}) + (\text{III}) \Rightarrow (\text{IV}), \quad (\text{IV}) \Rightarrow (\text{II}).
\]

In characteristic zero, (III) follows from the Hodge index theorem, so that the standard conjectures can be reduced only the only one conjecture (II), called standard conjecture of Lefschetz type.

0.3. In this article, we no longer impose any restrictions on the characteristic of the base field. We prove the following theorem (cf. Theorem 1.2.1 and 7.3.1).

**Theorem 0.3.1.** Let \( S \) be connected scheme of finite type over a field \( K \), and \( f : X \to S \) a projective lisse morphism.

We do one of the following two hypotheses:

(1) \( K \) is perfect and the homological equivalence coincides with the numerical equivalence for the projective lisse varieties over \( K \);

(2) The standard conjecture of Lefschetz type is true for lisse projective varieties over \( K \), and \( S \) is the complementary of a closed of codimension at least 2 in a projective variety \( \overline{S} \).

Let \( q \) be a natural integer and \( \xi \) a global section of \( R^{2q}f_{\text{ét}}^*\mathbb{Q}_\ell(q) \).

If there exists a point \( s \in S(K) \) such that \( \xi_s \in H^{2q}_{\text{ét}}(X_s_{\overline{K}}, \mathbb{Q}_\ell)(q) \) is the class of an algebraic

\[^2\] In [25], these four conjectures are named C,B,I,D respectively.

\[^3\] \( *_H \) designates right here “the Hodge involution” attached to an arbitrary polarization of \( X \) (cf. §2.2), and \( \langle \cdot, \cdot \rangle \) the Poincare duality pairing.
cycle, then it is the same for every point.

0.4. Alas, almost no progress has been made towards the standard conjectures since their formulation, so the theorem 0.3.1 has only one “academic” interest. To turn it into an “applicable” statement, we will appeal to the notion, introduced in [3], of motivated cycle. A motivated cycle is a cohomology class over a projective lisse variety $X$ of the form

$$\text{pr}_{X}^{X'}(\alpha \cup \ast_H(\beta)),$$

where $X'$ is an “auxiliary” projective lisse variety, $\alpha$ and $\beta$ are algebraic cycle classes on the product $X \times X'$, and $\ast_H$ is the algebraic Hodge involution attached to Segre product of a projective embedding of $X$ and of a projective embedding of $X'$.

For example, a motivated cycle on the point is an element of $\mathbb{Q}_{\ell}$ of the form $\langle \alpha \cup \ast_H(\beta) \rangle$. If $\text{char} K = 0$, these are rational numbers (by comparison with the rational Betti cohomology). It is not clear on the other hand if $\text{char} K > 0$; we denote $\mathbb{Q}_{\ell}(\ast)$ the smallest sub-field of $\mathbb{Q}_{\ell}$ containing these numbers.

We have thus the following unconditional statement (cf. Theorem 7.3.1).

**Theorem 0.4.1.** Let $S$ be a connected scheme of finite type over a field $K$, and $f : X \to S$ a projective lisse morphism. If $\text{char} K > 0$, we suppose that $S$ is the complement of a closed of codimension at least 2 in a projective variety $\overline{S}$.

Let $q$ be a natural integer and $\xi$ a global section of $R^{2q}f_{\text{ét}}^*\mathbb{Q}_\ell(q)$.

There exists a point $s \in S(K)$ such that $\xi_s \in H^{2q}_{\text{ét}}(X_s,\mathbb{Q}_\ell)(q)$ is a motivated cycle, up to multiplication by an element of $\mathbb{Q}_\ell^\times$, thus it is the same for every point.

The case of characteristic 0 already appeared in [3, § 5]; but the method proposed here (if $S$ is “almost” projective) is more effective than that of [3].

Under certain hypotheses (for example in the case of an abelian scheme over a lisse projective curve and for well-chosen polarizations), we show that the parallel transport from $s$ to $t$ is given by the formula $\ast_H t_s^* \ast_H t_{s,s}$ (where $t_s$ designates the inclusion of $X_s$ in $X$), and is a motivated correspondence (cf. Corollary 7.1.2).

0.5. Since $\text{char} K = 0$, the category of (pure) motives built using motivated correspondences is semi-simple Tannakian [3, § 4], thus the reductive motivic Galois group. We do not know if the same is true in any characteristic, but the techniques of monoidal splitting of [8,9] allows us again to define unconditionally reductive motivic Galois groups (see § 4 for the precise definition). One of the principal results of the article concerns their variation in family (cf. Theorem 8.4.1); it’s the following.

**Theorem 0.5.1.** Let $S$ be a connected lisse scheme over a separably closed field $K_0$, and $f : X \to S$ a projective lisse morphism. We suppose that $S$ is the complement of a closed of codimension at least 2 in a projective scheme $\overline{S}$ over $K_0$. Let $K$ be separably closed extension of $K_0$ in which $K_0(S)$ is embeddable. We denote by $\text{Exc}$ the set of $K$-points $s$ such that the motivic Galois group $G_{X_s}$ of the fibre $X_s$ does not contain the neutral component $H_{X_s}$ of the Zariski closure of the monodromy. Thus:

(0) $\text{Exc}$ does not contain any generic $K$-point of $S$;

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4We will refine in the following the notion of motivated cycles in order to minimize this field (cf. § 3).  
5For other applications of the theory of motivated cycles, we can consult [2],[4] and [5,III], as well as [6, Chapter 9,10] for a panorama of the theory and its applications.  
6$H_{X_s}$ is a semi-simple $\mathbb{Q}_\ell$-group.
(1) For every $s \notin \text{Exc}$, $H_{X_s}$ is a normal sub-group of $G_{X_s}$;
(2) Suppose that there exists $s_0$ such that the category of motives\footnote{There are motives built in terms of motivated correspondences.} cutout on the powers of $X_{s_0}$ is abelian semi-simple; hus the same is true for the corresponding category for $X_s$, for every $s \notin \text{Exc}$, and $G_{X_s}$ is independent of $s \notin \text{Exc}$ up to isomorphism;
(3) If in addition $G_{X_{s_0}}$ is a torus, thus for every $s \notin \text{Exc}$ we have an exact sequence
$$\{1\} \rightarrow H_{X_s} = G_{X_s}^{\text{der}} \rightarrow G_{X_s} \rightarrow T \rightarrow \{1\}$$
where $T$ is a quotient torus of $G_{X_{s_0}}$ (independent of $s$).

0.6. The above shows that the replacement of algebraic cycles by motivated cycles simplifies and made a certain number of problems accessible. However, it also gives rise to some new problems. For example, the appearance of field $\mathbb{Q}(\ell)$; or, the specialization: it is not clear that a motivated cycle over a lisse projective variety $X$ having good reduction specializes to a motivated cycle, because by definition, a motivated cycle involves an auxiliary variety $X'$ which could have bad reduction. This problem is analyzed in § 9.1 and resolved in a particular situation in § 9.2, thanks to the theorem of deformation the theorem 0.4.1 of motivated cycles.

0.7. The main applications in view of these results concern the abelian varieties in characteristic nonzero, and are subject of the following article [7].

1. Deformation of algebraic cycles and standard conjectures

1.1. We fix a field $K$, a separable closure $\overline{K}$ of $K$, and a prime number $\ell \neq \text{char } K$, for every projective lisse $K$-variety $Y$, we will write $H^i_{\ell}(Y)$ instead of $H^i_{\text{et}}(Y_{\overline{K}}, \mathbb{Q}_{\ell})$ for abbreviation, and $H^i_*(Y) = \bigoplus_j H^j_{\ell}(Y)$.
All our algebraic cycles will be with coefficients in $\mathbb{Q}$, and we will say sometimes abusively “algebraic cycle” instead of “(cohomology) class of algebraic cycle”. Taking rational coefficients rather than integer coefficients allows especially to apply Galois descent: a class of algebraic cycle over $Y_{\overline{K}}$ invariant under $\text{Gal}(\overline{K}/K)$ is the class of an algebraic cycle on $Y$.
We will use in an essential way the global theorem of invariant cycles.

Theorem 1.1.1 (Deligne). Let $S$ be a lisse connected scheme over $\overline{K}$, $f : X \rightarrow S$ a lisse projective morphism, and $\overline{X}$ a lisse projective compactification of $X$. Therefore for every $j$, the natural map
$$u_j : H^j_{\ell}(\overline{X}) \rightarrow H^0(S, R^jf_*\mathbb{Q}_{\ell})$$
is surjective.

It’s about showing that if $s \in S(\overline{K})$, the natural map
$$v_j : H^j_{\ell}(\overline{X}) \rightarrow H^j(X_s)^{\pi_1(S,s)}$$
is surjective. If $\text{char } \overline{K} = 0$, we go back to the case $\overline{K} = \mathbb{C}$, and the statement figures in [14,4.1.1]: it follows from the degeneration of the Leray spectral sequence, attached to the theory of weights. If $\overline{K} = \mathbb{F}_p$, the same argument applies, in taking “weights” in the sense of [15]. If $\text{char } \overline{K} = p$, we go back to the case of $\mathbb{F}_p$ by a standard specialization argument (the
In this paragraph, we prove the theorem 0.3.1 in the case (1), that is to say the following statement.

**Theorem 1.2.1.** We suppose $K$ perfect. We suppose that the homological equivalence coincide with the numerical equivalence for the projective lisse varieties over $K$.

Let $S$ be a connected scheme of finite type over $K$, and let $f : S \to S$ be a projective lisse morphism. Let $q$ be a natural integer and $\xi$ a global section of $R^2q_{\text{ét}}^* \mathbb{Q}_\ell(q)$. If there exists a point $s \in S(K)$ such that $\xi_s \in H^2q_{\text{ét}}(X_s)(q)$ is an algebraic cycle, then it is the same at every point.

By Galois descent, we reduce to the case $K = \overline{K}$. By joining $s$ to another given $K$-point $t$ by a chain of curves and by normalizing, we can also suppose that $S$ is a connected lisse affine curve.

To lighten the writing, we will dispense with writing the twists in the proofs of this paragraph.

**1.3.** We will first treat the case where $X$ is open dense of a lisse projective $K$-scheme $\overline{X}$. In this case, we have the following more precise statement.

**Proposition 1.3.1.** Let $S$ be a lisse connected scheme over $K = \overline{K}$, $f : X \to S$ a lisse projective morphism, and $\overline{X}$ a lisse projective compactification of $X$. Let $s,t$ be two $K$-points of $S$.

Suppose that on every finite product of factors $\overline{X}, X_s$ and $X_t$, the homological equivalence coincides with the numerical equivalence. Therefore

(i) The isomorphism “of parallel transport”

$$\Pi_{s,t} : H^*_\ell(X_s)^{\pi_1(S,s)} \cong H^*_\ell(X_t)^{\pi_1(S,t)}$$

between invariant subspaces under the monodromy is of motivic nature: this is the $\ell$-adic realisation of an isomorphism of sub-motives of $h(X_s)$ and of $h(X_t)$ respectively (these are Grothendieck motives for the homological or numerical equivalence);

(ii) Every global section $\xi$ of $R^2f_{\text{ét}}^* \mathbb{Q}_\ell(q)$ such that $\xi_s$ is an algebraic cycle comes from an algebraic cycle over $\overline{X}$.

**Proof.** Recall the argument, given in [3,5.1].

We denote $i_s : X_s \hookrightarrow \overline{X}$, $i_t : X_t \hookrightarrow \overline{X}$ the inclusions. The global theorem of invariant cycles says that the canonical map

$$u : H^*_\ell(\overline{X}) \to \bigoplus_j H^0(S, R^j f_\ast \mathbb{Q}_\ell)$$

is surjective. It follows that the maps

$$i_s^* : H^*_\ell(\overline{X}) \to H^*_\ell(X_s), \quad i_t^* : H^*_\ell(\overline{X}) \to H^*_\ell(X_t)$$

have the same kernel, namely $\text{Ker} \ u$, and that

$$\text{Im}(i_s^*) = H^*_\ell(X_s)^{\pi_1(S,s)}, \quad \text{Im}(i_t^*) = H^*_\ell(X_t)^{\pi_1(S,s)}.$$
Under the hypothesis that the homological equivalence coincides with the numerical equivalence on every product of copies of $\overline{X}, X_s, X_t$, the category of Grothendieck motives (relatively to the $\ell$-adic homological equivalence) cutout on such products is abelian and even Tannakian over $\mathbb{Q}$, and the $\ell$-adic cohomology defines an exact functor (and likewise a fibre functor) towards the $\mathbb{Q}_\ell$-vector spaces. There thus exists a motive $M$ whose realisation is the co-image of $u$ (i.e. the quotient of $H^*_\ell(\overline{X})$ by the kernel of $u$); in addition $\iota_\ast$ (respectively, $\iota_\ast^!$) induces an isomorphism of $M$ over its image, whose realisation is none other than $H^*_\ell(X_s)_{\pi_\ast(S,s)}$ (respectively, $H^*_\ell(X_t)_{\pi_\ast(S,s)}$). By composing the inverse of the first isomorphism with the second, we obtain an algebraic correspondence: it is $\Pi_{s,t}$. This establishes (i).

For (ii), we observe that the algebraic cycle $\xi_s$ provides a sub-motive $\cong 1$ of $\iota_\ast(M)(q)$. As the category of motives cutout on the product of copies of $\overline{X}$ and $X_s$ is semisimple, this is the image under $\iota_\ast$ of a sub-motives $\cong 1$ of motive of $\overline{X}$ twisted by $(q)$. Thus $\xi_s$ is the image under $\iota_\ast$ of an algebraic cycle over $\overline{X}$.

\[ \square \]

### 1.4

To prove the theorem 1.2.1, we will need an intermediary statement where we tolerate singular fibres.

**Proposition 1.4.1.** Let $\overline{S}$ be a connected lisse projective curve over a field $K = \overline{K}$, and let $\overline{f} : \overline{X} \to \overline{S}$ be a flat projective morphism. Let $S$ be an open of $\overline{S}$, and let $S_-$ the open of $S$, supposed non-empty, above which $\overline{f}$ is lisse. Denote $f$ and $f_-$ the restriction of $\overline{f}$ on $S$ and $S_-$ respectively.

Let $s \in S(K)$ such that the reduced irreducible components of fibres $\overline{X}_s$ are smooth divisors, and denote $\overline{X}_s$ the normalization of $\overline{X}_s$.

Suppose that on every finite product of factors of type $\overline{X}$ or components of $\overline{X}_s$, the homological equivalence coincides with the numerical equivalence.

Let $q$ be a natural integer and $\xi$ a global section of $R^{2q}f_\ell\ast\mathbb{Q}_\ell(q)$.

If the image $\overline{\xi}_s$ of $\xi$ in $H^{2q}_\ell(\overline{X}_s)(q)$ is an algebraic cycle, then the restriction $\xi_-$ of $\xi$ to $H^0(S_-, R^{2q}f_-\ell\ast\mathbb{Q}_\ell)(q)$ comes from an algebraic cycle on $\overline{X}$.

**Proof.** The proof is analogous to that of point (ii) of the proposition 1.3.1; after being brought back to $K = \overline{K}$, we must take into account, in addition, the following fact.

The kernel of $\overline{\iota}_\ast$ is contained in the kernel of the canonical map

$$u : H^*_\ell(\overline{X}) \to \bigoplus_i H^0(S_-, R^i f_-\ell\ast\mathbb{Q}_\ell).$$

Prove this fact. Let’s write $X_s = \sum e_j D_j$, with $e_j \in \mathbb{Z}_{\geq 0}$, $D_j$ integral lisse divisor of $X$, and denote $\iota_j$ the inclusion of $D_j$ in $\overline{X}$. We have $\overline{f}_\ast \eta_{\overline{S}} = \delta \sum e_j (\iota_j)_\ast(1)$ for suitable $\delta$ (the degree of the polarization of $\overline{S}$). We have thus

$$L_{\text{hor}} := \bigcap \overline{f}_\ast \eta_{\overline{S}} = \delta \sum e_j (\iota_j)_\ast \iota_j^\ast.$$

where $\ker \overline{\iota}_\ast \subset \ker L_{\text{hor}}$.

The operator $L_{\text{hor}}$ cancels on $\ker u$; we will denote

$$v : \bigoplus_j H^0(S_-, R^j f_-\ell\ast\mathbb{Q}_\ell) \to \bigoplus_j H^{j+2}_\ell(\overline{X})$$
the map that it induces, so that $L_{\text{hor}} = vu$. We conclude that $\ker \tilde{\iota}_s \subset \ker u$ by the following lemma.

**Lemma 1.4.2.**

(i) $v$ is injective and $uv = 0$.

(ii) $v$ and $\delta \cdot u$ are transposes relative to the Poincare duality pairing $\langle -, \cdot \rangle$.

