The Betti map

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The Betti map: a first glimpse

- Let $S$ be an irreducible variety over $\mathbb{C}$.
- Let $\pi : \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$.
- There exist positive integers $d_1 | \cdots | d_g$ such that $\mathcal{A} \to S$ is polarizable of type $D := \text{diag}(d_1, \cdots, d_g)$.
- Idea of the Betti map: we identify each closed fibre $\mathcal{A}_s$ with the real torus $\mathbb{T}^{2g}$ under the period matrices.
- More precisely: for any $s \in S(\mathbb{C})$, there exists an open neighborhood $\Delta \subseteq S^{\text{an}}$ of $s$ (which we may assume to be simply-connected), we can define the Betti map

$$b_\Delta : \mathcal{A}_\Delta = \pi^{-1}(\Delta) \to \mathbb{T}^{2g}$$

as follows:
The Betti map: a first glimpse ctd.

- We define a basis $\omega_1(s), \cdots, \omega_{2g}(s)$ of the period lattice of each fibre $s \in \Delta$ as holomorphic functions of $s$ (since $\Delta$ is simply connected).
- Each fibre $A_s = \pi^{-1}(s)$ can be identified with the complex torus $\mathbb{C}^g / \mathbb{Z}\omega_1(s) \oplus \cdots \oplus \mathbb{Z}\omega_{2g}(s)$, and each point $x \in A_s(\mathbb{C})$ can be expressed as the class of $\sum_{i=1}^{2g} b_i(x)\omega_i(s)$ for real numbers $b_1(x), \cdots, b_{2g}(x)$.

“Definition” of the Betti map: a first glimpse

Define $b_\Delta(x) := \text{class of tuple } (b_1(x), \cdots, b_{2g}(x)) \in \mathbb{R}^{2g}$ modulo $\mathbb{Z}^{2g}$.

- Note that the map $b_\Delta$ is not unique, but it is unique up to $\text{GL}_{2g}(\mathbb{Z}) \cong \text{Aut}(\mathbb{T}^{2g})$. (Late we will see that $b_\Delta$ is unique up to $\text{Sp}_{2g,D}(\mathbb{Z})$ if the basis is well-chosen.)
- The point is to compute the generic rank of $b_\Delta|X \cap A_\Delta$, for $X$ a closed irreducible subvariety of $A$. This rank is related to the relative Manin-Mumford conjecture (which reduces to a simpler conjecture on unlikely intersections via this rank).
Alternative constructions of the Betti map

There are a few alternative ways to construct the Betti map:

- Via 1-motives: André-Corvaja-Zannier
- Via arithmetic dynamics: Cantat-Gao-Habegger-Xie
- Via universal abelian varieties: Gao (coming up!)
- Ad hoc construction when \( \dim S = 1 \): Gao-Habbegger
Recollections on the moduli spaces

Let's recall a few facts about the moduli spaces from last time.

- Let $\mathbb{A}_{g,N,D}$ be the moduli space of abelian varieties of polarized type $D$ (thus of dimension $g$) with level-$N$-structure.
- As long as $N \geq 3$, $\mathbb{A}_{g,N,D}$ is a fine moduli space, thus admitting a universal family $\pi : \mathbb{A}_{g,N,D} \to \mathbb{A}_{g,N,D}$. (For simplicity we shall omit the level structure $N$ and polarization $D$ from the notations from now on.)
- The universal covering in the category of complex spaces for $\mathbb{A}_g$ is given by $u_G : \mathcal{H}_g^+ \to \mathbb{A}_g^{an}$, where
  $$\mathcal{H}_g^+ = \{ Z \in \text{Mat}_{g \times g}(\mathbb{C}) : Z = Z^T, \text{Im}(Z) > 0 \}$$ is the Siegel upper half space.
Uniformizing $\mathbb{A}_g$

- We denote by $\text{Sp}_{2g,D}$ the $\mathbb{Q}$-group defined by

$$\text{Sp}_{2g,D}(\mathbb{Q}) = \left\{ h \in \text{GL}_{2g}(\mathbb{Q}) : h \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} h^T = \begin{bmatrix} 0 & D \\ -D & 0 \end{bmatrix} \right\}.$$ 

- Let $\text{GSp}_{2g}$ be the image of $\mathbb{G}_m \times \text{Sp}_{2g}$ under the central isogeny $\mathbb{G}_m \times \text{SL}_{2g} \to \text{GL}_{2g}$.

