ON THE GEOMETRIC LANGLANDS CONJECTURES

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Minor thesis under the supervision of Prof. Dennis Gaitsgory

Contents

In this minor thesis, we will first give a brief overview of the (categorical) de Rham Geometric Langlands Conjecture, following various references including [Gai15] etc. In the end, we will also briefly mention the Betti Geometric Langlands Conjecture and a "restricted" Geometric Langlands Conjecture, but our exposition will primarily focus on the de Rham paradigm.

1. The de Rham Geometric Langlands Conjecture

1.0.1. Let k be an algebraically closed field of characteristic zero, and X a smooth and complete curve over k. Let G be a reductive group and $\check{G}$ its Langlands dual group, viewed as an algebraic group over k. The (categorical) de Rham Geometric Langlands Conjecture is a conjectural equivalence between two DG categories. The "geometric" (or "automorphic") side is the DG category $\text{D-mod}(\text{Bun}_G)$ of D-modules on the stack Bun_G of G-bundles on X. The "spectral" (or "Galois") side is given by quasi-coherent sheaves on the moduli stack $\text{LocSys}_{\check{G}}$ of $\check{G}$-local systems on X.

While our naive guess for the spectral side would be the DG category $\text{QCoh}(\text{LocSys}_{\check{G}})$, one can check that this guess is wrong as the corresponding geometric Langlands equivalence fails to be compatible with the the Eisenstein functor. More precisely, for P a parabolic of G with Levi quotient M, the Eisenstein functor on the geometric side

$$\text{Eis}_P : \text{D-mod}(\text{Bun}_M) \to \text{D-mod}(\text{Bun})_G)$$

and the Eisenstein functor on the spectral side

$$\text{Eis}_{\check{P},\text{spec}} : \text{QCoh}(\text{LocSys}_{\check{M}}) \to \text{QCoh}(\text{LocSys}_{\check{G}})$$

are conjectured to match up under the geometric Langlands equivalence (up to a twist by some line bundles), but since the functor Eis_P preserves compactness whereas $\mathrm{Eis}_{\check{P},\mathrm{spec}}$ does not, we get a contradiction. Hence $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ cannot be the correct spectral side. Instead, we consider a certain enlargement of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ to be our modified spectral side, namely the category

$$\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}),$$

using the theory of singular support of coherent sheaves developed in [AG15]. In summary, the (categorical) de Rham geometric Langlands conjecture predicts an equivalence

$$\mathbb{L}_G : \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\sim} \text{D-mod}(\mathrm{Bun}_G). \tag{1.0.2}$$

Moreover, $\mathbb{L}_G$ is expected to satisfy Property $\mathrm{Wh}^{\mathrm{ext}}$ (see 1.2.15), Property $\mathrm{He}^{\mathrm{naive}}$, Property $\mathrm{Ei}^{\mathrm{enh}}$ and Property $\mathrm{Km}^{\mathrm{prel}}$ (see [Gai15, § 4.4.5, 6.6.4, 10.3.5] respectively).

The idea for proving such an equivalence $\mathbb{L}_G$ is via embedding both sides of 1.0.2 into larger and more tractable categories and then compare the essential images.

1.1. **The geometric side.**

1.1.1. On the geometric side, the more tractable category is the *extended Whittaker category* $\mathrm{Whit}^{\mathrm{ext}}(G,G)$. In this section, we give a brief introduction to this category. First we start with the usual Whittaker category for G.

Consider the prestack that assigns to $S \in \mathrm{Sch}^{\mathrm{aff}}$ the groupoid of triples $(\mathcal{P}_G, U, \alpha)$ where

- $\mathcal{P}_G$ is a G-bundle on $S \times X$;
- U is a Zariski-open subset of $S \times X$ such that for each k-point of S, the corresponding open subset $\mathrm{pt} \times_S U \subset \mathrm{pt} \times_S (S \times X) \simeq X$ is non-empty;
- α is a datum of a reduction of $\mathcal{P}_G|U$ to the Borel subgroup B.

Let $\mathcal{P}_{B,U}$ be the B-bundle on U corresponding to α, and $\mathcal{P}_{T,U}$ the induced T-bundle.

Let $\mathrm{Bun}_G^{B\text{-gen}}$ be the prestack of "generic reductions to the Borel". We recall from [Gai15, 5.1.1] that this is the prestack that assigns to $S \in \mathrm{Sch}^{\mathrm{aff}}$ the quotient of the above groupoid of triples $(\mathcal{P}_G, U, \alpha)$ by the equivalence relation $\sim$, where $(\mathcal{P}^1_G, U^1, \alpha^1) \sim (\mathcal{P}^2_G, U^2, \alpha^2)$ iff $\mathcal{P}^1_G \simeq \mathcal{P}^2_G$ and $\alpha^1|_{U^1\cap U^2} = \alpha^2|_{U^1\cap U^2}$ under the identification of $\mathcal{P}^i_G$.