**Proof.** Let $t \in S_{-}(K)$. We have $L_{\text{hor}} = \delta \iota_t \iota_t^*$ and $\ker u = \ker \iota_t^*$. As $\iota_t^* \iota_t = 0$, we obtain $uv = 0$. Identify

$$\bigoplus_j H^0(S, R^j f_* \mathbb{Q}_\ell)$$

with $H^*_\ell(X_t)^{\pi_1(S,t)}$, $u$ is identified to $\iota_t^*$ and $v$ to its transpose $\iota_t$ multiplied by $\delta$. Since $u$ is injective, $v$ is surjective. This establishes (i) and (ii). \[\square\]

1.5. Reduction of Theorem 1.2.1 to the proposition 1.4.1

We can suppose $f$ purely of relative dimension $d$, that $K$ is algebraically closed, and that $S$ is an open (containing $s$ and another fixed $K$-point $t$) of a lisse projective curve $\overline{S}$. Choose an extension $\overline{f} : \overline{X} \to \overline{S}$ of $f$, with $\overline{f}$ projective flat. Failure to know if we can choose $\overline{X}$ lisse, we apply the theorem of de Jong [13, 4.1]. By virtue of this theorem, there exists a generically étale\(^8\) projective alteration $\pi : \overline{X}' \to \overline{X}$ (of generic degree $\mu$), with $\overline{X}'$ projective lisse, and such that the irreducible components of $\overline{X}_s$ and $\overline{X}_t$ are lisse divisors of $\overline{X}'$. We have thus the following cartesian diagram:

$$\begin{array}{ccc}
X' & \hookrightarrow & \overline{X}' \\
\pi \downarrow & & \downarrow \pi \\
X & \hookrightarrow & \overline{X} \\
\overline{f} \downarrow & & \downarrow \overline{f} \\
S & \hookrightarrow & \overline{S}
\end{array}$$

where the horizontal arrows are open immersions, the vertical arrows are projective, $\overline{f} = \overline{f} \circ \pi$ is generically lisse. We put $f' = f \circ \pi$, and we denote $S'$ the open dense of $S$ above which $f'$ is lisse (we do not assume that $s$ or $t$ is in $S_{-}(K)$).

As $\pi$ is a generically finite and flat morphism between lisse $K$-schemes of dimension $d + 1$, we have a trace morphism $R\pi_* \mathbb{Q}_\ell \to \mathbb{Q}_\ell$, whose composite with the canonical morphism $\mathbb{Q}_\ell \to R\pi_* \mathbb{Q}_\ell$ is the multiplication by $\mu$. We deduce two linear maps

$$H^0(S, R^{2q} f_* \mathbb{Q}_\ell) \xrightarrow{\pi_*} H^0(S, R^{2q} f'_* \mathbb{Q}_\ell) \xrightarrow{\pi_*} H^0(S, R^{2q} f_* \mathbb{Q}_\ell)$$

whose composite is again the multiplication by $\mu$. Denote by $\xi'$ the restriction of $\pi^* \xi$ to $H^0(S_{-}', R^{2q} (f')_* \mathbb{Q}_\ell)$.

---

\(^8\)This is where the hypothesis that $K$ is perfect in the theorem 1.2.1 is ultimately used.
We have a canonical morphism \( \tilde{\pi}_s \) which makes the diagram commute

\[
\begin{array}{ccc}
\tilde{X}'_s & \longrightarrow & X'_s \\
\downarrow \tilde{\pi}_s & & \downarrow \pi_s \\
\tilde{X}_s & \longrightarrow & X_s 
\end{array}
\]

and we have thus \( \tilde{\pi}_s^*(\tilde{\xi}_s) = (\pi^*\xi)_s \).

If \( \xi_s \) is algebraic, it is thus the same with \( \tilde{\pi}_s^*(\pi^*\xi)_s \).

To complete the reduction, it suffices thus to prove that if \( \xi'_s \) comes from an algebraic cycle \( \theta' \) on \( \tilde{X}' \), thus

\[
\xi_t = \mu^{-1} \cdot (\pi^*\theta')_t
\]

on \( \tilde{X} \) (the right-hand-side being clearly an algebraic cycle). As the two sides are clearly invariant under the monodromy, it suffices, by parallel transport, to prove this formula at an arbitrary \( K \)-point of \( S \). We can thus suppose \( t \in S_+(K) \), and thus the formula follows immediately from that \( \pi^*(\mu^{-1}\pi^*\theta) = \theta' \). This completes the proof of the reduction.

Leave to replace \( (f, \xi) \) by \( (f', \xi') \), this lemma reduces the theorem 1.2.1 to the proposition 1.4.1, which ends the proof of the theorem.

1.6. Here is another statement\(^9\) in the same vein as Theorem 1.2.1.

**Variant 1.6.1.** (We suppose that the homological equivalence coincides with the numerical equivalence over the base field \( K \) supposedly perfect.)

Let \( g: Y \to S \) be another lisse projective morphism. We give ourselves a finite sequence of points \( s_1, s_2, \cdots \) of \( S(K) \), and algebraic cycles \( \alpha_m \in H^{2p}(X_{s_m}, \mathbb{Q}_\ell)(p), \beta_m \in H^{2q}(Y_{s_m}, \mathbb{Q}_\ell)(q) \).

We suppose that there exists a unique morphism of local systems

\[
\chi: R^{2q}f_*\mathbb{Q}_\ell(p) \to R^{2q}g_*\mathbb{Q}_\ell(q)
\]

such that \( \chi(\alpha_m) = \beta_m \) for every \( m \). Therefore for every \( s \in S \), \( \chi_s \) is an algebraic correspondence.

**Proof.** The argument is analogous, based on the fact that under the standard conjecture accepted, the category of Grothendieck motives for the \( \ell \)-adic homological equivalence is abelian. We only give it here under the simplifying hypothesis that \( X \times_S Y \) is open dense of a projective lisse \( K \)-scheme \( \tilde{X} \times_S Y \). Let \( d \) be the relative dimension of \( f \), and put \( n = d + q - p \). By the global theorem of invariant cycles, the map

\[
H^{2n}_\ell(X \times_S Y)(n) \to \bigoplus_{j-i=2(n-d)} \text{Hom}_S(R^j f_*\mathbb{Q}_\ell(p), R^i g_*\mathbb{Q}_\ell(q))
\]

is surjective. For \( t = s \) (fixed) or one of the points \( s_m \), consider the morphism of motives

\[
i^*_t: h^{2n}(\tilde{X} \times_S \tilde{Y})(n) \to h^{2n}(X_t \times Y_t)(n) \cong \bigoplus_{j-i=2(n-d)} \text{Hom}(h^i(X_t)(p), h^j(Y_t)(q)).
\]

\(^9\)Which was suggested to us by Deligne
The kernel of $\iota_t$ is the same for $t = s$ and for $t = s_m$. It follows that
\[
\mathcal{K} = \ker((\pi^{2d-2p}_{X_t} \otimes \pi^{2q}_{X_t}) \circ \iota_t^*)
\]
is the same for $t = s$ and for $t = s_m$. The $\ell$-adic realisation of $h^{2n}(\overline{X \times_S Y})(n)/\mathcal{K}$ is nothing else than $\text{Hom}_S(R^{2p}f_\ast Q_\ell(p), R^{2q}f_\ast Q_\ell(q))$.

Otherwise, the evaluation map
\[
\text{Hom}_S(R^i f_\ast Q_\ell(p), R^j f_\ast Q_\ell(q)) \to \bigoplus_m H^{2q}_\ell(Y_{s_m})(q)
\]
given by
\[
\phi \mapsto \begin{cases} 
(\cdots, \phi(\alpha_m), \cdots) & \text{if } i = 2p, j = 2q \\
0 & \text{if not}
\end{cases}
\]
is the realisation of a morphism of motives
\[
\kappa : h^{2n}(\overline{X \times_S Y})(n)/\mathcal{K} \to \bigoplus_m h^{2q}(Y_{s_m})(q),
\]
and $\beta := (\cdots, \phi(\beta_m), \cdots)$ is a generator of the realisation of a sub-motive $\cong \mathcal{K}$ of $h^{2q}(Y_{s_m})(q)$.

As $\chi$ is the unique element of $\text{Hom}_S(R^{2q}f_\ast Q_\ell(p), R^{2q}f_\ast Q_\ell(q))$ such that $\kappa(\chi) = \beta$, it follows that $\chi$ is a generator of the realisation of a sub-motive $\cong \mathcal{K}$ of $h^{2n}(\overline{X \times_S Y})(n)/\mathcal{K}$. So $\chi_s$ is an algebraic correspondence.

\section{Reminders about the motivated cycles and correspondances}

\subsection{2.1}
Let $K$ be a field. We note by $\mathcal{P}_K$ the category of lisse projective schemes over $K$.

For every $X \in \mathcal{P}_K$ of dimension $d$, equipped with a polarization\footnote{Right here, a polarisation will be a $\mathbb{Q}_\ell^+$-multiple of a Gal($K/K$)-invariant class of ample invertible fibre over $X_X$; or, which amounts to the same, a $\mathbb{Q}_\ell^+$-multiple of the class of an ample divisor on $X$.} $\eta \in H^d_\ell(X)(1)$, we define the \textit{Lefschetz involution} $*_{L,\eta}$ on $\bigoplus_{r,s} H^s_\ell(X)(r)$ as being given by the
\[
L^{d-s}_\eta : H^s_\ell(X)(r) \xrightarrow{\cup \eta^{d-s}} H^{2d-s}(X)(d - s + r)
\]
if $s \leq \dim X$, and by the inverse of $L^{s-d}_\eta$ if $s > \dim X$ (cf. [3]).

We recall that $x \in H^j_\ell(X)(r)$ is said primitive (relatively to $\eta$) if $L^{d+1-j}_\eta x = 0$, and that every element $x \in H^j_\ell(X)(r)$ is written in a unique way
\[
x = \sum L^{k}_{\eta} x_{j-2k}
\]
where $x_{j-2k} \in H^{j-2k}_\ell(X)(r + k)$ is primitive (\textit{Lefschetz decomposition}).

\subsection{2.2}
We define thus the \textit{Hodge involution}\footnote{In Betti cohomology with real coefficients, this involution coincides with the classical Hodge operator $*$ on the cycles of type $(p,p)$ [3.1.1].} $*_H,\eta$ on $\bigoplus_{r,s} H^s_\ell(X)(r)$ by the formula
\[
*_H,\eta x = \sum (-1)^{(j-2k)(j-2k+1)/2} \frac{k!}{(d-j+k)!} L^{d-j+k}_\eta x_{j-2k}.
\]
It is this involution that will play a crucial role in the following (the coefficient $k!/(d-j+k)!$ having all its importance, see in particular the proposition 5.2.2).

The endomorphism $\ast_{H,\eta} L_\eta \ast_{H,\eta}$ is often denotes as $c\Lambda_\eta$. Thus
\[ (c\Lambda_\eta, h = \sum_{0}^{2d} (d - j)\pi_X, -L_\eta) \]
form a $sl_2$-triplet in the sense of [11, VIII.11.1] ($\pi_X$ denotes the Kunneth projectors of $X$): we have a representation Lie algebra $sl_2$ on $H^*_\ell(X)$ given by

\[
\begin{pmatrix}
0 & 1 \\
0 & 0 \\
\end{pmatrix} \mapsto c\Lambda_\eta, \quad \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix} \mapsto h, \quad \begin{pmatrix} 0 & 0 \\
-1 & 0 \end{pmatrix} \mapsto -L_\eta
\]

**Remark 2.2.1.** It is often desirable to consider the space of finite dimension $H^*_\ell(X)$ as $Q[L_\eta]$-module without wanting to sacrifice the twists. To do this, instead of fixing an isomorphism $Q(1) \cong Q_\ell$, we can consider all at once, which amounts to associating to $L_\eta$ a nilpotent action on $H^*_\ell(X)$ of a Lie algebra $L_\ell$ of dimension 1. For every $V \in \text{Vec}_{Q_\ell}$, write $V(\ell)$ for $V \otimes_K L_\ell$, as in [15, 1.6.14]. We thus find the “lost twists”: the nilpotent action in question is written $H^*_\ell(X)(1) \rightarrow H^*_\ell(X)$, and the Lefschetz isomorphism can be written canonically
\[ H^*_\ell(X)(r) \rightarrow H^{2d-s}_\ell(X)(d-s+r) \]
as above.

**Lemma 2.2.2.**

1. We have $Q[L_\eta, *_{L,\eta}] = Q[L_\eta, *_{H,\eta}] = Q[L_\eta, c\Lambda_\eta]$, and this $Q$-algebra contains the Kunneth projectors of $X$.
2. Let $X' \in \mathcal{P}_K$ equipped with a polarization $\eta'$. Thus $*_{\eta} \otimes *_{\eta'}$ is expressed as a polynomial (non commutative) in $L_\eta, L_{\eta'}$ and $*_{\eta \otimes 1 + 1 \otimes \eta'}$ with rational coefficients independant of $\ell$ (this is true for both the Hodge involution and the Lefschetz involution.)
3. We have $*_{H,\eta \otimes 1 + 1 \otimes \eta'} = (-1)^{ij} *_{H,\eta} \otimes *_{H,\eta'}$ on the Kunneth component $H^i(X) \otimes H^j(X')$.
4. Let $[\Delta_X]$ be the class of the diagonal of $X \times X$. Therefore
\[ Q[L_\eta \otimes 1 + 1 \otimes \eta, *_{H,\eta \otimes 1 + 1 \otimes \eta}][[\Delta_X]] \sup Q[L_\eta, *_{H,\eta}] . \]

**Proof.**

1. Cf. [3, 1.2].
2. Cf. [3, 1.3.2].
3.\(^{12}\) We have the decomposition of $sl_2$-modules $H^*_\ell(X) = \bigoplus_j S^{d-j} \otimes P^j(X)$, where $P^j(X)$ denotes the space of primitive elements of $H^j_\ell(X)$ and $S^{d-j}$ denotes the $(d-j)$-th power of the standard representations of $sl_2$. Otherwise, it follows from [11, VIII.1.5] that the action of Weyl element
\[ w := \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix} \in \text{SL}_2 \]
on the component $S^{d-j} \otimes P^j(X)$ is equal to that of $*_{H}$ down to the nearest sign $\epsilon(j) := \frac{1}{2} j(j+1)$.

\(^{12}\)Figure in [3, 1.3.2], but the proof is given (in 1.3.1) for only one component of the Clebsch-Gordan decomposition.
With the convention of the previous remark, the decomposition of Clebsch-Gordan attached to the Kunneth isomorphism provides a canonical isomorphism

\[ P^{j''}(X \times X') = \bigoplus_{j'' \in Q(j,j')} P^j(X) \otimes P^{j'}(X') \left( \frac{1}{2}(j + j' - j'') \right) \]

where

\[ Q(j,j') = \{ j'' | j + j' \leq j'' \leq d + d' - |j - j'|, j'' \equiv j + j' \mod 2 \} \]

(cf. [15, 1.6.14.4], with the dictionary \( P^{d-i} = P_{-i} \)).

On \( P^j(X) \otimes P^{j'}(X') \), \( *_{H,\eta} \otimes *_{H,\eta'} \) acts as \( \epsilon(j)\epsilon(j')w \), while on \( P^{j''}(X \times X') \) shifted in weight by \( (j + j' - j'') \), \( *_{H,\eta \otimes 1 + 1 \otimes \eta'} \) acts as \( \epsilon(j'' + (j + j' - j''))w = \epsilon(j + j')w \). So we find that these two operators coincide up to sign \( \epsilon(j + j')\epsilon(j)\epsilon(j') = (-1)^{j''} \) and we conclude noting that the cohomological degree of every element of \( S^{d-i} \otimes P^j(X) \) is of the same parity as \( j \) (the same applies to \( X' \) and \( X \times X' \)).

(4) In applying [25, 1.3.4], we find

\[ L_{\eta \otimes 1 + 1 \otimes \eta'}([\Delta_X]) = (L_{\eta} \otimes \text{Id} \otimes L_{\eta})([\Delta_X]) = 2L_{\eta}, \]
\[ \epsilon \Lambda_{\eta \otimes 1 + 1 \otimes \eta'}([\Delta_X]) = (\epsilon \Lambda_{\eta} \otimes \text{Id} \otimes \epsilon \Lambda_{\eta})([\Delta_X]) = 2^\epsilon \Lambda_{\eta}. \]

We conclude with the point (1). \( \Box \)

2.3. Fix a full sub-category \( \mathcal{V} \) of \( \mathcal{P}_K \), stable under products and finite disjoint sums, and passing to connected components. Let \( E \) be a field contained in \( \mathbb{Q}_l \).

A **motivated cycle** on \( X \in \mathcal{V} \) (in \( \ell \)-adic cohomology, modeled on \( \mathcal{V} \)), with coefficients in \( E \), is an element \( \xi \in \bigoplus_r H^{2r}_\ell(X)(r) \) of the form

\[ \xi = \text{pr}^{X'}_X (\alpha \cup \beta), \]

where \( X' \in \mathcal{V}, \alpha \) and \( \beta \) are \( E \)-linear combinations of classes of algebraic cycles on \( X \times X' \), and \( * = *_{L,\eta \otimes 1 + 1 \otimes \eta'} \) is the Lefschetz involution of \( X \times X' \) attached to arbitrary polarizations \( \eta \) and \( \eta' \) of \( X \) and \( X' \) respectively. (cf. [3]).