- Then $\text{GSp}_{2g}(\mathbb{R})^+$ (connected component of $\text{GSp}_{2g}(\mathbb{R})$ containing the identity) acts on $\mathcal{H}_g^+$ by the formula: for any $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \text{GSp}_{2g}(\mathbb{R})^+$ and $Z \in \mathcal{H}_g^+$, 

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} Z = (A' Z + B'(C' Z + D'))^{-1}.$$ 

- Well-known fact in the area: the action of $\text{GSp}_{2g}^{\text{der}}(\mathbb{R}) = \text{Sp}_{2g}(\mathbb{R})$ on $\mathcal{H}_g^+$ thus defined is transitive, and the uniformization $u_G$ is obtained by identifying via the following identification.

- $\text{Sp}_{2g,D}(\mathbb{R})$ acts transitively on $\mathcal{H}_g^+$, and the uniformization $u_G$ is obtained via $\mathbb{A}_g^\text{an} \simeq \text{Sp}_{2g,D}(1 + N \mathbb{Z}) \backslash \mathcal{H}_g^+$, where $\text{Sp}_{2g,D}(1 + N \mathbb{Z}) = \ker(\text{Sp}_{2g,D}(\mathbb{Z}) \to \text{Sp}_{2g,D}(\mathbb{Z}/N\mathbb{Z})).$
Uniformizing $\mathcal{A}_g$

- To obtain the uniformization of $\mathcal{A}_g$, we construct the following complex space $\mathcal{X}_{2g,a}^+$:
  - as a semi-algebraic space, $\mathcal{X}_{2g,a}^+ = \mathbb{R}^{2g} \times \mathcal{H}_g^+$;
  - the complex structure of $\mathcal{X}_{2g,a}^+$ is the one given by
    \[
    \mathcal{X}_{2g,a}^+ = \mathbb{R}^{g} \times \mathbb{R}^{g} \times \mathcal{H}_g^+ \sim \mathbb{C}^{g} \times \mathcal{H}_g^+
    \]
    \[
    (a, b, Z) \mapsto (Da + Zb, Z)
    \]
  - We then get a uniformization of $\mathcal{A}_g$ in the category of complex varieties $u : \mathcal{X}_{2g,a}^+ \rightarrow \mathcal{A}_g$.

- Similar to the discussion on $u_G$, there exists a $\mathbb{Q}$-group, which we call $P_{2g,a}^{\text{der}}$ such that $P_{2g,a}^{\text{der}}(\mathbb{R})$ acts transitively on $\mathcal{X}_{2g,a}^+$, and $u$ is obtained by identifying $\mathcal{A}_g^{\text{an}} \simeq \Gamma \backslash \mathcal{X}_{2g,a}^+$ for a suitable congruence subgroup $\Gamma = \mathbb{Z}^{2g} \rtimes \Gamma_{\text{Sp}_{2g}}$ of $P_{2g,a}^{\text{der}}(\mathbb{Z})$. 
The universal uniformized Betti map on $\mathcal{X}_{2g,a}^+$

First we define the *universal uniformized Betti map* on $\mathcal{X}_{2g,a}^+$, and then descend it to the pullback $\mathcal{A}_{\mathfrak{H}_g^+}$ of $\mathcal{A}_g/\mathcal{A}_g$ along $u_G : \mathfrak{H}_g^+ \rightarrow \mathcal{A}_g$ to $\mathfrak{H}_g^+$.

- Since we have defined $\mathcal{X}_{2g,a}^+ = \mathbb{R}^{2g} \times \mathfrak{H}_g^+$ with the prescribed complex structure in the previous slide, we can simply define the *universal uniformized Betti map* $\tilde{b}$ to be the natural projection

  $$\tilde{b} : \mathcal{X}_{2g,a}^+ \rightarrow \mathbb{R}^{2g}$$

- Then $\tilde{b}$ is semi-algebraic, and for the complex structure on $\mathcal{X}_{2g,a}^+$, it is clear that $\tilde{b}^{-1}(r)$ is complex analytic for each $r \in \mathbb{R}^{2g}$. 
The universal Betti map on $\mathcal{A}_{g}^{\pm}$

Now we’re ready to descend the universal uniformized Betti map on $\mathcal{X}_{2g, a}^{+}$ to the universal Betti map on $\mathcal{A}_{g}^{\pm}$.