Analogously, replacing B by N, we have the prestack $\mathrm{Bun}_G^{N\text{-gen}}$ of "generic reductions to the unipotent radical", which assigns to each $S \in \mathrm{Sch}^{\mathrm{aff}}$ the groupoid of quadruples $(\mathcal{P}_G, U, \alpha, \gamma)$, where $(\mathcal{P}_G, U, \alpha)$ is as before, and γ is a datum of a trivialization of the T-bundle $\mathcal{P}_{T,U}$.

We fix a square root $\omega_X^{\frac{1}{2}}$ of the canonical line bundle on X, and get a T-bundle

$$\check{\rho}(\omega_X) := 2\check{\rho}(\omega_X^{\frac{1}{2}}).$$

We define the prestack $\mathcal{Q}_G := \mathrm{Bun}_G^{N^\omega\text{-gen}}$ to be a twist of $\mathrm{Bun}_G^{N\text{-gen}}$ which assigns to S the data $(\mathcal{P}_G, U, \alpha, \gamma)$, where γ is modified to be an isomorphism with $\check{\rho}(\omega_X)|_U$.

We define the prestack $\mathcal{Q}_{G,G}$ as the quotient of $\mathrm{Bun}_G^{N^\omega\text{-gen}}$ by the action of $\mathbf{Maps}(X, Z^0_G)^{\mathrm{gen}}$, where Z^0_G is the connected component of the center of G.

Let $\mathrm{Ran}(X)$ be the Ran space of X. We recall that it is defined to be the prestack such that, for $S \in \mathrm{DG\,Sch}^{\mathrm{aff}}$, the ∞-groupoid $\mathrm{Maps}(S, \mathrm{Ran}(X))$ is the *set* of non-empty finite subsets of the set

$$\mathrm{Maps}(S, X_{\mathrm{dR}}) = \mathrm{Maps}((^{cl}S)_{\mathrm{red}}, X).$$

We refer the reader to [Gai15, 5.4.1] for the definitions of the open substack $(\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}}$ of $\mathcal{Q}_G \times \mathrm{Ran}(X)$ and the groupoid $\mathbf{N}$, and [Gai15, 5.5] for the character χ on $\mathbf{N}$.

Let $\text{D-mod}((\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}_G}, \chi}$ be the equivariant category of $\text{D-mod}((\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}})$ with respect to the groupoid $\mathbf{N}_{\mathcal{Q}_G}$ against the character χ.

Definition 1.1.2. The *Whittaker category* $\mathrm{Whit}(G)$ for G is the full subcategory of $\text{D-mod}(\mathcal{Q}_G)$ equal to the preimage of

$$\text{D-mod}((\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}_G}, \chi} \subset \text{D-mod}((\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}})$$

under the pullback functor

$$\text{D-mod}(\mathcal{Q}_G) \to \text{D-mod}((\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}})$$

i.e. $\mathrm{Whit}(G)$ is the fibre product

$$\mathrm{Whit}(G) := \text{D-mod}(\mathcal{Q}_G) \times_{\text{D-mod}((\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}})} \text{D-mod}((\mathcal{Q}_G \times \mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}_G}, \chi}$$

We denote by $\mathrm{Av}^{\mathbf{N}, \chi}$ the right adjoint of the fully faithful embedding

$$\mathrm{Whit}(G) \hookrightarrow \text{D-mod}(\mathcal{Q}_G).$$

Likewise, we define the variant category $\mathrm{Whit}(G, G)$ using the prestack $\mathcal{Q}_{G,G}$ instead of $\mathcal{Q}_G$, i.e.

Definition 1.1.3. The *Whittaker category* $\mathrm{Whit}(G, G)$ is the full subcategory of $\text{D-mod}(\mathcal{Q}_{G,G})$ equal to the preimage of

$$\text{D-mod}((\mathcal{Q}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}_{G,G}}, \chi} \subset \text{D-mod}((\mathcal{Q}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})$$

under the pullback functor

$$\text{D-mod}(\mathcal{Q}_{G,G}) \to \text{D-mod}((\mathcal{Q}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})$$

i.e. $\mathrm{Whit}(G, G)$ is the fibre product

$$\mathrm{Whit}(G, G) := \text{D-mod}(\mathcal{Q}_{G,G}) \times_{\text{D-mod}((\mathcal{Q}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})} \text{D-mod}((\mathcal{Q}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}_{G,G}}, \chi}$$

Let $\mathfrak{r}_G$ (resp. $\mathfrak{r}_{G,G}$) be the forgetful map $\mathcal{Q} \to \mathrm{Bun}_G$ (resp. $\mathcal{Q}_{G,G} \to \mathrm{Bun}_G$), which induce functors

$$\mathfrak{r}_G^\dagger : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathcal{Q}_G)$$

and

$$\mathfrak{r}_{G,G}^\dagger : \text{D-mod}(\mathrm{Bun}_G) \to \text{D-mod}(\mathcal{Q}_{G,G}).$$