Under the point (1) of previous lemma, it would be the same to replace the Lefschetz involution by the Hodge involution. \( \Box \)

We define from the notion of motivated correspondence (motivated cycle on a product), and we establish the stability by composition [3, 2.1].

**Lemma 2.3.1** ([3, 2.2]). The Lefschetz involutions and Hodge involutions (attached to an arbitrary polarization) as well as the Kunneth projectors of \( X \) are motivated correspondences of \( X \) towards \( X \).

2.4. Let \( \mathcal{W} \) be a full sub-category of \( \mathcal{V} \) stable under disjoint union and products, and passing to connected components.

Let \( \mathcal{M}_{\ell}(\mathcal{W})_\eta[E] \) be the category of pure motives constructed in [3] in terms of motivated correspondences (in \( \ell \)-adic cohomology, modeled on \( \mathcal{V} \)), with coefficients in \( E \), cutout on the objects of \( \mathcal{W} \); we take care that the auxiliary varieties \( X' \) intervening in the definition

\( ^{13} \)We omit the mention “with coefficients in \( E \) as well as the symbol \( [E] \) if \( E = \mathbb{Q} \).

\( ^{14} \)It does not follow immediately from the fact that these two involutions are proportional on each Lefschetz component, since we do not know if the Lefschetz components of a (class of) an algebraic cycle are algebraic cycles.
of motivated correspondances are not supposed to belong to \( \mathcal{W} \) but only to \( \mathcal{V} \) (so, even if every object of \( \mathcal{W} \) satisfies the standard conjecture of Lefschetz type, it is not clear that the morphisms of \( \mathcal{M}_{\ell}(\mathcal{W})_\mathcal{V} \) are algebraic correspondances). It depends a priori on \( \ell \).

This is a rigid \( E \)-linear pseudo-abelian \( \otimes \)-category; the constraint of commutativity is that induced by the \( \ell \)-adic realisation functor; with respect to the “naive” constraint given by the exchange of factors \( X \times Y \cong Y \times X \), so there is a sign given by the usual Koszul rule.\(^{15}\)

We have a natural functor

\[
\mathcal{W}^{op} \rightarrow \mathcal{M}_{\ell}(\mathcal{W})_\mathcal{V}[\mathbb{Q}]
\]

If \( K \) is of characteristic zero and \( E = \mathbb{Q} \), we show that \( \mathcal{M}_{\ell}(\mathcal{W})_\mathcal{V}[\mathbb{Q}] \) is in fact a semi-simple Tannakian category over \( \mathbb{Q} \), independent of \( \ell \) (cf. [3]). We denote \( \mathcal{M}(\mathcal{W})_\mathcal{V} \).

This allows for example to prove unconditionally the analogue of the variant 1.6.1 (in characteristic 0), replacing algebraic cycles by motivated cycles.

3. A refinement of the theory of motivated cycles

3.1. By definition, \( \text{End} \, 1 \) is the sub-\( E \)-algebra of \( \mathbb{Q}_\ell \) formed of \( E \)-linear combinations of “numbers” \( \langle \alpha, \ast_H \beta \rangle \), where \( \langle \alpha, \beta \rangle \) runs through the couples of (classes of) algebraic cycles on the objects \( X \) of \( \mathcal{V} \), and where \( \ast_H \) is the Hodge involution attached to an arbitrary polarisation\(^{16}\) of \( X \) (we can limit ourselves to \( \alpha = \beta \) by bilinearity).

If \( K \) is of characteristic zero, we know (via the Betti cohomology and the comparison isomorphisms) that the “numbers” \( \langle \alpha, \ast_H \alpha \rangle \) are rational numbers and independent of \( \ell \), and thus that \( \text{End} \, 1 = E \). We conjecture it is the same for arbitrary \( K \) (this would arise from the standard conjecture predicting that the \( \ast_H \alpha \) are classes of algebraic correspondances), but we don’t know now if, for \( E = \mathbb{Q} \), \( \text{End} \, 1 \) depends or not on \( \ell \).

The algebra \( \text{End} \, 1 \) grows a priori with \( \mathcal{V} \), and there is therefore an interest to choose the smallest possible \( \mathcal{V} \). Otherwise, the theorems of deformation of motivated cycles of [3]—and those we will see later—lead to choose in general \( \mathcal{V} \) bigger than \( \mathcal{W} \).

However, we notice that the auxiliary varieties \( X' \) often only intervene via a small direct factor of its cohomology. Hence the idea of refining the notion of motivated cycle as follows.\(^{17}\)

3.2. We give ourselves a class \( \tilde{\mathcal{V}} \) of couples \( (X, V) \) where the \( X \) is in \( \mathcal{P}_K \), and \( V \) is a graded subspace of \( H^\bullet_\mathcal{E}(X) \) such that the following condition is satisfied.

(Pol) The set \( \text{Pol}_\mathcal{V} \) of polarizations \( \eta \in H^2_\mathcal{E}(X)(1) \) such that \( V \) is direct factor of \( H^\bullet_\mathcal{E} \) as \( \mathbb{Q}[L_\eta] \)-module is non-empty.

For every \( \eta \in \text{Pol}_\mathcal{V} \), \( V \) is thus a sub-\( \mathbb{Q}[L_\eta, \ast_{H, \eta}] \)-module of \( H^\bullet_\mathcal{E}(X) \).

We suppose that

- \( \tilde{\mathcal{V}} \) contains \( (\text{Spec} \, K, \mathbb{Q}_\ell) \),
- \( \tilde{\mathcal{V}} \) is stable under sum: \( (X, V), (X', V') \in \tilde{\mathcal{V}} \Rightarrow (X \coprod X', V \oplus V') \in \tilde{\mathcal{V}} \),
- \( \tilde{\mathcal{V}} \) is stable under product: \( (X, V), (X', V') \in \tilde{\mathcal{V}} \Rightarrow (X \times X', V \otimes V') \in \tilde{\mathcal{V}} \).

Note that the condition (Pol) is stable under sum and product.

\(^{15}\)It follows that for any object \( M \) of \( \mathcal{M}_{\ell}(\mathcal{W})_\mathcal{V}[E] \), we have \( \bigwedge^r M = 0 \) as soon as \( r > \dim H^r_\mathcal{E}(M) \).

\(^{16}\)We can limit ourselves to one polarisation or take them all, it’s the same (cf. [3, Remark following 3.2.2]).

\(^{17}\)Following this paragraph being only a refinement without essential consequence for the continuation, the reader is invited not to dwell on it at first reading.
Definition 3.2.1. Let \((X, H^*_\ell(X)) \in \tilde{\mathcal{V}}\). A motivated cycle on \(X\) modeled over \(\tilde{\mathcal{V}}\), with coefficients in \(E\), is an element

\[
\xi \in \bigoplus_r H^{2r}_\ell(X)(r)
\]

of the form

\[
\xi = (pr_X^{X'})_* (\alpha \cup *H\beta),
\]

where \((X', V') \in \tilde{\mathcal{V}}\), and \(\alpha\) and \(\beta\) are \(E\)-linear combinations of classes of algebraic cycles on \(X \times X'\) which are\(^{18}\) in the subspace \(H^*_\ell(X) \otimes V'\) of \(H^*_\ell(X \times X')\), and \(*H = *_{H,\eta H \otimes 1+1 \otimes \eta'}\) is the Hodge involution of \(X \times X'\) attached to polarizations \(\eta_X, \eta_{X'}\) of \(X\) and \(X'\) respectively, with \(\eta_{X'} \in Pol\).

We denote \(A_{\text{mot}}(X)_{\tilde{\mathcal{V}}}[E]\) the set of motivated\(^{19}\) cycles on \(X\), with coefficients in \(E\), modeled on \(\tilde{\mathcal{V}}\).

Remark 3.2.2.
(i) \(H^*_\ell(X) \otimes V'\) is stable under cup-product with \((pr_X^{X'})^* x\) for every \(x \in H^*_\ell(X)\).
(ii) We can put \(*_L\) instead of \(*_H\), this is the same under point (1) of Lemma 2.2.2 and of fact that \(V\) is a \(\mathbb{Q}[L,*_H]\)-module.
(iii) Thanks to (Pol), it’s the same thing to ask (in the definition of a motivated cycle) that \(\beta\) is in \(H^*_\ell(X) \otimes V'\) or that \(*\beta\) is (for both \(*_L\) and \(*_H\)).
(iv) If \(\gamma' \in V'\) and \(\gamma'' \in V''\), thus \((*_\eta \otimes *_{\eta'})(\gamma' \otimes \gamma'')\) is in \(V' \otimes V''\). This comes from the point (2) of Lemma 2.2.2 and the fact that \(V\) is a \(\mathbb{Q}[L,*_H]\)-module.

It is deduced from the that the replacement of \((pr_X^{X'})_* (\alpha \cup *_{H,\eta H \otimes 1+1 \otimes \eta'}\beta)\) by \((pr_X^{X'})_* (\alpha \cup (*_{H,\eta} \otimes *_{H,\eta'})\beta)\) in the definition 3.2.1 still gives a motivated cycle.

Lemma 3.2.3.
(i) \(A_{\text{mot}}(X)_{\tilde{\mathcal{V}}}[E]\) is a sub-\(E\)-algebra of \(\bigoplus_r H^{2r}_\ell(X)(r)\) (relatively to cup-product), graded by the “degree” \(r\).
(ii) For every \(Y \in \mathcal{W}_K\), we have

\[
(pr_X^{XY})^* A_{\text{mot}}(X)_{\tilde{\mathcal{V}}} \subset A_{\text{mot}}(X \times Y)_{\tilde{\mathcal{V}}},
\]

\[
(pr_X^{XY})_* A_{\text{mot}}(X \times Y)_{\tilde{\mathcal{V}}} \subset A_{\text{mot}}(X)_{\tilde{\mathcal{V}}}
\]

Proof. (i) Let’s first show that every \(E\)-linear combination

\[
\alpha = \lambda (pr_X^{X'})_* (\alpha \cup *\beta) + \mu (pr_X^{X''})_* (\gamma \cup *\delta)
\]

of motivated cycles modeled on \(\tilde{\mathcal{V}}\) is of same type.
If the polarisations \(\eta_1, \eta_2\) of \(X\) brought into play in the first and the second terms respectively are equal, this combination is also written

\[
(pr_X^{X \cup X''})_* ((\lambda \alpha, \gamma) \cup *_{\eta H \otimes 1+1 \otimes \eta'} (\mu \beta, \delta))
\]

\(^{18}\)We neglect the twists here.

\(^{19}\)We omit the mention “with coefficients in \(E\) as well as the symbol \([E]\) if \(E = \mathbb{Q}\).
and it is clear that both \((\lambda \alpha, \gamma)\) and \((\mu \beta, \delta)\) are in \(H_\ell^t(X) \otimes (V' \oplus V'')\).

In the general case, we bring back, by additivity, to the case where \(X, X', X''\) are connected. Thus \(\lambda (\text{pr}_{X'}^X)_*(\alpha \cup \ast \beta)\), respectively \(\mu (\text{pr}_{X''}^X)_*(\gamma \cup \ast \delta)\), is rationally proportional to

\[(\text{pr}_{X}^X)_*(\Delta_X) \cup (\text{pr}_{X}^{X'I}_X)_*\alpha \cup (\text{pr}_{X}^{X''I}_X)_*(\ast \beta),\]

respectively

\[(\text{pr}_{X}^X)_*(\Delta_X) \cup (\text{pr}_{X}^{X''I}_X)_*\gamma \cup (\text{pr}_{X}^{X''I}_X)_*(\ast \delta))\]

(\(\lambda\alpha, \gamma\)) and \(\mu (\lambda \beta, \delta)\) is in \(H_\ell^t(X^2) \otimes V'\) considering the remark (i) as above, and likewise \((\text{pr}_{X''}^X)_*[\Delta_X] \cup (\text{pr}_{X'}^{X''I}_X)_*\gamma \) and \((\text{pr}_{X''}^{X''I}_X)_*(\ast \delta)\) are in \(H_\ell^t(X^2) \otimes V''\).

Let’s show the stability under \(\cup\). We have

\[(\text{pr}_{X}^X)_*(\alpha \cup \ast \beta) \cup (\text{pr}_{X''}^X)_*(\gamma \cup \ast \delta)\]

\[(\text{pr}_{X}^X)_*(\Delta_X) \cup (\text{pr}_{X}^{X'I}_X)_*(\alpha \cup \ast \beta) \otimes (\text{pr}_{X}^{X''I}_X)_*(\gamma \cup \ast \delta))\]

\[(\text{pr}_{X}^X)_*(\Delta_X) \cup (\text{pr}_{X}^{X'I}_X)_*(\alpha \otimes \gamma) \cup (\ast \beta \otimes \ast \delta))\]

\[(\text{pr}_{X}^X)_*(\Delta_X) \cup (\alpha \otimes \gamma) \cup (\ast \beta \otimes \ast \delta))\]

and we see again that \((\text{pr}_{X}^{X'I}_X)_*[\Delta_X] \cup (\alpha \otimes \gamma) \) and \(\ast \beta \otimes \ast \delta\) is in \(H_\ell^t(X) \otimes V' \otimes H_\ell^t(Y) \otimes V''\) (cf. remark (iv) as above).

\[\text{(ii) We have}\]

\[(\text{pr}_{XY}^X)_*(\text{pr}_{X}^X)_*(\alpha \cup \ast \beta) = (\text{pr}_{XY'}^X)_*(\text{pr}_{X'}^X)_*(\alpha \cup \ast \beta)\]

which is proportional to \((\text{pr}_{XY'}^X)_*(\text{pr}_{X'}^X)_*(\alpha \cup \ast \beta [Y])\) and \((\text{pr}_{X'}^X)_*(\alpha)\) is in \(H_\ell^t(X \times Y) \otimes V'\).

We have, on the other hand,

\[(\text{pr}_{X}^X)_*[\text{pr}_{XY'}^X)_*(\alpha \cup \ast \beta)] = (\text{pr}_{X}^X)_*(\alpha \cup \ast \beta),\]

and both \(\alpha\) and \(\beta\) are in \(H_\ell^t(X) \otimes H_\ell^t(Y) \otimes V'.\)

\[\Box\]

\[3.3.\] We define from the notion of motivated correspondance, and we establish the stability by composition, as in [3].

Let \(\mathcal{W}\) be a full subcategory of \(\mathcal{P}_K\), stable under products and finite disjoint union, and passing to connected components. We suppose that for every \(X \in \mathcal{W}, (X, H_\ell^t(X)) \in \mathcal{V}\).

We define thus as in [3], mutatis mutandis (making necessary alterations while not affecting the main point at issue), the category \(\mathcal{M}_t(\mathcal{W})_{\widehat{\mathcal{V}}}[E]\) of motives cutout on the objects of \(\mathcal{W}\), and modeled on \(\mathcal{V}\), with coefficients in \(E\) (we will sometimes denote \(\mathfrak{h}(X)\) the motive attached to \(X \in \mathcal{W}\)). This is a rigid \(E\)-linear (and even \(\text{End} \, 1\)-linear), pseudo-abelian \(\otimes\)-category.

We have a natural functor

\[\mathcal{W}^{op} \rightarrow \mathcal{M}_t(\mathcal{W})_{\widehat{\mathcal{V}}}[E].\]
In characteristic zero, for $E = \mathbb{Q}$, we show as in [3] (in utilising the Hodge index theorem) that this is a semi-simple Tannakian category over $\mathbb{Q}$, independant of $\ell$. We will denote simply $\mathcal{M}(\mathcal{W})$. 

4. Motivic Galois group

4.1. In characteristic nonzero, we ignore if $\mathcal{M}(\mathcal{W})\tilde{\mathcal{V}}$ is abelian in general. We ignore also if $\text{End} 1 = E$ in $\mathcal{M}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$. The following lemma is clear.

Lemma 4.1.1. The following conditions are equivalent.
(i) $\text{End} 1 = E$.
(ii) $E$ contains the “numbers” $\langle \alpha, *_H \alpha \rangle$, where $\alpha$ runs through the algebraic cycle classes on $X$ contained in $\mathcal{V}$, for every couple $(X, V) \in \tilde{\mathcal{V}}$, and $*_H$ is the Hodge involution on $X$ attached to an arbitrary polarisation in $\text{Pol}_V$.

These conditions are trivially satisfied if $E = \mathbb{Q}_\ell$.

Recall that $\mathcal{M}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$ is a rigid $E$-linear, pseudo-abelian $\otimes$-category, and the morphisms are $E$-linear combinations of motivated correspondances. Under the conditions of Lemma, the morphisms form an $E$-vector space of finite dimension.