- Recall that $\mathcal{A}_{g}^{an} \simeq \Gamma \setminus \mathcal{X}_{2g, a}^{+}$ as complex spaces, for a suitable congruence subgroup $\Gamma = \mathbb{Z}^{2g} \rtimes \Gamma_{\text{Sp}_{2g}}$ of $P_{2g, a}^{\text{der}}(\mathbb{Z})$.
- The family of abelian varieties $\mathcal{A}_{g}^{\pm}$ can be identified with the quotient space $(\mathbb{Z}^{2g} \rtimes \{1\}) \setminus \mathcal{X}_{2g, a}^{+}$.
- Taking the quotient by $\mathbb{Z}^{2g}$ on both sides of the universal uniformized Betti map $\tilde{b}$, we obtain the universal Betti map

$$b : \mathcal{A}_{g}^{\pm} \to \mathbb{T}^{2g},$$

where $\mathbb{T}^{2g}$ is the real torus of dimension $2g$. 
Summary for the universal (uniformized) Betti map

The above discussion can be summarized into the following properties for $\tilde{b}$ and $b$:

**Proposition 0**

The Betti map $b_\Delta$ satisfies the following properties:

(i) Both $\tilde{b}$ and $b$ are real-analytic, and $\tilde{b}$ is moreover semi-algebraic.

(ii) For each $r \in \mathbb{R}^{2g}$ (resp. each $t \in \mathbb{T}^{2g}$), we have that $\tilde{b}^{-1}(r)$ (resp. $b^{-1}(t)$) is complex analytic.

(iii) For each $\tau \in \mathcal{H}_g^+$, the restriction $b|_{(A_{\mathcal{H}_g^+})_\tau}$ is a group isomorphism.
Betti map on arbitrary abelian schemes

- Since any abelian scheme $\pi : \mathcal{A} \to S$ can be equipped with a polarization of type $D$ for some $D = \text{diag}(d_1, \cdots, d_g)$ with $d_1 | \cdots | d_g$, thus up to taking a finite cover of $S$ and taking the appropriate base change of $\mathcal{A} \to S$, there exists a Cartesian diagram

$$
\begin{array}{c}
\mathcal{A} \\
\downarrow \pi_S
\end{array}
\xrightarrow{\iota}
\begin{array}{c}
\mathcal{A}_g \\
\downarrow \pi
\end{array}
\begin{array}{c}
\mathcal{A}|_{\Delta_0} \\
\downarrow \pi_S
\end{array}
\xrightarrow{\iota_S}
\begin{array}{c}
\mathcal{A}_g|_{\Delta_0}
\end{array}
$$

The morphism $\iota$ is called the **modular map**.

- Let $\Delta_0$ be a simply-connected open subset in $\mathbb{A}^\text{an}_g$. Fix a component $\tilde{\Delta}_0$ of $u_G^{-1}(\Delta_0)$ under the uniformization map $u_G : \mathcal{H}_g^+ \to \mathbb{A}_g$.
- Since $\Delta_0$ is simply-connected, $u_G|_{\tilde{\Delta}_0}$ is an isomorphism in the category of complex spaces.
- Thus the universal Betti map $b$ induces a map $b_{\Delta_0} : \mathcal{A}_g|_{\Delta_0} \to \mathbb{T}^{2g}$ by identifying $\mathcal{A}_g|_{\Delta_0} = \pi^{-1}(\Delta_0)$ with $\mathcal{A}_{\mathcal{H}_g^+}|_{\tilde{\Delta}_0}$.

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Betti map on arbitrary abelian schemes ctd.

- For any \( s \in S(\mathbb{C}) \), we can find a simply-connected \( \Delta_0 \) such that \( \iota_S(s) \in \Delta_0 \). Let \( \Delta \) be a component of \( \iota_S^{-1}(\Delta_0) \) containing \( s \).
- Let \( \mathcal{A}_\Delta = \pi_S^{-1}(\Delta) \). Define \( b_\Delta : \mathcal{A}_\Delta \to \mathbb{T}^{2g} \) to be the composite of \( \iota \) and \( b_{\Delta_0} \).