We define the functors of *Whittaker coefficients* as

$$\mathrm{coeff}_G := \mathrm{Av}^{\mathbf{N}, \chi} \circ (\mathfrak{r}_G)^\dagger : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G)$$

and

$$\mathrm{coeff}_{G,G} := \mathrm{Av}^{\mathbf{N}, \chi} \circ (\mathfrak{r}_{G,G})^\dagger : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G, G)$$

1.1.4. Let $\mathfrak{ch}(G) := \mathrm{Spec}(k[\Lambda^{\mathrm{pos,sat}_G}])$ be the toric variety which classifies maps of monoids $\Lambda^{\mathrm{pos,sat}_G} \to \mathbb{A}^1$, with $\mathbb{A}^1$ viewed as a monoid with respect to multiplication. Here Λ is the weight lattice of G, and $\Lambda^{\mathrm{pos}} \subset \Lambda$ (resp. $\Lambda^{\mathrm{pos},\mathbb{Q}} \subset \Lambda^{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$) is the sub-monoid of weights that can be expressed as integral (resp. rational) non-negative combinations of simple roots. Denote by $\Lambda^{\mathrm{pos,sat}_G} := \Lambda \cap \Lambda^{\mathrm{pos},\mathbb{Q}}$ the saturation of Λ^{pos}.

We define the prestack $\mathcal{Q}^{\mathrm{ext}}_{G,G}$ by making the following modification to the definition of $\mathcal{Q}_{G,G}$: for a quadruple $(\mathcal{P}_G, U, \alpha, \gamma)$, let γ instead be a section over U of the scheme $\mathfrak{ch}(G)_{\check{\rho}(\omega_X)|_U \otimes \mathcal{P}_T^{-1}}$.

Let $\mathfrak{r}^{\mathrm{ext}}_{G,G}$ be the forgetful map $\mathcal{Q}^{\mathrm{ext}}_{G,G} \to \mathrm{Bun}_G$.

Definition 1.1.5. The *extended Whittaker category* $\mathrm{Whit}^{\mathrm{ext}}(G,G)$ is the preimage of

$$\text{D-mod}((\mathcal{Q}^{\mathrm{ext}}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}^{\mathrm{ext}}_{G,G}}, \chi^{\mathrm{ext}}} \subset \text{D-mod}((\mathcal{Q}^{\mathrm{ext}}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})$$

under the pullback functor

$$\text{D-mod}(\mathcal{Q}^{\mathrm{ext}}_{G,G}) \to \text{D-mod}((\mathcal{Q}^{\mathrm{ext}}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})$$

i.e. $\mathrm{Whit}(G,G)^{\mathrm{ext}}$ is the fibre product

$$\mathrm{Whit}(G,G)^{\mathrm{ext}} := \text{D-mod}(\mathcal{Q}^{\mathrm{ext}}_{G,G}) \times_{\text{D-mod}((\mathcal{Q}^{\mathrm{ext}}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})} \text{D-mod}((\mathcal{Q}^{\mathrm{ext}}_{G,G} \times \mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}^{\mathrm{ext}}_{G,G}}, \chi^{\mathrm{ext}}}$$

The fully faithful forgetful functor

$$\mathrm{Whit}(G,G)^{\mathrm{ext}} \to \text{D-mod}(\mathcal{Q}^{\mathrm{ext}})$$

admits a right adjoint which we denote by $\mathrm{Av}^{\mathbf{N},\chi^{\mathrm{ext}}}$.

Definition 1.1.6. We define the functor of *extended Whittaker coefficient*

$$\mathrm{coeff}^{\mathrm{ext}}_{G,G} : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}^{\mathrm{ext}}(G,G)$$

to be

$$\mathrm{coeff}^{\mathrm{ext}}_{G,G} := \mathrm{Av}^{\mathbf{N},\chi^{\mathrm{ext}}} \circ (\mathfrak{r}^{\mathrm{ext}}_{G,G})^{\dagger}.$$

We have the following key conjecture on the geometric side of the Langlands conjecture:

Conjecture 1.1.7. The functor $\mathrm{coeff}^{\mathrm{ext}}_{G,G}$ is fully faithful.

Theorem 1.1.8. [Gai15, Theorem 8.2.10] *Conjecture 1.1.7 holds for $G = \mathrm{GL}_n$.*

1.2. **The spectral side.**

1.2.1. On the spectral side, the more tractable category is $\mathrm{Glue}(\check{G})_{\mathrm{spec}}$. First we need to introduce various Eisenstein and constant term functors on the spectral side.

To define the (enhanced) constant term functor on the spectral side, we need to first recall a few notations from [Gai15, 6.4].