The fact that the Kunneth projectors are motivated,20 every object of $\mathcal{M}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$ admits a canonical decomposition according to “the weights” [3, 4.4]. The constraint of commutativity is given by the Koszul rule according to the weights, so that the $\ell$-adic étale realisation is a faithful exact $\otimes$-functor $H^*_\ell : \mathcal{M}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E] \to \text{Vec}_{\mathbb{Q}_\ell}$, and that the rank (i.e. the Tannakian dimension) of every object of $\mathcal{M}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$ is a natural integer: this is the dimension of its $\ell$-adic realisation.

4.2. Let $\mathcal{N}$ be the largest monoidal ideal $\mathcal{M}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$. According to [8, 7.1.1], it is formed of motivated correspondances $\alpha \in V$ of $X$ towards $Y$ (with $(X \times Y, V) \in \tilde{\mathcal{V}}$) such that $\langle \alpha, \beta \rangle = 0$ for every motivated correspondances $\beta \in V$ of $X$ towards $Y$. 21

A variant of the argument of Jannsen [22] (or a direct application of the abstract version [8, Erratum] of theorem of Jannsen) gives the following proposition.

Proposition 4.2.1. The quotient $\overline{\mathcal{M}}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$ of $\mathcal{M}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$ by $\mathcal{N}$ is a semi-simple Tannakian category over $E$, and the functor of projection

$$\pi : \mathcal{M}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E] \to \overline{\mathcal{M}}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$$

is conservative, i.e. reflects the isomorphisms.

In addition, for every object $M$ of $\mathcal{M}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$, the image $\overline{M}$ in $\overline{\mathcal{M}}_\ell(\mathcal{W})\tilde{\mathcal{V}}[E]$, the kernel of

$$\text{End} M \to \text{End} \overline{M}$$

is a nilpotent ideal.

In applying [8, §8.2.4], we also have the following proposition.

---

20Modeled on $\tilde{\mathcal{V}}$ since $\tilde{\mathcal{V}}$ contains $(X, H^*_\ell(X))$ for every $X \in \mathcal{W}$.

21This is thus the analogue, for the motivated correspondances considered from the ideal of algebraic correspondances numerically equivalent to 0.
Proposition 4.2.2. Every object $M$ of $\mathcal{M}_\ell(W)[E]$ of rank $r$ verifies $\Lambda^{r+1}M = 0$.

Finally, the theorem of splitting [8, Theorem 2] applies and gives the following theorem.

Theorem 4.2.3. The $\otimes$-functor of projection $\pi : \mathcal{M}_\ell(W)[E] \to \mathcal{M}_\ell(W)[E]$ admits a $\otimes$-section $\sigma$ (automatically compatible with the constraint of commutativity). It is unique up to conjugation by a $\otimes$-isomorphism.

4.3. As in [9], we can take advantage of the existence of such a section $\sigma$ to define the motivic Galois $\mathbb{Q}_\ell$-group attached to every object $M$ of $\mathcal{M}_\ell(W)[E]$: this is the $\mathbb{Q}_\ell$-group $G_M$ of automorphisms of fibre functor $H^*_\ell \circ \sigma$ restricted to the Tannakian sub-category of $\mathcal{M}_\ell(W)[E]$ generated by $M$. This Tannakian sub-category is in fact graded (by the weights), which translates to the existence of a canonical central cocharacter

$$G_{m,E} \xrightarrow{\iota^w} G_M$$

called cocharacter of weights.

The group $G_M$ is a reductive sub-group of $\text{GL}(H^*_\ell(M))$, the group of automorphisms of the $\ell$-adic realisations of $M$. In particular, since $M = h(X)$ is the object of $\mathcal{M}_\ell(W)[E]$ attached to a variety $X \in W$, we denote by $G_X$ the group thus obtained.  

Denote by $(X)$ the smallest full sub-category of $\mathcal{P}_K$ containing $X$ and stable under disjoint sums and products, and passing to connected components. Thus $G_X$ is none other than the automorphism group of fibre functor $H^*_\ell \circ \sigma$ on $\mathcal{M}_\ell(W)[E]$. It is thus described as: this is the largest subgroup of $\text{GL}(H^*_\ell(X))$ which fixes the motivated cycles (corresponding to projective embedding $\mathcal{P}_K$ containing $X$ and stable under disjoint sums and products, and passing to connected components).

Of course, in characteristic 0, all these complications disappear: we can take $E = \mathbb{Q}$ and we have $\mathcal{M}(W)[E] = \mathcal{M}(W)[E]$.

5. Global invariant cycles and Hodge involution

5.1. Let $\mathcal{S}$ be a connected lisse projective curve over $K$, and let $\mathcal{f} : X \to \mathcal{S}$ be a generically lisse projective morphism. Let $S$ be an open non-empty of $\mathcal{S}$ above which the restriction $f : X \to S$ of $\mathcal{f}$ is lisse. For every $t \in S(K)$, we denote $\iota_t$ the inclusion morphism of $X_t$ in $X$.

Choose projective embeddings $\mathcal{S} \hookrightarrow \mathbb{P}^N$, $X \hookrightarrow \mathbb{P}^M \times \mathcal{S}$. We denote by $\eta_\mathcal{S} \in H^2_\mathcal{S}(\mathcal{S})(1)$ and $\eta_\mathcal{T} \in H^2(X)(1)$ the corresponding classes, such that $\eta_X := \mathcal{f}^* \eta_\mathcal{S} + \eta_\mathcal{T}$ is a polarisation of $X$ (corresponding to projective embedding $X \hookrightarrow \mathbb{P}^M \times \mathbb{P}^N \hookrightarrow \mathbb{P}^{MN+M+N}$). We denote by $\eta_{X_t}$ the polarisation induced by $\eta_T$ on every fibre $X_t$ of $f$.

5.2. In the following of this paragraph, we suppose $K = \mathcal{K}$ and we omit to write the twists to lighten.

On $H^*_\ell(X)$, the cup product with $\mathcal{f}^* \eta_\mathcal{S}$, let $L_{\text{hor}}$, is none other than $\delta \iota_{t*} \iota^*_t$, where $\delta$ is of degree of $\eta_\mathcal{T}$. It is of square zero.

We will denote by $L_{\text{ver}}$ the cup-product operator with $\eta_\mathcal{T}$. The operators $L_{\text{hor}}$ and $L_{\text{ver}}$

\footnote{Somewhat abusive notation; this group depends on $\sigma$, but depends only on conjugation.}

\footnote{with coefficients in $E$ or in $\mathbb{Q}_\ell$.}
commute, and we denote \( L = L_{\text{ver}} + L_{\text{hor}} \) the cup product with the polarisation \( \eta = \bar{f}^* \eta_{\bar{z}} + \eta_T \) of \( \overline{X} \). We have the relative Lefschetz decomposition

\[
H^0(S, R^q f_* \mathbb{Q}_\ell) = \bigoplus_{k \leq d-q} H^0(S, L_{\text{ver}}^k P^{q-2k} f_* \mathbb{Q}_\ell),
\]

noting \( P^q f_* \mathbb{Q}_\ell \) the primitive part of \( R^q f_* \mathbb{Q}_\ell \) relative to the relative polarisation \( \eta_f (L^{d+1-q} P^q f_* \mathbb{Q}_\ell = 0) \).

The theorem of global invariant cycles show that for every \( q \), the canonical map

\[
u : H^*_\ell(X_t) \to \bigoplus_q H^0(S, R^q f_* \mathbb{Q}_\ell)
\]
is surjective. It is compatible with the operator \( L_{\text{ver}} \). On the other hand the natural map

\[
v : \bigoplus_q H^0(S, R^{2d-q} f_* \mathbb{Q}_\ell) \to \bigoplus_q H^0(S, L_{\text{ver}}^k P^{q-2k} f_* \mathbb{Q}_\ell)
\]
induced by \( L_{\text{hor}} \), such that \( L_{\text{hor}} = vu \).

**Lemma 5.2.1.** On \( \bigoplus_q H^0(S, R^q f_* \mathbb{Q}_\ell) \), the form \( \langle \cdot, \cdot \rangle \) induced by the Poincare duality pairing \( R^q f_* \mathbb{Q}_\ell \otimes R^{2d-q} f_* \mathbb{Q}_\ell \to \mathbb{Q}_\ell \) is non-degenerate.

**Proof.** On \( H^*_\ell(X_t) \), \( \langle \cdot, \cdot \rangle \) is non-degenerate and \( \pi_1(S, t) \)-invariant, thus its restriction to \( \pi_1(S, t) \)-invariants is non-degenerate because the action of \( \pi_1(S, t) \) is semi-simple, [15, 3.4.13].

**Proposition 5.2.2.** For every \( x \in H^0(S, P^q f_* \mathbb{Q}_\ell) \) and every \( k \leq d-q \), we have

\[
*_{H, \eta} v \ast_{H, \eta_T} L^k x \equiv L^k_{\text{ver}} *_{H, \eta} v *_{H, \eta_T} x = (d + 1 - q)L^k_{\text{ver}}(u *_{L, \eta} v *_{L, \eta_T} x),
\]

congruence taking place modulo \( \mathbb{Q}[L].v(x) \). A fortiori,

\[
u *_{H, \eta} v \ast_{H, \eta_T} L^k_{\text{ver}} x = L^k_{\text{ver}} u *_{H, \eta} v *_{H, \eta_T} x
\]

\[
= (d + 1 - q)L^k_{\text{ver}}(u *_{L, \eta} v *_{L, \eta_T} x) \in H^0(S, L^k_{\text{ver}} P^q f_* \mathbb{Q}_\ell)
\]

and

\[
u *_{H, \eta} v *_{H, \eta_T} = *_{H, \eta_T} u *_{H, \eta} v.
\]

**Proof.** We have

\[
v *_{H} L^k_{\text{ver}} x = (-1)^{(q+1)/2} \frac{k!}{(d-q-k)!} L^{d-q-k} v(x).
\]

Otherwise, \( L^d_{\text{ver}} x = 0 \), hence \( L^{d+1-q} v(x) = 0 \), hence we get that the Lefschetz decomposition of \( v(x) \) (concerning \( \eta \)) has only two terms:

\[
v(x) = x'_{q+2} + Lx'_q,
\]
by designating by $x'_m$ a suitable primitive element in $H^m_\ell(X)$. We have thus
\[
*_{H^v} *_{H} L^k_{\text{ver}} x
= (-1)^{q(q+1)/2} \frac{k!}{(d - q - k)!}
\times \left[ (-1)^{(q+2)(q+3)/2} \frac{(d - q - k)!}{(k - 1)!} L^{k-1} x'_{q+2} + (-1)^{q(q+1)/2} \frac{(d + 1 - q - k)!}{k!} L^k x'_q \right]
= -k L^{k-1} x'_{q+2} + (d + 1 - q - k) L^k x'_q
\equiv (d + 1 - q) L^k x'_q \mod \mathbb{Q}[L] v(x).
\]
For $k = 0$, there is no term in $L^{k-1}$, and we actually have
\[
*_{H^v} *_{H} x = (d + 1 - q) x'_q.
\]
In applying $L^k$ to this last equality, and in combining to the previous congruence, we obtain
the 1st congruence of the proposition. Otherwise, we have
\[
L^k(*_{L^v} *_{L} x) = L^k *_{L} (L^{d-q} v(x)) = L^k *_{L} (L^{d+1-q} x'_q) = L^k x'_q,
\]
which gives $L^k_{\text{ver}} *_{H^v} *_{H} x = (d + 1 - q) L^k *_{L^v} *_{L} x$.

The second assertion easily follows. \qed

5.3. The following result is the key technique of this article.

**Theorem 5.3.1.** We suppose that one of the following two conditions is satisfied:
(a) For every $q$, the restriction homomorphism
\[
H^0(\mathcal{S}, R^q I_* \mathbb{Q}_\ell) \to H^0(S, R^q f_* \mathbb{Q}_\ell)
\]
is injective;
(b) The linear map
\[
u : H^*_\ell(X) \to \bigoplus_q H^0(S, R^q f_* \mathbb{Q}_\ell)
\]
admits a section $\nu'$ compatible with the action of $L_{\text{ver}}$.

We have thus the following properties.
(1) For every $q$, $u *_{H,q} v *_{H,q} = *_{H,q} u *_{H,q} v$ is an automorphism of $H^0(S, R^q f_* \mathbb{Q}_\ell)$ which respects the Lefschetz decomposition.
(2) $*_{H,q}(\text{Im } v)$ relive $\text{Im } u$ in $H^*_\ell(X)$. The graded subspace
\[
V^*_\ell := \text{Im } v \oplus *_{H,q}(\text{Im } v)
\]
of $H^*_\ell(X)$ is stable under $L$, and is direct factor of $H^*_\ell(X)$ as $\mathbb{Q}[L]$-module: a supplement is given by the orthogonal relative to the form $\langle \cdot, *_{H} \cdot \rangle$. The sum $\text{Im } v \oplus *_{H,q}(\text{Im } v)$ is orthogonal relative to this form.
(3) In the case (b), we have
\[
u *_{H,q} v *_{H,q} = \text{Id}_{H^0(S, R^q f_* \mathbb{Q}_\ell)}.
\]
In addition, the orthogonal projector $\pi_{V^*_\ell}$ onto $V^*_\ell$ breaks down into orthogonal idempotent sum
\[
u *_{H,q} u *_{H,q} + *_{H,q} v *_{H,q} u
\]
reflecting the decomposition $\text{Im } v \oplus *_{H,q}(\text{Im } v)$.

(4) If we are simultaneously in the cases (a) and (b), then $V_\gamma = \text{Im } v \oplus \text{Im } v'$.

Proof. It is clear that $u *_{H} v *_{H}$ induces an endomorphism of $H^0(S, R^q f_* \mathbb{Q}_\ell)$. The previous proposition allows us to, in (1), limit ourselves to considering its restriction to $H^0(S, P^q f_* \mathbb{Q}_\ell)$.

Let’s start with the case (a). The hypothesis (a) attached to the surjectivity of $u$ reflects the decomposition $\text{Im } v$. It should also be noted that under the strong Lefschetz theorem for $H^2(S, j_* R^q f_* \mathbb{Q}_\ell)$, thus

$$E^p_{2q} = H^p(S, R^q f_* \mathbb{Q}_\ell) \Rightarrow H^{p+q}(\overline{X}_R)$$

degenerates at $E_2$. We will denote by $F^p$ the associated decreasing filtration, such that

$$F^p / F^{p+1} = \bigoplus H^p(S, R^q f_* \mathbb{Q}_\ell).$$

We denote $j : S \hookrightarrow \overline{S}$ and $i : \overline{S} \setminus S \hookrightarrow \overline{S}$ the inclusions. According to the theorem of local invariant cycles [15, 3.6.1], the morphism

$$R^q \overline{f}_* \mathbb{Q}_\ell \rightarrow j_* R^q f_* \mathbb{Q}_\ell$$

is an epimorphism, hence a short exact sequence

$$0 \rightarrow i_* i^! R^q \overline{f}_* \mathbb{Q}_\ell \rightarrow R^q \overline{f}_* \mathbb{Q}_\ell \rightarrow j_* R^q f_* \mathbb{Q}_\ell \rightarrow 0.$$

Considering (a), $i_* i^! R^q \overline{f}_* \mathbb{Q}_\ell = 0$. It follows that

$$F^0 / F^1 \cong \bigoplus H^0(S, R^q f_* \mathbb{Q}_\ell),$$

$$F^1 / F^2 \cong \bigoplus H^1(S, j_* R^q f_* \mathbb{Q}_\ell) \quad \text{(where } j : S \hookrightarrow \overline{S} \text{ denotes the inclusion)},$$

$$F^2 = \text{Im } i_* = \text{Im } v = \text{Im } L_{\text{hor}} \cong \bigoplus H^2(S, j_* R^q f_* \mathbb{Q}_\ell).$$

It should also be noted that under the strong Lefschetz theorem for $\eta_{\overline{f}}$, $L^{d+1-q}$ induces an isomorphism

$$H^1(\overline{S}, j_* R^{q-1} f_* \mathbb{Q}_\ell) \xrightarrow{\cong} H^1(\overline{S}, j_* R^{2d+1-q} f_* \mathbb{Q}_\ell)$$

and $L^{d+2-q}$ induces an isomorphism

$$H^2(\overline{S}, j_* R^{q-2} f_* \mathbb{Q}_\ell) \xrightarrow{\cong} H^2(\overline{S}, j_* R^{2d+2-q} f_* \mathbb{Q}_\ell)$$

(5.2) (these isomorphisms are the same as those induced by $L^{d+1-q}_{\text{ver}}$ and $L^{d+2-q}_{\text{ver}}$ respectively).

To prove (1), it’s about showing, under the previous proposition, that the map

$$w_q : x \in H^0(S, P^q f_* \mathbb{Q}_\ell) \mapsto u v L^{d-q}_{\text{ver}} x$$

defines an automorphism of $H^0(S, P^q f_* \mathbb{Q}_\ell)$. We have

$$L^{d+1-q}_{\text{ver}} w_q(x) = u L^{d+1-q} v L^{d-q}_{\text{ver}} x = w v L^{d-q}_{\text{ver}} x = 0$$

(cf. the Lemma 1.4.2), thus $u v L^{d-q}_{\text{ver}} x \in H^0(S, P^q f_* \mathbb{Q}_\ell)$, and we have to prove the injectivity of $w_q$.