The following is just a corollary of Proposition 0.

**Proposition 1**

The Betti map \( b_\Delta \) satisfies the following properties:

(i) The map \( b_\Delta \) is real analytic.
(ii) For each \( t \in \mathbb{T}^{2g} \), we have that \( b_\Delta^{-1}(t) \) is complex analytic.
(iii) For each \( s \in \Delta \), the restriction \( b_\Delta|_{\mathcal{A}_s} \) is a group isomorphism.

Note that \( b_\Delta \) is not unique as we can choose different components of \( u_G^{-1}(\Delta_0) \), but \( b_\Delta \) is unique up to \( \text{Sp}_{2g}(\mathbb{Z}) \) because \( \mathbb{A}_g \cong \Gamma_{\text{GSp}_{2g}} \backslash \mathcal{H}_g^+ \) for some congruence subgroup \( \Gamma_{\text{GSp}_{2g}} \) of \( \text{Sp}_{2g}(\mathbb{Z}) \).
**Definition of Betti rank**

Let $X$ be an irreducible subvariety of $\mathcal{A}$ and let $x \in X^{\text{sm}}(\mathbb{C})$. The **Betti rank** of $X$ at $x$ is defined to be

$$\text{rank}_{\text{Betti}}(X, x) := \text{rank}_{\mathbb{R}}(db_{\Delta}|_{X^{\text{sm,an}}})_x$$

where $\Delta$ is an open neighborhood of $\pi(x)$ in $S^{\text{an}}$ and $b_{\Delta}$ is the Betti map.

- The **Betti form** is a closed semi-positive smooth $(1, 1)$-form $\omega$ on $\mathcal{A}^{\text{an}}$ with the property $[N]^*\omega = N^2\omega$ such that the following property holds true:

  For any subvariety $X$ of $\mathcal{A}$ and any $x \in X^{\text{sm}}(\mathbb{C})$, we have

  $$(\ast) \quad \text{rank}_{\text{Betti}}(X, x) = 2 \dim X \Leftrightarrow (\omega|_{X}^{\dim X})_x \neq 0.$$
Constructing the Betti form

There are several ways to construct the Betti form $\omega$:

- Via a formula of Mok’s
- Via arithmetic dynamics
- Via the Betti map (we will use this construction)

Construction of Betti form

Denote by $(a, b) = (a_1, b_1; \cdots; a_g, b_g)$ the coordinates of $T^{2g}$. Let $\Delta$ be a simply-connected open subset of $S^{an}$ and $b_\Delta : A_\Delta \to T^{2g}$ the Betti map defined earlier. We define the 2-form $\omega_\Delta$ on $A_\Delta$ by

$$\omega_\Delta := b_\Delta^{-1}(2(Dda)^T \wedge db) = b_\Delta^{-1}\left(2\sum_{j=1}^{g} d_j da_j \wedge db_j\right).$$

Note that $\omega_\Delta$ is well-defined, because two different choices of $b_\Delta$ differ from an element in $Sp_{2g,D}(\mathbb{Z})$, and $2(Dda)^T \wedge db$ is preserved by $Sp_{2g,D}(\mathbb{R})$. (Can check that these $\omega_\Delta$ glue together to a 2-form $\omega$ on $A^{an}$. This $\omega$ is the desired Betti form.)
The Betti form ctd.

- To check that $\omega$ is a $(1,1)$-form and is semi-positive, one can do an explicit computation by the change of coordinates $(b_\Delta, \pi) : \mathcal{A}_\Delta \to \mathbb{T}^{2g} \times \Delta$ from part (i) of Proposition 1.

- In fact, we can prove the following statement via some computation in [DGH]: consider the uniformization $u : \mathbb{C}^g \times \mathcal{H}_g \to \mathcal{A}^{an}_{g,D}$ and the Betti form $\omega$ on $\mathcal{A}^{an}_{g,D}$, we have

$$u^* \omega = \sqrt{-1} \partial \bar{\partial} (2(\text{Im } w)^T (\text{Im } Z)^{-1}(\text{Im } w)),$$

where $(w, Z)$ denotes the coordinates on $\mathbb{C}^g \times \mathcal{H}_g$.