Consider the derived stack $\mathrm{LocSys}_{\check{P}}$ and the diagram

$$\mathrm{LocSys}_{\check{G}} \xleftarrow{\mathsf{p}_{\check{P},\mathrm{spec}}} \mathrm{LocSys}_{\check{P}} \xrightarrow{\mathsf{q}_{\check{P},\mathrm{spec}}} \mathrm{LocSys}_{\check{M}}$$

Since $\mathsf{q}_{\check{P},\mathrm{spec}}$ is quasi-smooth, we have a well-defined functor

$$\mathsf{q}_{\check{P},\mathrm{spec}}^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}}).$$

On the other hand, since the morphism $\mathsf{p}_{\check{P},\mathrm{spec}}$ is schematic and proper, we have a well-defined *continuous* functor

$$\mathsf{p}_{\check{P},\mathrm{spec}}^{!} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}}),$$

which is right adjoint to $(\mathsf{p}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}}$.

Lemma 1.2.2. [Gai15, Lemma 6.4.3]

(a) *The functor* $\mathsf{q}_{\check{P},\mathrm{spec}}^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}})$ *sends the subcategory* $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{M}}}(\mathrm{LocSys}_{\check{M}})$ *to the subcategory* $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}}}(\mathrm{LocSys}_{\check{P}})$.

(b) *The functor* $(\mathsf{p}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$ *sends the subcategory* $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}}}(\mathrm{LocSys}_{\check{P}})$ *to the subcategory* $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$.

Definition 1.2.3. The *spectral Eisenstein functor*

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{M}}}(\mathrm{LocSys}_{\check{M}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$$

is given by

$$\mathrm{Eis}_{\check{P},\mathrm{spec}} := (\mathsf{p}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}} \circ \mathsf{q}_{\check{P},\mathrm{spec}}^{\mathrm{IndCoh},*}.$$

The *spectral constant term functor*

$$\mathrm{CT}_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{M}}}(\mathrm{LocSys}_{\check{M}})$$

is given by

$$\mathrm{CT}_{\check{P},\mathrm{spec}} := (\mathsf{q}_{\check{P},\mathrm{spec}})_*^{\mathrm{IndCoh}} \circ \mathsf{p}_{\check{P},\mathrm{spec}}^{!}.$$

By construction, $\mathrm{CT}_{\check{P},\mathrm{spec}}$ is the right adjoint of $\mathrm{Eis}_{\check{P},\mathrm{spec}}$.

1.2.4. Consider the groupoid

$$\begin{array}{ccc} & \mathrm{LocSys}_{\check{P}} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}} & \\ \swarrow^{p_2} & & \searrow^{p_1} \\ \mathrm{LocSys}_{\check{P}} & & \mathrm{LocSys}_{\check{P}} \end{array}$$

over $\mathrm{LocSys}_{\check{P}}$. Since the maps p_1 and p_2 are schemeatic and proper, we have an adjoint pair of continuous functors

$$(p_i)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}}) \to \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}}) : p_i^!.$$

Let

(1.2.5)
$$\mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}})_{\Delta} \hookrightarrow \mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}})$$

be the full subcategory consisting of objects that are set-theoretically supported on the image of the diagonal embedding $\mathrm{LocSys}_{\check{P}} \hookrightarrow \mathrm{LocSys}_{\check{P}} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}}$. Let $(p_{i,\Delta})_*^{\mathrm{IndCoh}}$ denote the restriction of $(p_i)_*^{\mathrm{IndCoh}}$ to the subcategory 1.2.5. Let $p_{i,\Delta}^!$ be the right adjoint of $(p_{i,\Delta})_*^{\mathrm{IndCoh}}$.

The structure of groupoid on $\mathrm{LocSys}_{\check{P}} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{LocSys}_{\check{P}}$ endows the endo-functor $(p_{2,\Delta})_*^{\mathrm{IndCoh}} \circ p_{1,\Delta}^!$ of $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}})$ with a mondad structure, and we denote this monad by $\mathsf{F}_{\check{P}}$. Consider the category $\mathsf{F}_{\check{P}}$-mod$(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}))$ of $\mathsf{F}_{\check{P}}$-modules in $\mathrm{IndCoh}(\mathrm{LocSys}_{\check{P}})$.

Let $\mathrm{CT}^{\mathrm{enh}}_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) \to \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}}}(\mathrm{LocSys}_{\check{P}}))$ be the *enhanced spectral constant term functor* defined in [Gai15, 6.5.7], which is obtained from the functor $\mathsf{p}^!_{\check{P},\mathrm{spec}}$ via the factorization

$$\mathsf{p}^!_{\check{P},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) \to \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}}}(\mathrm{LocSys}_{\check{P}})) \xrightarrow{\mathbf{oblv}_{\mathsf{F}_{\check{P}}}} \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}}}(\mathrm{LocSys}_{\check{P}}).$$

Likewise, we also have an *enhanced spectral Eisenstein functor* $\mathrm{Eis}^{\mathrm{enh}}_{\check{P},\mathrm{spec}}$, which is the left adjoint of $\mathrm{CT}^{\mathrm{enh}}_{\check{P},\mathrm{spec}}$.