Suppose thus that $w_q(x) = 0$, i.e. $y := v L^{d-q}_{\text{ver}} x \in F^1$. We have $L^{d+1-q} y = v L^{d-q}_{\text{ver}} x \in F^2$ and $L^{d+2-q} y = 0$. According to the isomorphism (5.2), this implies $y \in F^2$, and in fact $y$ is identified to an element of $H^2(S, j_* R^{q-2} f_* \mathbb{Q}_\ell)$. But then by virtue of (5.3), $y = 0$, hence finally $x = 0$. This establishes (1) in the case (a).
Let’s go to the case (b), and establish the formula (5.1). For every \( q \leq d \), the Lefschetz isomorphism of source \( H^q(X) \) is written

\[
*_{L} = L^{d+1-q} = L_{\text{ver}}^{d+1-q} + (d + 1 - q)L_{\text{hor}}I_{\text{ver}}^{d-q}.
\]

Put

\[
W'_q = \text{Ker}[H^q(X) \xrightarrow{u} H^0(S, R^d f_\ast Q_\ell)],
\]

\[
W_q = \text{Ker}[H^q(X) \xrightarrow{u} H^0(S, R^d f_\ast Q_\ell)] \xrightarrow{L_{\text{ver}}^{d+1-q}} H^0(S, R^{2d+2-q} f_\ast Q_\ell)],
\]

such that \( W_q/W'_q \) is identified to \( H^0(S, P^q f_\ast Q_\ell) \).

Establish the following points.

(i) \( L_{\text{hor}}L_{\text{ver}}^{d-q}(W'_q) = 0 \).

(ii) \( L_{\text{ver}}^{d+1-q}(W_q) \subset *_{L} W'_q \).

(iii) \( L_{\text{hor}}L_{\text{ver}}^{d-q}(W'_q) \subset *_{L} W_q \).

(iv) \( (d + 1 - q) *_{L} L_{\text{hor}}L_{\text{ver}}^{d-q} \) induces the identity automorphism on \( W_q/W'_q \).

(i) This point comes from that \( W'_q \subset \text{Ker } L_{\text{hor}} \).

(ii) Considering (i) and the formula (5.4), we have \( L_{\text{ver}}^{d+1-q}(W'_q) \subset *_{L} W'_q \). So \( L_{\text{ver}}^{d+1-q}(W_q) \) induces a homomorphism

\[
W_q/W'_q = H^0(S, P^q f_\ast Q_\ell) \to H_\ell^{2d+2-q}(X)/ *_{L} W'_q
\]

which is zero by definition of \( P^q \), considering that \( u \) admits a section \( v' \) compatible with the action of \( L_{\text{ver}} \).

(iii) This point results from (ii) and from the formula (5.4).

(iv) By (i) and (iii) it is clear that \( (d + 1 - q) *_{L} L_{\text{hor}}L_{\text{ver}}^{d-q} \) induces a well-defined endomorphism of \( W_q/W'_q \). To see that this is the identity, it is to show that for every \( x \in W_q \),

\[
L^{d+1-q}(x) - (d + 1 - q)L_{\text{hor}}I_{\text{ver}}^{d-q}(x) = L_{\text{ver}}^{d+1-q}(x) \in *_{L} W'_q,
\]

which results from (ii).

These points being established, let \( x \in W_q \) be a lifting of \( \xi \in W_q/W'_q = H^0(S, P^q f_\ast Q_\ell) \). Considering the proposition 5.2.2, to establish the 1st assertion of (3)–that is to say (5.1)–under (b), is to prove that

\[
u((d + 1 - q) *_{L} (L_{\text{hor}}L_{\text{ver}}^{d-q}x)) = \xi,
\]

this is to say that \( (d + 1 - q) *_{L} (L_{\text{hor}}L_{\text{ver}}^{d-q}x) \) and \( x \) has same image modulo \( W'_q \). This is the point (4) of previous lemma. This also establishes (1) in the case (b).

The point (1) of theorem being now established, let’s deduce the point (2). From (1) follows that \( *_{H, \eta} \text{Im } v \) lifts \( \text{Im } u \) in \( H_\ell^q(X) \). Prove that the graded subspace \( V_\ell := \text{Im } v \oplus *_{H, \eta} \text{Im } v \) of \( H_\ell^q(X) \) is stable under \( L \).

It’s about seeing that for every \( x \in H^0(S, P^q f_\ast Q_\ell) \), and every couple \((h, k)\) of natural integers with \( k \leq d-q \), we have \( L^h *_{H} L^{d-q-k}v(x) \in V_\ell \). Or by the proposition 5.2.2 and in resuming these notations, we have \( v(x) = x'_{q+2} + Lx'_q \) and

\[
*_{H} L^{d-q-k}v(x) = (-1)^{q(q+1)/2}(d + 1 - q)(d - q - k)!L^k x'_q + \text{elements of } Q[L]v(x),
\]
hence $L^kx_q'$ ∈ $V_\mathcal{T}$ for every k. So $L^k *_H L^d-q-kv(x)$ is in $V_\mathcal{T}$.

Then check that the form $\langle \cdot, *_H \cdot \rangle$ is non-degenerate on $V_\mathcal{T}$. As $\text{Im } v$ and $*_H \text{Im } v$ are orthogonal for this form, it suffices to see that

$$\langle v(x), *_H (\text{Im } v) \rangle \Rightarrow x = 0.$$  

Or $\langle v(x), *_H \text{Im } v \rangle = \langle x, u *_H (\text{Im } v) \rangle$ and we conclude from the fact that $u *_H v *_H$ is an automorphism of $\text{Im } u$, and that $\langle \cdot, \cdot \rangle$ is non-degenerate on $\bigoplus H^0(S, R^q f_* \mathbb{Q}_\ell)$ (Lemma 5.2.1). The first assertion of (3) follows directly from that, under (b), this last automorphism is the identity, and from Lemma 5.2.1.

It remains, to finish, to prove (4). The first congruence of the proposition 5.2.2 show that $*_H v *_H$ is a section of $F^0/F^2 \rightarrow F^0/F^1 \cong \bigoplus_{q} H^0(S, R^q f_* \mathbb{Q}_\ell)$ compatible with the action of $L_{\text{ver}}$. The formula (5.3) shows on the other hand that such a section is unique. This is thus $v' \bmod \text{Im } v$.

This completes the proof of the theorem. □

**Remark 5.3.2.** If $K = \mathbb{C}$, we can replace the $\ell$-adic cohomology by the de Rham cohomology (algebraic or transcendental) with complex coefficients. We thus put the Hodge decomposition and the antilinear Hodge-* operator of the Kahler theory. On the component of Hodge type $(p, q)$, * is the complex conjugate of $(\sqrt{-1})^{q-p} *_H$ (cf. [3,1.1]). So we can see that, in this context, replace in the theorem $*_H$ by *.

### 5.4. Commentaries on the hypotheses of Theorem 5.3.1

(1) The condition (a) is evidently satisfied in the case where $S$ is projective ($S = \overline{S}$). It is also for example for the Lefschetz pencil of degree sufficiently large, at least in dimension relative even or in characteristic distinct from 2 (cf. [SGA 7 II, Expose XVIII]).

(2) It is easy to see that condition (b) is not superfluous to the point (3) of Theorem (replace for example $L_{\text{ver}}$ by $L_{\text{ver}} + \epsilon L_{\text{hor}}$ and $L_{\text{hor}}$ by $(1 - \epsilon)L_{\text{hor}}$, with $\epsilon \in \overline{\mathbb{Q}} \cap [0,1]$.) We ignore however, same in the case where $S = \overline{S}$, if we can still find a relative polarisation $\eta_{\overline{S}}$ satisfying the condition (b).

(3) The condition (b) keep a sense if dim $S > 1$, and is stable under base change $S' \rightarrow S$.

### 6. The case of abelian schemes

In this paragraph, we study the condition (b) of theorem 5.3.1 in the case of abelian schemes.

#### 6.1. Suppose that $f : X = \overline{X} \rightarrow S = \overline{S}$ is an abelian scheme of relative dimension $d$, of base a lisse projective variety of dimension $n$. We have thus, according to Lieberman, a canonical decomposition of the Leray filtration

$$H^i_f(X) = \bigoplus_{j=0,\ldots,2n} H^j(S, R^{i-j}f_* \mathbb{Q}_\ell)$$

where $H^j(S, R^{i-j}f_* \mathbb{Q}_\ell)$ is the part where, for every $N \in \mathbb{Z}$, the endomorphism of multiplication by $N$ acts by homothety of rapport $N^{i-j}$. This decomposition, given by algebraic correspondances, is stable under cup-product. Let $\eta_f$ be the class of a relatively ample invertible fibre, symmetric, and rigidified along the zero section. We know thus $[N]^* \eta_f = N^2 \eta_f$ (thus $\eta_f$ is in the component $H^2(S, f_* \mathbb{Q}_\ell)$ of the canonical decomposition of $H^2(X)$), and we
conclude that the decomposition above is stable under $L_{\text{ver}}$.

6.2. Suppose moreover that $\dim S = 1$, i.e. that $f : X \to S$ is an abelian scheme over a lisse projective curve, so we are both in the cases (a) and (b) of theorem 5.3.1. We thus have a canonical decomposition

$$H^i(X) = H^2(S, R^{i-2}f_*\mathbb{Q}_\ell) \oplus H^1(S, R^{i-1}f_*\mathbb{Q}_\ell) \oplus H^0(S, R^if_*\mathbb{Q}_\ell)$$

and the projection on each factor is induced by an algebraic correspondence; if the even Kunneth projector $\pi_{\text{even}}^X = \oplus \pi_{\text{even}}^{2j}$ is algebraic (for example if $K$ is algebraic over a finite field [24]), the canonical splitting of the Leray filtration of $H^*_\ell(X) = \bigoplus H^i_\ell(X)$ thus obtained is given by algebraic correspondences.

In addition, the part $H^\text{even}(S, \oplus R^j f_*\mathbb{Q}_\ell)$ is orthogonal to $H^1(S, \oplus R^j f_*\mathbb{Q}_\ell)$ relative to the Poincaré pairing $\langle \cdot, \cdot \rangle$.

The point (4) of theorem 5.3.1 shows that $V_f = H^\text{even}(S, \oplus R^j f_*\mathbb{Q}_\ell)$, and we say that the orthogonal of $V_f$ relative to the form $\langle \cdot, \ast H^\cdot \rangle$ is $H^1(S, \oplus R^j f_*\mathbb{Q}_\ell)$.

**Proposition 6.2.1.** In the situation of an abelian scheme $f : X \to S$ over a projective lisse curve $S$, and of a relative polarization $\eta_f$ given by the class of a relatively ample, symmetric, and rigidified, invertible fibre along the zero section, the conditions (a) and (b) are met, and we have moreover:

(c) if $\pi_{\text{even}}^X$ is given by an algebraic correspondance, the same is true of the orthogonal projector $\pi_{V_f}$ on $V_f = H^\text{even}(S, \oplus R^j f_*\mathbb{Q}_\ell)$;

(d) every endomorphism of $V_f^\perp = H^1(S, \oplus R^j f_*\mathbb{Q}_\ell)$ derived from an element of $\mathbb{Q}[L_\eta, \ast_{H_\eta}]$ is given by an algebraic correspondance.

**Proof.** It remains to be proven that the property (d). According to the point (1) of Lemma 2.2.2, it suffices to show that the restriction of $\ast L$ to $V_f^\perp$ is given by an algebraic correspondance.

We will briefly borrow a few results of the theory of relative Chow motives, in the particular case of abelian schemes $X \to S$ [16,26]. The relative Chow motive $R(X/S)$ admits a decomposition “in the sense of Lieberman”

$$R(X/S) = \bigoplus_{0}^{2d} R^j(X/S)$$

where the endomorphism of multiplication by $N$ acts by homothety of rapport $N^j$ on $R^j(X/S)$. In addition, we have an isomorphism “in the sense of Lefschetz”

$$R^j(X/S) \xrightarrow{\cup c_1(L)^{d-j}} R^{2d-j}(X/S)(d-j).$$

Let $\Psi_j$ be the algebraic correspondance (modulo rational equivalence) which induces the inverse isomorphism.

We have a functor of relative Chow motives to the bounded derived category $D^b(S, \mathbb{Q}_\ell)$ of $\mathbb{Q}_\ell$-sheaf in the sense of [17], and the previous isomorphism are reflected in the level of $D^b(S, \mathbb{Q}_\ell)$:

$$Rf_*\mathbb{Q}_\ell = \bigoplus_{0}^{2d} R^i f_*\mathbb{Q}_\ell[-j], \quad R^i f_*\mathbb{Q}_\ell[-j] \xrightarrow{\cup c_1(L)^{d-j}} R^{2d-j} f_*\mathbb{Q}_\ell(d-j)[j-2d],$$
and we deduce from the fact that in the decomposition which follows after passage to $R \Gamma$
\[ R \Gamma(Rf_* \mathbb{Q}_\ell) \cong \bigoplus_i H^i(X)[-i] = \bigoplus_{i,j} H^{i-j}(S, R^j f_* \mathbb{Q}_\ell)[-i], \]
the algebraic correspondence $\Psi_j$ (viewed as “absolute” correspondence on $X$) is the inverse
of $\bigcup c_1(\mathcal{L})^{d-j}$ on each factor $H^{i-j}(S, R^j f_* \mathbb{Q}_\ell)$. On $H^1(S, R^j f_* \mathbb{Q}_\ell)$, this is thus $*_L$ since $L$
coincides with $L_{\text{ver}}$ on this factor. □

6.3. The case of an abelian scheme $f : X \to S$ over a lisse affine curve, equipped with
a relatively ample invertible fibre, symmetric, and rigidified along the zero section $\mathcal{L}$, is
important for these applications (cf. [7, 6.2]), but a lot more problematic. Let’s quickly
indicate how we could approach it, in the spirit of the above.

In [27] (see also [28]), Künemann showed that up to replacing $S$ by a finite covering (not
necessarily separable), $f$ extends to a strictly semi-stable fibration $\overline{f} : \overline{X} \to \overline{S}$
over the lisse compactification of $S$. Its construction, which relies on the work of Mumford, Chai
and Faltings, is explicit, although non canonical. In addition, up to replacing $\mathcal{L}$ by a power, it
extends into a relatively ample invertible fibre $L$ over $X$. The cup product with the class of $L$
definest an operator $L_{\text{ver}}$ on $H^*_X(\overline{X})$, and moreover also on $\bigoplus_j H^0(S, R^j f_* \mathbb{Q}_\ell)$, in a manner
compatible with the canonical surjection
\[ u : H^*_X(\overline{X}) \to \bigoplus_j H^0(S, R^j f_* \mathbb{Q}_\ell). \]

**Question 6.3.1.** Does, for a suitable choice of polarizations, $u$ admit a compatible section
with $L_{\text{ver}}$?

To construct such a section, we can seek to employ as above the operator of multiplication
$[N_f]$ by a suitable integer $N$ over the abelian scheme $X \to S$
A first difficulty which presents itself is that $[N_f]$ does not extend in general to an $S$-
endomorphism of $X$ (this is already seen in the case of a non isotrivial elliptic pencil). The
endomorphism $[N_f]$ defines in fact a rational $S$-map of $X$ towards itself in the sense of [EGA
I, 7.1.2]. Its graph $[N_f]$ is an integral closed subscheme of $X \times_S X$ (cf. [EGA IV, 20.4.1]),
that we consider to be relative correspondence. The theory of relative correspondances in
this general context is set up in [12], where it is proved that the relative correspondances
of degree 0 for $X/S$ acts on the object $Rf_* \mathbb{Q}_\ell$ of the bounded derived category $D^b(S, \mathbb{Q}_\ell)$ of
$\mathbb{Q}_\ell$-sheaves in the sense of [17], thus acts on the Leray spectral sequence of $\overline{f}$.

7. **Deformation of motivated cycles**

7.1. Exploit the theorem 5.3.1 in the framework of motivated cycles. We place ourselves
in the situation and the notations of § 5.1. We fix two $K$-points $s, t$ of $S$.

**Choice of $\mathcal{W}, \mathcal{V}$:** we suppose that $\mathcal{W}$ contains $X_s, X_t$. We suppose that $\mathcal{V}$ contains, besides
the $(Y, H^i(Y))$ for every $Y \in \mathcal{W}$:

(i) $(\overline{X}, V_\mathcal{F})$ if the orthogonal projector $\pi_{\mathcal{V}_\mathcal{F}}$ on $V_\mathcal{F}$ is given by an algebraic correspondence;

\[ \text{The theory of [12] requires only that } \overline{X} \to S \text{ is projective, that } \overline{X} \text{ is lisse over } k, \text{ and that } S \text{ is quasi-projective over } k. \]
Lemma 7.1.1. $t_i^* H \circ t_s$ is a motivated correspondence modeled over $\tilde{\mathcal{V}}$.