1.2.6. We are now ready to define the gluing category $\mathrm{Glue}(\check{G})_{\mathrm{spec}}$.

More precisely, let the category $\mathrm{Par}(G)$ opposite to the poset of standard parabolics of G be the index category. The assignment

$$P \mapsto \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}))$$

can be upgraded to a lax diagram of DG categories, parametrized by $\mathrm{Par}(G)$, in the following way:

For $P_1 \subset P_2$, let

$$\mathsf{p}_{P_1/P_2,\mathrm{spec}} : \mathrm{LocSys}_{\check{P}_1} \to \mathrm{LocSys}_{\check{P}_2}$$

be the corresponding map, which induces an adjoint pair of functors

$$(\mathsf{p}_{P_1/P_2,\mathrm{spec}})^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}_1}}(\mathrm{LocSys}_{\check{P}_1}) \rightleftarrows \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}_2}}(\mathrm{LocSys}_{\check{P}_2}) : \mathsf{p}^!_{P_1/P_2,\mathrm{spec}},$$

and thus the pair of functors (which we still denote by the same notations)

$$(\mathsf{p}_{P_1/P_2,\mathrm{spec}})^{\mathrm{IndCoh}}_* : \mathsf{F}_{\check{P}_1}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}_1}}(\mathrm{LocSys}_{\check{P}_1})) \rightleftarrows \mathsf{F}_{\check{P}_2}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}_2}}(\mathrm{LocSys}_{\check{P}_2})) : \mathsf{p}^!_{P_1/P_2,\mathrm{spec}}$$

that commute with the forgetful functors

$$\mathbf{oblv}_{\mathsf{F}_{P_i}} : \mathsf{F}_{\check{P}_i}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}_i}}(\mathrm{LocSys}_{\check{P}_i})) \to \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}_i}}(\mathrm{LocSys}_{\check{P}_i}).$$

On the other hand, we also have functors [Gai15, 6.5.6]

$$\Xi_{\check{P}_i} : \mathsf{F}_{P_i}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}_i})) \rightleftarrows \mathsf{F}_{P_i}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}_i}}(\mathrm{LocSys}_{\check{P}_i})) : \Psi_{\check{P}_i}.$$

Consider the composite functor

$$\Psi_{\check{P}_1} \circ \mathsf{p}^!_{P_1/P_2,\mathrm{spec}} \circ \Xi_{\check{P}_2} : \mathsf{F}_{P_2}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}_2})) \to \mathsf{F}_{P_1}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}_1})).$$

Thus the assignment

$$P \mapsto \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}))$$

upgrades to a lax diagram of DG categories, parametrized by $\mathrm{Par}(G)$, and we denote the resulting lax limit category by

$$\mathrm{Glue}(\check{G})_{\mathrm{spec}}.$$

For each parabolic P, we denote by

$$\mathrm{ev}_{\check{P},\mathrm{spec}} : \mathrm{Glue}(\check{G})_{\mathrm{spec}} \to \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}))$$

the corresponding evaluation functor.

Definition 1.2.7. We define the canonical functor

$$\mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}}) : \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Glue}(\check{G})_{\mathrm{spec}}$$

by the collection of functors $\{\Psi_{\check{P}} \circ \mathrm{CT}^{\mathrm{enh}}_{\check{P},\mathrm{spec}}\}$ for each parabolic P,

$$\mathrm{ev}_{\check{P},\mathrm{spec}} \circ \mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}}) := \Psi_{\check{P}} \circ \mathrm{CT}^{\mathrm{enh}}_{\check{P},\mathrm{spec}},$$

i.e. $\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) \xrightarrow{\mathrm{CT}^{\mathrm{enh}}_{\check{P},\mathrm{spec}}} \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{P}}}(\mathrm{LocSys}_{\check{P}})) \xrightarrow{\Psi_{\check{P}}} \mathsf{F}_{\check{P}}\text{-mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}}))$.

We have the following key conjecture on the spectral side of the Langlands conjecture:

Conjecture 1.2.8. The functor

$$\mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}}) : \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Glue}(\check{G})_{\mathrm{spec}}$$

is fully faithful.

Unlike the geometric side conjecture 1.1.7, which is only known for $G = \mathrm{GL}_n$, the spectral side conjecture is proved for all reductive groups.

Theorem 1.2.9. [Gai15, Theorem 9.3.8] *(Arinkin-Gaitsgory) Conjecture 1.2.8 holds for all reductive groups G.*

1.2.10. To relate the geometric side to the spectral side, we first express the extended Whittaker category $\mathrm{Whit}^{\mathrm{ext}}(G, G)$ defined in 1.1.5 as a glued category.

The prestack $\mathcal{Q}_{G,P}$ is defined similarly as the prestack $\mathcal{Q}_{G,G}$, and we refer the reader to [Gai15, 7.1.1] for the precise definition. Similarly, we also have the forgetful map $\mathfrak{r}_{G,P} : \mathcal{Q}_{G,P} \to \mathrm{Bun}_G$. The groupoid $\mathbf{N}_{\mathcal{Q}_{G,P}}$ is also defined similarly (see [Gai15, 7.1.2]), with character χ_P.

Definition 1.2.11. The *degenerate Whittaker category* $\mathrm{Whit}(G, P)$ is the full subcategory of $\text{D-mod}(\mathcal{Q}_{G,P})$ equal to the preimage of

$$\text{D-mod}((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}_{G,P}},\chi_P} \subset \text{D-mod}((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}})$$

under the pullback functor

$$\text{D-mod}(\mathcal{Q}_{G,P}) \to \text{D-mod}((\mathcal{Q}_{G,P} \times \mathrm{Ran}(X))_{\mathrm{good}})$$

i.e. $\mathrm{Whit}(G, P)$ is the fibre product

$$\mathrm{Whit}(G,P) := \text{D-mod}(\mathcal{Q}_{G,P}) \times_{\text{D-mod}((\mathcal{Q}_{G,P}\times\mathrm{Ran}(X))_{\mathrm{good}})} \text{D-mod}((\mathcal{Q}_{G,P}\times\mathrm{Ran}(X))_{\mathrm{good}})^{\mathbf{N}_{\mathcal{Q}_{G,P}},\chi_P}$$

Note that when $P = G$, Definition 1.2.11 recovers the definition 1.1.2 for $\mathrm{Whit}(G, G)$. As in the case of $\mathrm{Whit}(G, G)$, the fully faithful forgetful functor

$$\mathrm{Whit}(G, P) \to \text{D-mod}(\mathcal{Q}_{G,P})$$

admits a right adjoint, denoted by $\mathrm{Av}^{\mathbf{N},\chi_P}$.

For a parabolic P of G with Levi quotient M, we have a canonical locally closed embedding of prestacks ([Gai15, 8.3.1])

$$\mathbf{i}_P : \mathcal{Q}_{G,P} \to \mathcal{Q}^{\mathrm{ext}}_{G,G},$$

which corresponds to the locally closed subscheme $\mathring{\mathfrak{ch}}(M) \subset \mathfrak{ch}(G)$. The functor $(\mathfrak{i}_P)^\dagger$ gives rise to a functor

$$(\mathfrak{i}_P)^\dagger : \mathrm{Whit}^{\mathrm{ext}}(G, G) \to \mathrm{Whit}(G, P),$$

which admits a partially defined left adjoint $(\mathbf{i}_P)_\dagger$ that is well-defined on the full subcategory

$$\mathrm{Whit}(G,P) \subset \text{D-mod}(\mathcal{Q}_{G,P}).$$

Thus we also have a fully faithful functor

$$(\mathbf{i}_P)_\dagger : \mathrm{Whit}(G,P) \to \mathrm{Whit}^{\mathrm{ext}}(G,G).$$

The extended Whittaker category $\mathrm{Whit}^{\mathrm{ext}}(G,G)$ can be realized as a glued category from the degenerate Whittaker categories $\mathrm{Whit}(G,P)$. We still take the index category to be $\mathrm{Par}(G)$. For each parabolic P, we consider the degenerate Whittaker category $\mathrm{Whit}(G,P)$, and extend the assignment

$$P \mapsto \mathrm{Whit}(G,P)$$

to a lax diagram of DG categories, parametrized by $\mathrm{Par}(G)$, in the following way: we send an inclusion $P_1 \subset P_2$ to the functor $(\mathbf{i}_{P_1})^\dagger \circ (\mathbf{i}_{P_2})_\dagger : \mathrm{Whit}(G,P_2) \to \mathrm{Whit}(G,P_1)$.

Let $\mathrm{Glue}(G)_{\mathrm{geom}}$ be the resulting lax limit category.

Lemma 1.2.12. [Gai15, 9.2.3] *The naturally defined functor*

$$\mathrm{Whit}^{\mathrm{ext}}(G,G) \to \mathrm{Glue}(G)_{\mathrm{geom}}$$

corresponding to the collection of functors $(\mathbf{i}_P)^\dagger$ *is an equivalence.*

Now that we have realized both the geometric side enlarged category $\mathrm{Whit}^{\mathrm{ext}}(G,G) \simeq \mathrm{Glue}(G)_{\mathrm{geom}}$ and the spectral side enlarged category $\mathrm{Glue}(\check{G})_{\mathrm{spec}}$ as glued categories, we can relate the two sides.

Quasi-Theorem 1.2.13. [Gai15, 9.4.2]

(a) *The assignment that sends a parabolic* P *to the functor*

$$\mathbb{L}^{\mathrm{Whit}}_{G,P} : \mathsf{F}_{\check{P}}\text{–mod}(\mathrm{QCoh}(\mathrm{LocSys}_{\check{P}})) \to \mathrm{Whit}(G,P)$$

extends to a strict natural transformation of the corresponding lax diagrams.