**Proof.** If $X \in \mathcal{W}$, this is clear by composition of motivated correspondances, but we don’t assume that $X \in \mathcal{W}$. We have the formula (cf. [25, 1.3.4])

$$t_i^* \circ *_H \circ t_s = (t_s \times t_i)^*(_H) = (pr^{X_s \times X_t}_{X_s \times X_t})_*[\Gamma (t_s \times t_i) \cup ([X_s \times X_t] \otimes *_H)]
$$

(where $\Gamma$ denotes the graph). We know by the point (4) of Lemma 2.2.2 that $*_H, \eta$ is written as non-commutative polynomials with rational coefficients in $L X^2, \otimes$. This will allow us to conclude in the case (ii).

In the case (i), we remark that $\Gamma(t_s \times t_i) \in H^*_H(X_s \times X_t) \otimes H^*_H(X \times \bar{X})$ is contained in

$$H^*_e(X_s \times X_t) \otimes \langle t_s, *_H \rangle H^*_e(X_s) \otimes \langle t_t, *_H \rangle H^*_e(X_t) \subset H^*_e(X_s \times X_t) \otimes V \otimes V$$

thus equal to

$$\Gamma(t_s, t_i)[\Gamma(t_s, t_i)].$$

As $\Gamma$ is an orthogonal projector with regard to the form $\langle \cdot, \cdot \rangle$, we can thus replace $[\Delta_X]$ by $\Gamma(t_s \times t_i)[\Delta_X]$ which is none other than $\Gamma$ (cf. [25, 1.3.4]). This is thus by hypothesis an algebraic cycle, which allows us to conclude. □

**Corollary 7.1.2 (of theorem 5.3.1).** We suppose that we are in the situation (b) of theorem 5.3.1: $u$ admits a section compatible with the action of $L_{\text{ver}}$. Thus

1. The orthogonal projector of $H^*_e(X_s)$ on $H^*_e(X_s) \Gamma_1(S_{\bar{\mathcal{P}}}, s)$ is induced by $\Gamma(t_s \times t_i) \otimes t_s *_H t_s *_H$ which is a motivated correspondence modeled on $\tilde{\mathcal{V}}$;

2. The isomorphism “of parallel transport”

$$\Pi_{s,t} : H^*_e(X_s) \Gamma_1(S_{\bar{\mathcal{P}}}, s) \cong H^*_e(X_t) \Gamma_1(S_{\bar{\mathcal{P}}}, t)$$

is induced by

$$\Gamma(t_s \times t_i) \otimes t_s *_H t_s *_H$$

which is a motivated correspondence modeled on $\tilde{\mathcal{V}}$.

**Proof.** This follows from the point (3) of Theorem 5.3.1 (by Galois descent of $\bar{\mathcal{K}}$ to $K$) and from the previous lemma. □

**Remark 7.1.3.**

1. It is possible that the corollary 7.1.2. is true even if $\dim S > 1$, by dividing $\Gamma(t_s \times t_i) \otimes t_s *_H t_s *_H$ by $(\dim S)!$; we have not proved it.

2. The corollary admits an obvious variant where we limit ourselves, in the hypothesis and the conclusion, to the even part of this cohomology. It is sufficient for the application to the parallel transport of motivated cycles.

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$^{25}$Relative to the Poincare pairing $\langle \cdot, \cdot \rangle$. 
7.2. We gives ourselves a lisse projective morphism $f : X \to S$ of base a connected lisse curve. We fix two $K$-points $s,t$ of $S$.

We suppose that there exists a finite covering $S' \to S$ and an open immersion $S' \subseteq S$ of $S'$ in a connected lisse projective curve, such that the inverse image $f'$ of $f$ above of $S'$ extends to a projective morphism $\overline{f} : \overline{X} \to \overline{S}'$, with $\overline{X}'$ lisse.

Choice of $\mathcal{W}, \mathcal{V}$: we suppose that $\mathcal{W}$ contains $X_s, X_t$. We suppose that $\mathcal{V}$ contains, besides $(Y, H^*_t(Y))$ for every $Y \in \mathcal{W}$, $(\overline{X}, H^*_t(\overline{X}))$.

**Corollary 7.2.1.** We suppose that $K$ is perfect or else $S' \to S$ is étale. We suppose in addition that

(b) the canonical morphism $H^\text{even}_t(\overline{X}) \to H^0(S', R^\text{even}_t \mathcal{F}_{\mathcal{W}}, \mathbb{Q}_\ell)$ admits a section compatible with cup-product with the relative polarization (which is the case for example if $\overline{f}'$ is an abelian scheme).

Let $q$ be a natural integer and $\xi$ a global section of $R^2\mathcal{F}_\mathcal{W}, \mathbb{Q}_\ell(q)$. The following conditions are equivalent:

(i) $\xi_s \in H^{2q}_t(X_s)(q)$ is a motivated cycle on $X_s$ modeled over $\mathcal{V}$,

(ii) $\xi_t$ is a motivated cycle on $X_t$ modeled over $\mathcal{V}$.

Proof. This follows from point (2) of the previous corollary (considering the remark 7.1.3(2)) applied to $f'$ and to $K$-points $s', t'$ of $S'$ lifting $s,t$.

7.3. Suppose for now that $S$ is an irreducible scheme of finite type over a field $K$, and let $f : X \to S$ be a lisse projective morphism. We suppose that $S$ is the complement of a closed codimension at least 2 in the projective $K$-scheme $\overline{S}$. We fix two $K$-points $s,t$ of $S$.

Choice of $\mathcal{W}, \mathcal{V}, E$: we suppose that $\mathcal{W}$ contains $X_s, X_t$.

We suppose that $\mathcal{V}$ contains (besides the $(Y, H^*_t(Y))$ for every $Y \in \mathcal{W}$), a couple of the form $(X \times_S T, V)$, where

- $T$ is a lisse projective connected curve equipped with a morphism $T \to S$ whose image passes through $s$ and $t$,
- $V = V_{f \times_S T}$ if the orthogonal projector $\pi_{V_{f \times_S T}}$ on $V_{f \times_S T}$ is given by an algebraic correspondence, and $V = H^*_t(X \times_S T)$ if not.

We suppose that $\text{End} \mathbf{1} = E$ (cf. Lemma 4.1.1).

**Theorem 7.3.1.** Let $q$ be a natural integer and $\xi$ a global section of $R^2\mathcal{F}_\mathcal{W}, \mathbb{Q}_\ell(q)$. The following conditions are equivalent:

(i) $\xi_s \in H^{2q}_t(X_s)(q)$ is an $E$-linear combination of motivated cycles on $X_s$ modeled over $\mathcal{V}$,

(ii) $\xi_t$ is $E$-linear combination of motivated cycles on $X_t$ modeled over $\mathcal{V}$,

If in addition $S = \overline{S}$ is a curve and $X \in \mathcal{W}$, these conditions are equivalent to:

(iii) $\xi$ is image under $u$ of an $E$-linear combination of motivated cycles on $X \otimes E$ modeled over $\mathcal{V}$.

This result implies the case (2) of Theorem 0.3.1, as well as Theorem 0.4.1 in characteristic nonzero (applying it to irreducible components of the base, and by Galois descent of $K$ to $K$ (suppose perfect) of motivated cycles (cf. [3, Scolie 2.5])).
Proof. Let’s start by showing the existence of a morphism $T \to S$ as above. If $\dim S \leq 1$, we have $S = \overline{S}$ and it suffices to normalize.

Suppose $\dim S > 1$. We can suppose $\overline{S}$ is reduced, and $S$ dense in $\overline{S}$, so that $\overline{S}$ is integral. Cutting by linear sections of codimension $\dim S - 1$ the blow-up of $\overline{S}$ at $s$ and $t$, immersed in a suitable projective space, we obtain by Bertini\textsuperscript{26} irreducible curves whose images do not meet the boundary $\overline{S} \setminus S$ (but meet the exceptional divisor), and we obtain $T$ in normalizing the reduced scheme associated to one of these curves.

We come back to the case where $S = \overline{S} = T$ is a lisse projective curve passing through $s$ and $t$. It suffices to prove (i)$\Rightarrow$(iii).

We are thus in the case (a) of Theorem 5.3.1 (with $j = 2q$), which we use the notations. So, $\ast_{H}u \ast_{H}v$ is an automorphism of $H^{0}(S, R^{2q}f_{*}\mathbb{Q}_{\ell})$.

By Cayley-Hamilton, there exists a polynomial $P$ and with a constant coefficient zero such that $P(\ast_{H}u \ast_{H}v)$ is the identity automorphism of $H^{0}(S, R^{2q}f_{*}\mathbb{Q}_{\ell})$; its coefficients are in the field generated by the traces of powers of $\ast_{H}u \ast_{H}v$. By the lemma 7.1.1, by the Lefschetz trace formula, and by the choice of $E$, these coefficients are thus in $E$.

Let’s write $P(\ast_{H}u \ast_{H}v)$ under the form $\ast_{H}u \phi v$, where $\phi$ is $E$-linear combination of motivated correspondances of $X$ to itself. We have thus

$$\xi = \ast_{H}u \phi v(\xi) = \ast_{H}(\phi _{t,s})(\xi_s).$$

Hence the result, considering the previous lemma. \qed

Remark 7.3.2. $P(\ast_{H}t_{s}^{\ast} \ast_{H}t_{s}s)$ is the orthogonal projector on $H_{\ell}^{*}(X_{s})_{\pi_{1}(S,s)}$ (relative to the Poincare pairing $\langle \cdot, \cdot \rangle$, or has the form $\langle \cdot, \ast_{H} \cdot \rangle$).

8. Variation of motivic Galois group in a family

8.1. Let $K_{0}$ be a separably closed field and $K$ a separably closed extension of $K_{0}$.

Let $S$ be a connected lisse $K_{0}$-scheme, and $f : X \to S$ a lisse projective morphism. We suppose that the field of functions of $S$ admits an embedding in $K$.

We suppose we are in one of the following two situations:

(a') $S$ is the complement of a closed of codimension at least 2 in a normal projective scheme $\overline{S}$,

(b') $K$ is algebraically closed, and for every $n \in \mathbb{Z} > 0$, there exists a finite covering $S_{n} \to S$ and an open immersion $S_{n} \hookrightarrow \overline{S}_{n}$ of $S_{n}$ in a connected lisse projective curve, such that the inverse image $f_{n}$ above of $S_{n}$ of the $n$-th fibre power $f^{\times s^n} = f \times_{S} \cdots \times_{S} f$ extends to a projective morphism $\overline{f}_{n} : \overline{X}_{n} \to \overline{S}_{n}$, with $\overline{X}_{n}$ lisse, and such that the canonical homomorphism $H_{\ell}^{*}(\overline{X}_{n}) \to \bigoplus H^{0}(S_{n}, R^{i}(f_{n})_{*}\mathbb{Q}_{\ell})$ admits a section compatible with cup-product with a relative polarisation.

8.2. Choice of $\mathcal{W}, \widetilde{V}, E$: we suppose that $\mathcal{W}$ contains the $K$-fibres of $f$.

In the case (a'), we suppose that $\widetilde{V}$ contains the couples of the form $(X \times_{S} T_{K}, V)$, where

- $T$ is a lisse projective connected curve arbitrarily drawn over $S$,
- $V = V_{f \times S T_{K}}$ if the orthogonal projector $\pi_{V_{f} \times S T_{K}}$ onto $V_{f \times S T_{K}}$ is given by an algebraic correspondance, and $V = H_{\ell}^{*}(X \times_{S} T_{K})$ if not.

In the case (b'), we suppose that $\widetilde{V}$ contains the couples $(\overline{X}_{n}, H_{\ell}^{*}(\overline{X}_{n}))$. We choose $E$ such that $\text{End} 1 = E$ in $\mathcal{M}_{\ell}(\mathcal{W})_{\widetilde{V}[E]}$, for example $E = \mathbb{Q}_{\ell}$.

\textsuperscript{26}See [30] in the case where $K$ is a finite field.
According to § 4.3, we can attach to every $K$-fibre $X_s$ its motivic Galois group $G_{X_s}$, which is a reductive subgroup of $\text{GL}(H^*_\ell(X_s))$: this is the largest subgroup of $\text{GL}(H^*_\ell(X))$ which fixes the motivated cycles fixed by $\sigma \circ \pi$ on the powers of $X_s$ (notations $\sigma, \pi$ of theorem 4.2.3).

We are interested here in the variation of $G_{X_s}$ with $s$. The case of characteristic zero has been treated in detail in [3, § 5] (see also [2]) and is subject to stronger results than the ones that follow, where we have more particularly in view the case of characteristic $p$.

8.3. For every $K$-point $s$ of $S$, denote by $H_{X_s}$ the connected monodromy group, i.e. the neutral component of the Zariski closure of the image of $\pi_1(S, s)$ in $\text{GL}(H^*_\ell(X_s))$.

As in [3] denote by $\text{Exc}$ the set of “exceptional” $K$-points $s$ such that $H_{X_s} \not\subset G_{X_s}$.

The terminology is justified by the fact that $\text{Exc}$ contains no generic $K$-points of $S$ (i.e. $K$-points whose image in $S$ is the generic point; they exist since $K_0(S)$ is supposedly embeddable in $K$). We have the following more general fact.

Lemma 8.3.1. Let $Z$ be the $K_0$-Zariski closure of the image of $s$ in $S$. Suppose that the image of $\pi_1(Z, s)$ in $\pi_1(S, s)$ is of finite index. Thus $s \notin \text{Exc}$.

Proof. The fibre $X_s$ as well as its powers is defined over the field of functions $K_0(Z)$ of integral $K_0$-scheme $Z$. Every motivated cycle over $X_s$ is thus invariant under a subgroup of finite index of $\text{Aut}(K/K_0(Z))$ (cf. [3, § 2.5, scolie]), thus under the action of a subgroup of finite index of $\pi_1(Z, s)$. The hypothesis made on $s$ implies thus that it is invariant under $H_{X_s}$. Since $G_{X_s}$ is the stabilizer of certain motivated cycles on the powers of $X_s$, we have $H_{X_s} \subset G_{X_s}$. \hfill \Box

Denote $\mathcal{W}'$ the smallest full sub-category of $\mathcal{W}$ stable under disjoint sums and products, and passing to connected components, and containing the $X_s$ with $s \notin \text{Exc}$.

8.4. We are in one of the situations (a') or (b') of § 8.1.

Theorem 8.4.1.

(1) For every $s \notin \text{Exc}$, $H_{X_s}$ is a normal subgroup of $G_{X_s}$, contained in the derived group $G_{X_s}^{\text{der}}$.

(2) Suppose that there exists $s_0$ such that

$$\mathcal{M}_\ell((X_{s_0}))_{\overline{\nu}}[E] = \overline{\mathcal{M}_\ell((X_{s_0}))_{\overline{\nu}}}[E].$$

Thus

$$\mathcal{M}_\ell(\mathcal{W}')_{\overline{\nu}}[E] = \overline{\mathcal{M}_\ell(\mathcal{W}')_{\overline{\nu}}}[E],$$

i.e. $\mathcal{M}_\ell(\mathcal{W}')_{\overline{\nu}}[E]$ is semi-simple Tannakian, and $G_{X_s}$ is independent of $s \notin \text{Exc}$ up to isomorphisms. (3) If in addition there exists $s_1$ such that $G_{X_{s_1}}$ is a torus, thus $H_{X_s} = G_{X_s}^{\text{der}}$ for every $s \notin \text{Exc}$. If $s_1 = s_0$, we even have an exact sequence

$$\{1\} \to H_{X_s} \to G_{X_s} \to T \to \{1\}$$

where $T$ is the quotient torus $G_{X_{s_0}}/(G_{X_{s_0}} \cap H_{X_{s_0}})$.

With the previous lemma, this result implies, in the case (a'), the theorem 0.5.1 of the introduction.
Proof. We can freely replace $S$ by a finite étale covering, so that we can suppose, and we will suppose, that the Zariski closure of the image of $\pi_1(S, s)$ in $\text{GL}(H^*_t(X_s))$ is connected, thus equal to $H_{X_s}$ (this condition is independent of $s$).

(1) Fix $s \notin \text{Exc}$. According to [15,3.4.13], $H_{X_s}$ is a reductive group, and according to [23, Theorem 6], it is even semi-simple, thus $H_{X_s} = H^\text{der}_{X_s} \subset G^\text{der}_{X_s}$.

Use the following normality criterion [1, Lemma 1].