(b) *The resulting functor*

$$\mathbb{L}^{\mathrm{Whit}^{\mathrm{ext}}}_{G,G} : \mathrm{Glue}(\check{G})_{\mathrm{spec}} \to \mathrm{Glue}(G)_{\mathrm{geom}} \simeq \mathrm{Whit}^{\mathrm{ext}}(G,G)$$

is compatible with the actions of $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$.

Moreover, Quasi-Theorem 1.2.13 implies the following:

Quasi-Theorem 1.2.14. *The functor*

$$\mathbb{L}^{\mathrm{Whit}^{\mathrm{ext}}}_{G,G} : \mathrm{Glue}(\check{G})_{\mathrm{spec}} \to \mathrm{Whit}^{\mathrm{ext}}(G,G)$$

is fully faithful.

1.2.15. The geometric Langlands functor $\mathbb{L}_G$ is expected to satisfy the following property.

Property $\mathrm{Wh}^{\mathrm{ext}}$: The functor $\mathbb{L}_G$ is said to satisfy Property $\mathrm{Wh}^{\mathrm{ext}}$ if the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Glue}(\check{G})_{\mathrm{spec}} & \xrightarrow{\mathbb{L}^{\mathrm{Whit}^{\mathrm{ext}}}_{G,G}} & \mathrm{Whit}^{\mathrm{ext}}(G,G) \\ {\scriptstyle \mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}})}\uparrow & & \uparrow{\scriptstyle \mathrm{coeff}^{\mathrm{ext}}_{G,G}} \\ \mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}}) & \xrightarrow{\mathbb{L}_G} & \text{D-mod}(\mathrm{Bun}_G) \end{array} \tag{1.2.16}$$

Combining Conjecture 1.1.7 on the geometric side (recall this is only proven for $G = \mathrm{GL}_n$), Theorem 1.2.9 on the spectral side, and Quasi-Theorem 1.2.13 relating the two sides, we obtain the following:

Corollary-of-Conjecture 1.2.17. [Gai15, 9.4.8]

(a) *Property* $\mathrm{Wh}^{\mathrm{ext}}$ *determines the equivalence* $\mathbb{L}_G$ *uniquely; and if* $\mathbb{L}_G$ *exists, it satisfies property* $\mathrm{He}^{\mathrm{naive}}$ *(see* [Gai15, § 4.4.5]*).*

(b) *The equivalence* $\mathbb{L}_G$ *exists if and only if the essential image of*

$$\mathrm{IndCoh}_{\mathrm{Nilp}^{\mathrm{glob}}_{\check{G}}}(\mathrm{LocSys}_{\check{G}})$$

in $\mathrm{Whit}^{\mathrm{ext}}$ *under the functor* $\mathbb{L}^{\mathrm{Whit}^{\mathrm{ext}}}_{G,G} \circ \mathrm{Glue}(\mathrm{CT}^{\mathrm{enh}}_{\mathrm{spec}})$*, and the essential iamge of*

$$\text{D-mod}(\mathrm{Bun}_G)$$

under the functor $\mathrm{coeff}^{\mathrm{ext}}_{G,G}$*, coincide.*

2. Other Geometric Langlands Conjectures

2.1. **The Betti Geometric Langlands Conjecture.** The Betti Geometric Langlands Conjecture [BZN16] is formulated in the setting where the ground field $k = \mathbb{C}$, where the geometric side is given by a certain category

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$$

of topological sheaves on Bun_G, and the spectral side is given by

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X))$$

where $\mathrm{LocSys}_{\check{G}}(X)$ is a certain Betti version of the stack LocSys. More precisely,

Conjecture 2.1.1. [BZN16, Conjecture 1.5] There is an equivalence

$$\mathrm{Shv}^{\mathrm{all}}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_{\check{G}}(X)) \tag{2.1.2}$$

compatible with the actions of $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(X))$ on both sides.

We shall give a brief comparison of the Betti GLC with the "restricted" version of the GLC in Remark 2.2.10.

2.2. **The "restricted" Geometric Langlands Conjecture.** The recent work [AGK+20] formulates a geometric Langlands conjecture in the setting where the ground field k is of arbitrary characteristic. The spectral side is given by a restricted version $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)$ of the stack of local systems, i.e. consider

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)).$$

The geometric side is $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$, which in particular embeds into the geometric sides of the Betti and de Rham contexts when $k = \mathbb{C}$. More precisely,

Conjecture 2.2.1. [AGK+20, Conjecture 14.2.4] There exists a canonical equivalence

$$\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)), \tag{2.2.2}$$

compatible with the action of $\mathrm{QCoh}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X))$ on both sides.

Here the left-hand-side is the category of ind-constructible sheaves on Bun_G with nilpotent singular support.

Remark 2.2.3. In the de Rham setting, Conjecture 2.2.1 is a formal consequence of the de Rham geometric Langlands Conjecture 1.2.17(a)

$$\text{D-mod}(\text{Bun}_G) \simeq \text{IndCoh}_{\text{Nilp}^{\text{glob}}_{\check{G}}}(\text{LocSys}_{\check{G}})$$

To see this, we tensor both sides of $\mathbb{L}_G$ with $\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))$ over $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$ and get:

$$\begin{array}{ccc} \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \otimes_{\text{QCoh}(\text{LocSys}_{\check{G}}(X))} \text{D-mod}(\text{Bun}_G) & \xrightarrow{\sim} & \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \\ \sim\downarrow & & \downarrow\sim \\ \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \otimes_{\text{QCoh}(\text{LocSys}_{\check{G}}(X))} \text{IndCoh}_{\text{Nilp}}(\text{LocSys}_{\check{G}}(X)) & \xrightarrow{\sim} & \text{IndCoh}_{\text{Nilp}}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \end{array}$$

where the top horizontal arrow follows from [AGK+20, Proposition 10.5.9].

Recall also from 1.2.17 that Property Wh^{ext} determines the equivalence $\mathbb{L}_G$ uniquely. In particular, assuming property Wh^{ext}, the restricted version of the Geometric Langlands Correspondance is also uniquely determined.

Remark 2.2.4. The inverse implication for Remark 2.2.3 also holds, i.e. under Hypothesis 2.2.5 below, the restricted version Geometric Langlands equivalence 2.2.2 implies the de Rham geometric Langlands equivalence 1.0.2. This implication follows from the following key observation 2.2.7 below.

Hypothesis 2.2.5. *There exists a functor*

$$\mathbb{L} : \text{IndCoh}_{\text{Nilp}}(\text{LocSys}_{\check{G}}(X)) \to \text{D-mod}(\text{Bun}_G)$$

that preserves compactness and is compatible with the actions of $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$ *on both sides.*

Remark 2.2.6. Hypothesis 2.2.5 holds if one assumes that [Gai15, Quasi-Theorems 6.7.2 and 9.5.3] hold unconditionally.

Lemma 2.2.7. [AGK+20, 14.4.4] *Assume that the functor* $\mathbb{L}$ *from Hypothesis 2.2.5 induced an equivalence*

(2.2.8)

$$\text{IndCoh}_{\text{Nilp}}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \simeq \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \otimes_{\text{QCoh}(\text{LocSys}_{\check{G}}(X))} \text{IndCoh}_{\text{Nilp}}(\text{LocSys}_{\check{G}}(X)) \xrightarrow{\text{Id}\otimes\mathbb{L}}$$

(2.2.9)

$$\to \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \otimes_{\text{QCoh}(\text{LocSys}_{\check{G}}(X))} \text{D-mod}(\text{Bun}_G) \simeq \text{Shv}_{\text{Nilp}}(\text{Bun}_G).$$

Then the functor $\mathbb{L}$ *itself is an equivalence.*

Likewise, the Betti geometric Langlands equivalence also implies the restricted version of the geometric Langlands equivalence.

Remark 2.2.10. In the Betti setting, Conjecture 2.2.1 is a formal consequence of the Betti geometric Langlands Conjecture 2.1.2. To see this, similar to the de Rham setting, we tensor both sides of 2.1.2 with $\text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X))$ over $\text{QCoh}(\text{LocSys}_{\check{G}}(X))$ and get:

$$\begin{array}{ccc} \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \otimes_{\text{QCoh}(\text{LocSys}_{\check{G}}(X))} \text{Shv}^{\text{all}}_{\text{Nilp}}(\text{Bun}_G) & \xrightarrow{\sim} & \text{Shv}_{\text{Nilp}}(\text{Bun}_G) \\ \sim\downarrow & & \downarrow\sim \\ \text{QCoh}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \otimes_{\text{QCoh}(\text{LocSys}_{\check{G}}(X))} \text{IndCoh}_{\text{Nilp}}(\text{LocSys}_{\check{G}}(X)) & \xrightarrow{\sim} & \text{IndCoh}_{\text{Nilp}}(\text{LocSys}^{\text{restr}}_{\check{G}}(X)) \end{array}$$

where the top horizontal arrow follows from [AGK+20, Corollary 10.5.11].

The inverse implication in the Betti setting (analogous to Remark 2.2.4) should also hold, assuming some Betti analogue of Hypothesis 2.2.5.

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References

[AG15] D. Arinkin and D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, Selecta Math. (N.S.) **21** (2015), no. 1, 1–199. MR 3300415

[AGK+20] D. Arinkin, D. Gaitsgory, D. Kazhdan, S. Raskin, N. Rozenblyum, and Y. Varshavsky, *The stack of local systems with restricted variation and geometric langlands theory with nilpotent singular support*, 2020.

[BZN16] David Ben-Zvi and David Nadler, *Betti geometric langlands*, 2016.

[Gai15] Dennis Gaitsgory, *Outline of the proof of the geometric Langlands conjecture for GL_2*, Astérisque (2015), no. 370, 1–112. MR 3364744

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