Lemma 8.4.2. Let $V$ be a vector space over a field $F$ of characteristic zero, and let $H \subset G$ be closed algebraic subgroups of $\text{GL}(V)$. Suppose that for every character $\chi$ of $H$, and for every space of mixed tensors $T^{m,n}V = (V^*)^\otimes m \otimes V^\otimes n$, the subspace $T^{m,n}V$ where $H$ acts through $\chi$ is stable under $G$. Thus $H$ is normal in $G$. According to Chevalley, $H$ is the stabilizer of a right $D \subset T^{m,n}V$ suitable. Let $\chi$ be the character through which $H$ acts on $D$. For every commutative $F$-algebra $R$ and every element $g \in G(R)$, $g$ stabilizes $T^{m,n}(V_R)^\chi$ by hypothesis. Thus for every $h \in H(R), ghg^{-1}$ acts through homothety on $T^{n,n}(V_R)^\chi$, thus stabilizes $D_R$, which implies $ghg^{-1} \in H(R)$.

Let’s apply this to

$$H_{X_s} \subset G_{X_s} \subset \text{GL}(H^*_t(X_s)) \quad \text{and} \quad T^{m,n}H^*_t(X_s) \cong H(X_s)^{\otimes m+n}(md).$$

As $H_{X_s}$ is semi-simple, $\chi$ is trivial, and we are brought back to prove that for every integer $n > 0$, $H^*_t(X_s)^{\pi_1(S,s)}$ is stable under $G_{X_s}$, thus is cut by an idempotent $\mathbb{Q}_l$-linear combination of motivated correspondences modeled on $\tilde{V}$ in the image of $\sigma \circ \pi$. To simplify the notations, take $n = 1$.

In the case (a'), we start by replacing $S$ by a lisse projective connected curve $T$ traced over $S$ and passing through $s$. For that, it suffices to cut $S$ with $\dim S - 1$ quadrics passing through $s$, sufficiently general in a fixed projective embedding of $S$. This is a lisse connected curve according to [SGA 4, XI, 3], and it does not cut the boundary $\overline{S} \setminus S$ which is of codimension at least 2. In addition, by the Lefschetz theorem, $\pi_1(T,s) \to \pi_1(S,s)$ is surjective, so that $H(X_s)^{\pi_1(S,s)} = H(X_s)^{\pi_1(T,s)}$.

Likewise, in the case (b'), we can replace $S$ by the finite étale covering $S_n$.

We thus suppose in the following of the proof of point (1) that $S$ is a curve, and that we are in the situaton of Corollary 7.2.1 (case (b)) or the Theorem 7.3.1 (case (a')). By the remark 7.3.2 and the Lemma 7.1.1, the orthogonal projector $e = P(\star H^*_t \star H \: t_{ss})$ on $H(X_s)^{\pi_1(S,s)}$ is an idempotent in $\mathcal{M}_t((X_s))\varphi(h(X_s), h(X_s))$ (taking $P(x) = x$ in the case (b')). The same is true of its image $e'$ under $\sigma \circ \pi$ and it’s about seeing that $\text{Im} e' = H^*_t(X_s)^{\pi_1(S,s)}$. \hfill \Box

The fact that $e$ is an idempotent, $\mathbb{Q}_l[e]$ is a semi-simple $\mathbb{Q}_l$-subalgebra of $\mathcal{M}_t((X_s))\varphi(h(X_s), h(X_s))$. According to the complement of Malcev at theorem of splitting of Wedderburn (cf., [8, 12.2]), $\mathbb{Q}_l[e]$ is automatically conjugate to $\mathbb{Q}_l[e']$ by a unipotent element $u \in \mathcal{M}_t((X_s))\varphi(h(X_s), h(X_s))$. Since $s \notin \text{Exc}$, we have $\gamma u = u \gamma$ for every $\gamma \in \pi_1(S,s)$. Hence $\gamma e' = \gamma u e' = u e' = e$, which means that $e'H^*_t(X_s) = H^*_t(X_s)^{\pi_1(S,s)}$.

(2) Let’s fix level $s \notin \text{Exc}$. As $\pi_1(S,s)$ acts by conjugation through $H_{X_s}$ on $G_{X_s}$, that defines a system of subgroups $G_t \subset \text{GL}(H^*_t(X_s))$ for every $K$-point $t$ of $S$ (system locally constant in an appropriate way: $\pi_0(G_s)$ is constant, and the Lie $G_s$ form a local sub-system

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27 As $K = \overline{K}$, $K_0(S)$ remains embeddable in $K$.
28 The converse is true if $G$ is connected (cf. [1, Lemma 1]).
of $\bigoplus_j \text{End}(R^j f_s \mathbb{Q}_\ell)$. We have $G_s = G_{X_s}$ and $H_{X_t} < G_t$ according to (1).

Show that $G_{X_{s_0}} \subset G_{s_0}$: as they are reductive groups, it suffices to examine the tensors they fix. An element $\theta_{s_0}$ of $T^{m,n} H^*_\text{t}(\mathbb{Q}_\ell)$ fixed by $G_{s_0}$ is a $\pi_1(S, s_0)$-invariant tensor whose parallel transport at $s$ is fixed under $G_{X_s}$, thus is a $\mathbb{Q}_\ell$-linear combination of motivated cycles modeled on $\tilde{\mathcal{V}}$. By the corollary 7.2.1 (case (b')) or the theorem 7.3.1 (case (a')) $\theta_{s_0}$ is already a $\mathbb{Q}_\ell$-linear combination of motivated cycles modeled on $\tilde{\mathcal{V}}$, thus is fixed by $G_{X_{s_0}}$ since

$$\mathcal{M}_\ell(X_{s_0}, \tilde{\mathcal{V}}[E] = \overline{\mathcal{M}}_\ell((X_{s_0}), \tilde{\mathcal{V}}[E] \quad (\sigma = \pi = 1d).$$

Consider thus the group

$$G'_{s_0} := (\text{pr}_{G_{s_0}/H_{X_{s_0}}})^{-1}(G_{X_{s_0}}/(G_{X_{s_0}} \cap H_{X_{s_0}})).$$

This is a reductive sub-group of $G_{s_0}$ containing $H_{s_0}$. Again, we can form a system of sub-groups $G'_t \subset G_t$ for every $K$-point $t$ of $S$, locally constant in an appropriate sense, taking the value given at $s_0$.

An element $\theta'_s$ of $T^{m,n} H^*_\text{t}(\mathbb{Q}_\ell)$ fixed by $G'_s$ is a $\pi_1(S, s)$-invariant tensor whose parallel transport at $s_0$ is fixed under $G'_{s_0}$, thus is in particular a $\mathbb{Q}_\ell$-linear combination of motivated cycles modeled on $\tilde{\mathcal{V}}$. By the corollary 7.2.1 (case (b')) or the theorem 7.3.1 (case (a')), $\theta_s$ is already a $\mathbb{Q}_\ell$-linear combination of motivated cycles modeled over $\tilde{\mathcal{V}}$. And reciprocally. We conclude that $\mathcal{M}_\ell((X_{s_0}), \tilde{\mathcal{V}}[\mathbb{Q}_\ell]$ is semi-simple Tannakian, of Tannakian group $G'_{s_0}$, and thus that $G'_s = G_s = G_{X_s}$. This shows also that $G_{X_s}$ is independent of $s \notin \text{Exc}$, up to isomorphism.

More generally, let $s_1, \ldots, s_n$ be points out of $\text{Exc}$. By recurrence on $n$, we consider the composite morphism $X_{s_1} \times \cdots \times X_{s_{n-1}} \times X \xrightarrow{pr_n} X \to S$, we obtain that $\mathcal{M}_\ell((X_{s_0}, \ldots, X_{s_n}), \tilde{\mathcal{V}}[\mathbb{Q}_\ell]$ is semi-simple Tannakian, and the same is true of $\mathcal{M}_\ell(W'[\mathbb{Q}_\ell]$.

It remains to deduce that the same is again true of $\mathcal{M}_\ell(W'[\mathbb{Q}_\ell)]$ is zero. It’s about showing that for every $X \in W'$, and every nonzero element $\alpha \in A_{\text{mot}}(X, \tilde{\mathcal{V}}[E], there exists $\beta \in A_{\text{mot}}(X, \tilde{\mathcal{V}}[E]$ such that $\langle \alpha, \beta \rangle \neq 0$. Because $\mathcal{N}_\ell(W'[\mathbb{Q}_\ell] = 0$, there exists $\beta' \in A_{\text{mot}}(X, \tilde{\mathcal{V}}[\mathbb{Q}_\ell]$ such that $\langle \alpha, \beta' \rangle \neq 0$. We obtain the existence of $\beta$ in utilising the density of $A_{\text{mot}}(X, \tilde{\mathcal{V}}[E]$ in $A_{\text{mot}}(X, \tilde{\mathcal{V}}[\mathbb{Q}_\ell$ equipped with the $\ell$-adic topology ($E$ is dense in $\mathbb{Q}_\ell$).

(3) With the notations of (2), it suffices to show that $H_{s_1} = G_{s_1}^{\text{der}}$, that is to say that the action of $G_{s_1}$ on $T^{m,n} H^*_\text{t}(X_{s_1}, \pi_1(S, s_1)$ is abelian. For this it suffices to show that $G_{s_1}$ acts through $G_{X_{s_1}}$. An element $\theta_{s_1}$ of $T^{m,n} H^*_\text{t}(X_{s_1}, \pi_1(S, s_1)$ fixed by $G_{X_{s_1}}$ is a $\mathbb{Q}_\ell$-linear combination of motivated cycles modeled on $\tilde{\mathcal{V}}$, thus its parallel transport at $s$ is again, thus is fixed by $G_s = G_{X_s}$ by virtue of (2). Thus $H_{X_s} = G_{X_s}^{\text{der}}$.

If $s_1 = s_0, T = G_{X_{s_0}}/(G_{X_{s_0}} \cap H_{X_{s_0}})$ is a quotient torus of $G_{X_{s_0}}$, and we saw above that $G_{X_s} = G'_{s}$ fits into an exact sequence

$$\{1\} \to H_{X_s} \to G'_s \to G_{X_{s_0}}/(G_{X_{s_0}} \cap H_{X_{s_0}}) \to \{1\}.$$  

**Corollary 8.4.3.** If $S$ is projective integral, the set of points of $S$ which are images of a $K$-point $t$ such that $\mathcal{M}_\ell((X_t), \tilde{\mathcal{V}}[\mathbb{Q}_\ell$ is semi-simple Tannakian is stable by generalisation.

**Proof.** Let $s$ and $t$ be two $K$-points of $S$. We suppose that $\mathcal{M}_\ell((X_t), \tilde{\mathcal{V}}[\mathbb{Q}_\ell$ is semi-simple Tannakian and that $s$ is a $K$-points of the $K_0$-closure $Z$ of the image of $s$ in $S$. If $s$ falls into
the lisse part of $Z$, it suffices to apply the point (2) of Theorem (case (a')) to this lisse part. In general, we are brought back to the case by normalizing and by proceeding by successive specialisations. □

Remark 8.4.4. The Theorem 8.4.1 and the corollary 8.4.3 will be used in [7] in the case of abelian schemes in the case (a'). It would also be useful for the applications of [7] in the case (b'). We come up against the difficulty raised in §6.3.

9. Specialization of motivated cycles

9.1. Problem of specialization
Suppose that $K$ is the field of fractions of a complete discrete valuation ring $\mathfrak{o}$. We denote $k$ the residue field.

We say that $X \in \mathcal{P}_K$ have good reduction if this is the generic fibre of a lisse projective $\mathfrak{o}$-scheme $X$. By the proper smooth base change theorem, we have thus a canonical isomorphism between the cohomology of $X_K$ and that of the geometric special fibre $X_k$:

$$H^\ast(X_K, \mathbb{Q}_\ell) \cong H^\ast(X_k, \mathbb{Q}_\ell).$$

Conjecture 9.1.1. For every motivated cycle $\xi \in H^{2r}(X_K, \mathbb{Q}_\ell)(r)$, its image in $H^{2r}(X_k, \mathbb{Q}_\ell)(r)$ is a motivated cycle on $X_k$.

The difficulty comes from the bad reduction eventuality of the auxilliary variety $X'$ involved in the presentation of $\xi$ as motivated cycle. Explain this in more detail in the case where $X'$ is the generic fibre of a semi-stable strictly projective $\mathfrak{o}$-scheme (it can be reduced to this case thanks to the results of de Jong [13], but whatever); the same is true of $Z = X \times X'$.

Remember that an $\mathfrak{o}$-scheme $\mathfrak{Z}$ locally of finite presentation is said strictly semi-stable [13,32] if $\mathfrak{Z}$ is regular and flat over $\mathfrak{o}$, if its generic fibre $Z = \mathfrak{Z}_K$ is lisse, and if its special fibre $Y = \mathfrak{Z}_k$ is a reduced divisor whose components are lisse and meet transversely (for every $m > 0$, the disjoint sum $Y^{[m]}$ of $m$-tuple intersections of irreducible components of $Y$ is thus lisse over $k$).

If $\mathfrak{Z}$ is proper over $\mathfrak{o}$, we then have the “spectral sequence of weights” connecting the $\ell$-adic étale cohomology of $Z_K$ to that of $Y^{[m]}_K$, for every $\ell$ distinct from the residue characteristic of $\mathfrak{o}$:

$$E_1^{p,q} = \bigoplus_{i \geq \max(0,-p)} H^{q-2i}(Y^{[p+2i+1]}_K, \mathbb{Q}_\ell)(r-i) \Longrightarrow H^{p+q}(Z_K, \mathbb{Q}_\ell)(r)$$

(Rapoport and Zink [31], reviewed by Saito [32, 2.2.4]). According to [29] (or [21]), this spectral sequence degenerates at $E_2$.

In [32, §§2.3, 2.4], Saito showed the functoriality of this spectral sequence (“pullback”), its compatibility with direct images (“push-forward”) and with “cup-products” by the Chern classes of a vector bundle on $\mathfrak{X}$, and deduced the compatibility of this spectral sequence with algebraic correspondances (through the construction of a canonical “strictly semi-stable resolution” of fibre products of strictly semi-stable fibrations).

Suppose in addition $Z$ projective over $\mathfrak{o}$ of relative dimension $d$. We can thus take for $\mathcal{E}$ an ample invertible bundle, and we obtain the compatibility of the spectral sequence of weights with the Lefschetz isomorphism: by putting $r = d - p - q$, we have a morphism of spectral sequences
(the vertical arrows are isomorphisms [15, §4.1]). Both on \( E_1^{p,q} \) and on \( E_2^{p,q} = E_\infty^{p,q} \), the vertical arrows are Lefschetz involutions.

As O. Gabber pointed out to us, we cannot say that the spectral sequence of weights are compatible with the Lefschetz involution \( *_L \); indeed, \( d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q} \) commutes with \( \bigcup c_1(E)^{d-p-q} \) (which is \( *_L \) on \( E_1^{p,q} \)), but we have \( *_L = \bigcup c_1(E)^{d+1-p-q} \) on the term \( E_1^{p+1,q} \).

For this reason, returning to the case \( Z = X \times X' \), \( X \) having good reduction, \textit{it is not clear that the specialization of} \( \text{pr}_X^{X'}_* (\alpha \cup *_L \beta) \) \textit{is of the form}
\[
\text{pr}_Y^{Y''[1]} (\tilde{\alpha} |_{Y \times Y''[1]} \cup *_L \tilde{\beta} |_{Y \times Y''[1]}),
\]
for the suitable liftings \( \tilde{\alpha} \) and \( \tilde{\beta} \) of \( \alpha \) and \( \beta \) in \( H(3) \).

\textbf{9.2. A theorem of specialisation}

We will overcome this difficulty in a sufficiently particular situation, thanks to a process of deformation.

Again let \( K \) be a field of fractions of a complete discrete valuation ring \( \mathfrak{o} \), with algebraically closed residue field \( k \).

Let \( f : X \to S \) be a lisse projective morphism, with \( S \) projective flat over \( \mathfrak{o} \), with connected lisse generic fibre and reduced special fibre. Let \( s \) and \( t \) be two \( \mathfrak{o} \)-points of \( S \). Let \( \xi \in H^0(S_K, R^2f_!(f_K)_* \mathbb{Q}_l)(q) \). We suppose that \( \xi_K \) is algebraic.

We denote by \( S_i \) the normalization of irreducible components of the special fibre of \( S \) (these are lisse projective curves), and \( f_i : X_i \to S_i \) the lisse projective morphism from \( f \).

\textit{Choice of} \( \mathcal{W}_K, \tilde{\mathcal{V}}_K, \mathcal{W}_k, \tilde{\mathcal{V}}_k, E \): we suppose that \( \mathcal{W}_K \subset \mathcal{P}_K \) contains \( X_{sK}, X_{tK} \). We suppose that \( \tilde{\mathcal{V}}_K \) contains

(i) \( (X_K, V_{iK}) \) if the orthogonal projector \( \pi_{V_{iK}} \) onto \( V_{iK} \) is given by an algebraic correspondence,
(ii) \( (X_K, H^*_l(X_K)) \) if not.

We suppose that \( \mathcal{W}_k \subset \mathcal{P}_k \) contains \( X_{sK}, X_{tK} \) as well as the fibres of \( f_i \) at points of intersection \( S_i \cap S_j \). We suppose that \( \tilde{\mathcal{V}}_k \) contains

(i) \( (X_i, V_{f_i}) \) if the orthogonal projector \( \pi_{V_{f_i}} \) onto \( V_{f_i} \) is given by an algebraic correspondence,
(ii) \( (X_i, H^*_l(X_i)) \) if not.

We suppose that \( E = \text{End} \mathbf{1} \), for both \( \mathcal{M}_l(\mathcal{W}_K)_{\tilde{\mathcal{V}}_K}[E] \) and \( \mathcal{M}_l(\mathcal{W}_k)_{\tilde{\mathcal{V}}_k}[E] \).

\textbf{Theorem 9.2.1.} \( \xi_{tK} \) is an \( E \)-linear combination of motivated cycles modeled on \( \tilde{\mathcal{V}}_K \), and its specialisation \( \xi_{tK} \) is an \( E \)-linear combination of motivated cycles modeled on \( \tilde{\mathcal{V}}_K \).

Suppose in addition that \( \mathfrak{S}_K \) is a curve and that the condition (b) of theorem 5.3.1 is satisfied by both \( f_K \) and the \( f_i \) (which is the case in particular if \( f \) is an abelian scheme). Thus we can replace “\( E \)-linear combination of motivated cycles” by “motivated cycles”.

Proof. That $\xi_{tK}$ is an $E$-linear combination of motivated cycles modeled on $\overline{V}$ follows from Theorem 7.3.1. As $\mathfrak{S}$ is a connected special fibre reduced over perfect $k$, we have a surjective homomorphism of specialisation

$$\pi_1(\mathfrak{S}_K, t_K) \rightarrow \pi_1(\mathfrak{S}_k, t_k)$$

(cf. [SGA 1, Exposé 2.4]), which is compatible with the specialisation isomorphism

$$H^2_\ell(\mathfrak{X}_{ik}) \rightarrow H^2_\ell(\mathfrak{X}_{ik})$$

Taking the inverse image over the normalized $S_i$ of a component which contains $t_k$, we find that $\xi$ induces a $\pi_1(\mathfrak{S}_k, t_k)$-invariant element of $H^2_\ell(\mathfrak{X}_{ik})(q)$, hence a section $\xi_i$ of $H^0(S_i, R^{2\alpha}f_{i*}(Q_\ell))(q)$. Step by step, passing by the points of intersection of components of $\mathfrak{S}_k$ if necessary, we connect $s_k$ and $t_k$ and we apply the theorem of deformation (Theorem 7.3.1) (or the corollary 7.2.1 in the case where (b) is verified). \qed

9.3. Fix full sub-categories $\mathcal{W}' \subset \mathcal{W}$ of $\mathcal{P}_K$, stable under products and finite disjoint sums, and passing to connected components; hence $\mathbb{Q}$-vector space $A_{mot}(X)_{\mathcal{W}'_K}$ for $X \in \mathcal{W}'$. We suppose that:

1. for every variety $X \in \mathcal{W}'_K$, there exists a system of generators $\alpha$ of the $\mathbb{Q}$-algebra $A_{mot}(X)_{\mathcal{W}'_K}$, and for every $\alpha$, a connected geometrically lisse projective curve $S^\alpha$ over $K$, two $K$-points $s^\alpha$ and $t^\alpha$ of $S^\alpha$, and a projective lisse morphism $f^\alpha : X^\alpha \rightarrow S^\alpha$ such that
   (i) $X^2_{s^\alpha}$ and $X^2_{t^\alpha}$ are in $\mathcal{W}'_K$,
   (ii) there exists algebraic correspondances
   $$\tau^\alpha : H^*_{\ell}(X_{s^\alpha}) \rightarrow H^*_{\ell}(X), \quad \sigma^\alpha : H^*_{\ell}(X) \rightarrow H^*_{\ell}(X_{t^\alpha}), \quad \sigma^\alpha \circ \tau^\alpha = \text{Id},$$
   such that $\sigma^\alpha(\alpha)$ extends to a global section $\xi^\alpha$ of $R^{2\alpha}f^\alpha_*\mathbb{Q}_\ell$,
   (iii) the fibre at $s^\alpha$ of $\xi^\alpha$ is an algebraic cycle;

2. every object of $\mathcal{W}'_K$ has good reduction;
3. for every $\alpha$, we suppose given an extension $f^\alpha : X^\alpha \rightarrow \mathfrak{S}^\alpha$ of $f^\alpha$ to a projective lisse morphism of projective flat $\mathfrak{o}$-schemes, the special fibre of $\mathfrak{S}^\alpha$ being reduced.

Choice of $\mathcal{W}_K, \overline{\mathcal{W}}_K, \mathcal{W}_k, \overline{\mathcal{W}}_k, E$: as above, but for every $f^\alpha$.

Remark 9.3.1. By the theorem 7.3.1, we have a canonical faithful $\otimes$-functor

$$\mathcal{M}_\ell(\mathcal{W}'_K)_{\mathcal{W}'_K}[E] \rightarrow \mathcal{M}_\ell(\mathcal{W}_K)_{\overline{\mathcal{W}}_K}[E].$$

Corollary 9.3.2. Under these hypotheses, there exists a canonical faithful $\otimes$-functor $\mathcal{M}_\ell(\mathcal{W}_K)_{\overline{\mathcal{W}}_K}[E] \rightarrow \mathcal{M}_\ell(\mathcal{W}_K)_{\overline{\mathcal{W}}_K}[E]$. A fortiori, if $\text{char } K = 0$, we obtain a canonical exact faithful $\otimes$-functor of semi-simple Tannakian categories (with the notation of the proposition 4.2.1), called “specialisation functor”

$$\mathcal{M}_\ell(\mathcal{W}'_K)_{\mathcal{W}'_K}[E] \rightarrow \overline{\mathcal{M}}_\ell(\mathcal{W}_K)_{\overline{\mathcal{W}}_K}[E].$$

Proof. As $\mathfrak{S}^\alpha$ is proper over $\mathfrak{o}$, $s^\alpha$ and $t^\alpha$ extends into $\mathfrak{o}$-points $s^\alpha$ and $t^\alpha$ of $\mathfrak{S}^\alpha$. We apply the theorem 9.2.1 to each $f^\alpha$. \qed
**Remark 9.3.3.** The theorem of specialisation (Theorem 9.2.1) and its corollary (Corollary 9.3.2) will be applied in [7], in the case of abelian schemes and where \( \mathcal{S}_K \) is a curve. We are thus in the case (b) and, at least if \( k = \mathbb{F}_p \), in the case (i) of choice of \( \mathcal{V} \), etc. Otherwise, as \( \mathcal{S}_K \) is a curve, the theorem of stable reduction for the curves allows us to get rid of the hypothesis that the special fibre \( \mathcal{S}_k \) is reduced.

9.4. In the same vein as the conjecture 9.1.1 the question of behavior of the standard conjecture of Lefschetz type arises by specialisation in the strictly semi-stable case: with the notations of §9.1.

**Conjecture 9.4.1.** Suppose that the Lefschetz involution of \( Z = 3_K \) is algebraic. Thus the same is true of the Lefschetz involution of components \( Y_i \) of the special fibre \( 3_k \).

The problem is similar, albeit less subtle. It comes from that, if we obtain by specialisation the algebraic correspondances on \( Y_i \), it is only on the sub-quotients \( E_2 \) and \( E_1 \) that we are assured (by Saito) that they define \( *_L \).

The problem is serious, as shown in the following result.

**Scolie 9.4.2.** In the particular case of abelian pencil ovr a semi-stable projective \( \sigma \) curve, with \( k = \mathbb{C} \), the conjecture 9.4.1 implies the Hodge conjecture for abelian varieties.

**Proof.** In [3,§6], we have proven that the Hodge conjecture for complex abelian varieties is true if for \( X \), the total space of an arbitrary compact abelian pencil \( f : X \rightarrow S \) of relative dimension \( d > 1 \), the standard conjecture of Lefschetz type is true—or becomes true even to replace the lisse projective curve \( S \) by an etale covering, and to change \( f \) by another isogenous abelian pencil.

Prove thus that the conjecture 9.4.1 for a compact abelian pencil \( f : X \rightarrow S \) of relative dimension \( d > 1 \) principally polarized and equipped with a (symplectic) level structure \( \nu \geq 3 \), implies the standard conjecture of Lefschetz type for \( X \).

Let \( \mathcal{A} \) be the (lisse connected) moduli space of corresponding abelian varieties; we have a morphism of \( S \) to \( \mathcal{A} \), such that \( f \) is the inverse image of a universal abelian scheme over \( \mathcal{A} \). As \( d > 1 \), the boundary \( \partial \mathcal{A} \) of \( \mathcal{A} \) in its Baily-Borel compactification \( \mathcal{A}^* \subset \mathbb{P}^{N_2} \) is of (complex) codimension at least 2.

An embedding \( S \hookrightarrow \mathbb{P}^{N_1} \) being fixed, denote \( S_1 \) the image isomorphic to \( S \) in the “diagonal” embedding

\[
S \hookrightarrow S \times \mathcal{A}^* \subset \mathbb{P}^{N_2} \times \mathbb{P}^{N_1} \subset \mathbb{P}^{N=N_2N_1+N_2+N_1},
\]

and we denote \( \mathcal{O}(1) \) the sheaf of hyperplane sections of \( S \times \mathcal{A}^* \) relative to this projective embedding in \( \mathbb{P}^{N} \) and \( \mathcal{I} \) the ideal of \( S_1 \) in \( S \times \mathcal{A}^* \).

Let \( \epsilon : Bl \rightarrow S \times \mathcal{A}^* \) be the blow-up of \( S_1 \) in \( S \times \mathcal{A}^* \), \( E \) the exceptional divisor (which does not cut the strict transform of \( S \times \partial \mathcal{A} \)), \( \mathcal{O}_{Bl}(1) \) the invertible sheaf \( \mathcal{O}(-E) \). For \( \delta \) sufficiently large, the invertible sheaf \( \epsilon^*\mathcal{O}(\delta) \otimes \mathcal{O}_{Bl}(1) \) is very ample; a basis of global sections define an embedding \( Bl \hookrightarrow \mathbb{P}^{M} \). Noticing that \( \epsilon_*(\epsilon^*\mathcal{O}(\delta) \otimes \mathcal{O}_{Bl}(1)) = \mathcal{O}(\delta) \otimes \mathcal{I} \), we deduce that the linear system of hypersurfaces of degree \( \delta \) of \( \mathbb{P}^{N} \) passing through \( S_1 \) defines an embedding (locally closed) of \( (S \times \mathcal{A}^*) \setminus S_1 \) in the projective space \( \mathbb{P}^{M} \), composite of the canonical isomorphism \( (S \times \mathcal{A}^*) \setminus S_1 \cong Bl \setminus E \) and the embeddings \( Bl \setminus E \hookrightarrow Bl \hookrightarrow \mathbb{P}^{M} \) as above.

According to Bertini, a general system of \( (\dim \mathcal{A} - 2) \) hypersurfaces of degree \( \delta \) of \( \mathbb{P}^{N} \) passing through \( S_1 \) cuts \( (S \times \mathcal{A}^*) \setminus S_1 \) transversally, but avoid \( S \times \partial \mathcal{A} \). On the other hand, it is clear...
that such a system cuts $S_1$ transversally.
We conclude that it defines a connected lisse projective surface $V$ in $S \times \mathcal{A}$ which contains \( S_1 : S \cong S_1 \subset V \subset S \times \mathcal{A} \subset \mathbb{P}^N \).

We denote by $O_{\mathcal{V}}(1)$ the sheaf of hyperplane sections of $V$ relative to embedding in $\mathbb{P}^N$. Let also $\delta'$ be a sufficiently large integer, such that $O_{\mathcal{V}}(\delta') \otimes O(S_1)$ is very ample, and let $S_2$ be a lisse hyperplane section of $V$ of degree $\delta'$, cutting $S_1$ transversally. Consider the reducible curve $S_1 + S_2$ at ordinary quadratic points meeting of $S_1$ and $S_2$ as a hyperplane section of $V$ in an embedding $V \hookrightarrow \mathbb{P}^{M'}$ associated to $O_{\mathcal{V}}(\delta') \otimes O(S_1) : S_1 + S_2 = V \cap \mathcal{H}_0$. Let $\mathcal{H}$ be another hyperplane of $\mathbb{P}^{M'}$, transverse to $S_1$ and to $S_2$, but not passing by $S_1 \cap S_2$. Consider the linear pencil of hyperplanes passing by $\mathcal{H}_0$ and $\mathcal{H}$. Its axis cuts $V$ transversally at a finite set $\Sigma$ of points. If $\tilde{V}$ denotes the blow-up of $V$ along $\Sigma$, we have a morphism $g : \tilde{V} \to \mathbb{P}^1$, with $g^{-1}(0) = S_1 + S_2$. According to Bertini again, every fibre of $g$ except a finite number are connected lisse projective curves.

The universal abelian scheme over $\mathcal{A}$ defines by reciprocal image an abelian scheme $\tilde{f} : \tilde{X} \to \tilde{V}$, whose restriction to $S \cong S_1 \subset g^{-1}(0)$ is the original pencil. To apply the conjecture 9.4.1, it’s about showing that the generic fibre of composite morphism $g \circ \tilde{f} : \tilde{X} \to \mathbb{P}^1$ (which is the total space of an abelian pencil of base $g^{-1}(\eta)$) satisfies the standard conjecture of Lefschetz type. As it is construction of algebraic cycles, it is also free to replace “generic point” (in the sense of Grothendieck) by “Weil generic point”, i.e. a point $t \in \mathbb{P}^1(\mathbb{C})$ transcendental over a field of definition of $g \circ \tilde{f}$.

According to Lefschetz, $\pi_1(V) \to \pi_1(S \times \mathcal{A})$ is surjective,\(^\text{29}\) in the same way $\pi_1(g^{-1}(t)) \to \pi_1(\tilde{V})$ for $t$ sufficiently general. On the other hand, if $\varepsilon' : \tilde{V} \to V$ denotes the blow-up morphism, $\varepsilon'_* : \pi_1(\tilde{V}) \to \pi_1(V)$ is an isomorphism (Van Kampen theorem brings us back to replace $V$ by little balls around the points of blow-up center, and to shrink in consequence; since then, the exceptional divisor—a disjoint union of straight-lines —is a deformation retract of $\tilde{V}$ shrunk, which is thus contractible).

In composing these epimorphisms, we see thus that for every $s \in g^{-1}(t)$, the image of $\pi_1(g^{-1}(t), s)$ in $\text{Sp} H^1(\tilde{f}^{-1}(s), \mathbb{Q})$ by the monodromy representation of abelian pencil is Zariski-dense. Or the algebra of invariants of $\text{Sp} H^1(\tilde{f}^{-1}(s), \mathbb{Q})$ in

\[ H(\tilde{f}^{-1}(s), \mathbb{Q}) \cong \bigwedge H^1(\tilde{f}^{-1}(s), \mathbb{Q}) \]

is generated by the polarization. We conclude by applying to this abelian pencil the following proposition.

\[ \text{Proposition 9.4.3.} \] Let $f : X \to S$ be a compact abelian pencil, such that $H^*_c(X_{\sigma})^{\pi_1(S, \sigma)}$ is generated by algebraic cycles defined over $\mathbb{C}(S)$, $\sigma$ denoting a generic geometric point of $S$. Thus $X$ verifies the standard conjecture of Lefschetz type.

\[ \text{Proof.} \] It’s about demonstrating that $*_L$ on $H(X, \mathbb{Q})$ is given by an algebraic correspondence, or, equivalently, that $*_L$ on $H(X, \mathbb{Q})$ is a $\mathbb{Q}$-linear combination of algebraic correspondences. Let’s resume the notations of proposition 6.2.1. On the component $V^\perp_f$ of $H^*_c(X)$, it follows from the property (d) of §6.4.1.

Let $\imath_\eta : X_\sigma \hookrightarrow X$ be the inclusion of the generic fibre. Thus

\[ V_f = \text{Im} \imath_{\sigma,*} \oplus *_H(\text{Im} \imath_{\sigma,*}) \quad \text{and} \quad \imath_{\sigma,*}(*_H(\text{Im} \imath_{\sigma,*})) = H^*_c(X_{\sigma})^{\pi_1(S, \sigma)}. \]

\[ ^{29}\text{Of course this is about } \pi_1 \text{ discrete.} \]
Choose a $\mathbb{Q}_\ell$-basis $(\alpha_1, \cdots, \alpha_r)$ of algebraic cycles over $X_\sigma$ which generates $H^*_\ell(X_\sigma)^{\pi_1(S, \sigma)}$ over $\mathbb{Q}_\ell$. By specialisation on the one hand, and by image under $\iota_\sigma^*, \ast$ on the other hand, we obtain a family of $2r$ algebraic cycles which generate $V_f$. As $\langle \cdot, \cdot \rangle$ is non-degenerate on $V_f$, it follows that everything endomorphism of $V_f$ (in particular, the restriction of $\ast_L$) is a $\mathbb{Q}_\ell$-linear combination of algebraic correspondances.