- The symmetric real matrix representing $u^* \omega$ is

$$\begin{bmatrix}
1 & -(\text{Im } w)^T (\text{Im } Z)^{-1} \\
-(\text{Im } Z)^{-1}(\text{Im } w) & (\text{Im } Z)^{-1}(\text{Im } w)(\text{Im } w)^T (\text{Im } Z)^{-1}
\end{bmatrix} \otimes (\text{Im } Z)^{-1}.$$

- By some computation of Kühne's, the above constructed $\omega$ satisfies Property ($\ast$).
Assume $\ell \geq 3$ is even. There exists a tautological relatively ample line bundle $\mathcal{L}_{g,D}$ on $\mathcal{A}_{g,D}/\mathcal{A}_{g,D}$, i.e. for each $s \in \mathcal{A}_{g,D}(\mathbb{C})$, we have $((\mathcal{A}_{g,D})_s, (\mathcal{L}_{g,D})_s)$ is the polarized abelian variety parametrized by $s$.

Note that $[-1]^* \mathcal{L}_{g,D} = \mathcal{L}_{g,D}$.

**Proposition 2 (Mok ’91)**

The cohomology class of the Betti form $\omega$ on $\mathcal{A}^{\text{an}}_{g,D}$ coincides with the first Chern class $c_1(\mathcal{L}_{g,D})$ of $\mathcal{L}_{g,D}$. 
Non-degenerate subvarieties

Let $\omega$ be the Betti form on $A$ constructed above.

**Definition**

An irreducible subvariety $X$ of $A$ is said to be **non-degenerate** if one of the following equivalent conditions holds true:

(i) $\text{rank}_{\text{Betti}}(X, x) = 2 \dim X$ for some $x \in X^{\text{sm}}(C)$;

(ii) $(\omega|_X^{\wedge \dim X})_x \neq 0$ for some $x \in X^{\text{sm}}(C)$.

Remark: by the previous Proposition 2 and condition (ii), non-degeneracy should be interpreted as some *bigness* condition of the appropriate line bundle.

**Lemma 3**

$X$ is nondegenerate $\Rightarrow \iota|_X$ is generically finite and $\dim X \leq g$.

Alert: The converse is in general false.
However, the converse holds true if the geometric generic fibre of $A \to S$ is a simple abelian variety (this is a Theorem of Gao’s)
Proof of the lemma
Recall the Cartesian diagram from before, with the modular map \( \iota \),

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\iota} & \mathcal{A}_g \\
\downarrow{\pi} & & \downarrow \\
\mathcal{S} & \xrightarrow{\iota_S} & \mathbb{A}_g
\end{array}
\]

- The Betti map \( b_\Delta \) factors through \( \iota \), so \( \text{rank}_{\text{Betti}}(X, x) \leq 2 \dim \iota(X) \).
- Thus by part (i) of the definition (of non-deg subvar), \( \iota|_X \) must be generically finite if \( X \) is non-degenerate.
- On the other hand, since the target of \( b_\Delta \) is \( \mathbb{T}^{2g} \), so \( \text{rank}_{\text{Betti}}(X, x) \leq 2g \). So by part (i) of the definition again, we have \( \dim X \leq g \) if \( X \) is non-degenerate.

Thus \( \mathcal{C}_g - \mathcal{C}_g \) defined in the next few slides is a degenerate subvariety of \( \text{Jac}(\mathcal{C}_g/\mathcal{M}_g) \), because its dimension is \( > g \).
Product with a non-degenerate subvariety

**Lemma 4**

Let $X$ and $Y$ be irreducible subvarieties of $A$ such that $\pi|_X$ and $\pi|_Y$ are both dominant. If $X$ is non-degenerate, then $X \times_\mathcal{S} Y$ is a non-degenerate subvariety of $A \times_\mathcal{S} A$.

**Proof Sketch:**

- By generic smoothness, assume wlog that $\mathcal{S}$ is smooth and that both $X^{\text{sm}} \to \mathcal{S}$ and $Y^{\text{sm}} \to \mathcal{S}$ are smooth morphisms. We have $\dim X \times_\mathcal{S} Y = \dim X + \dim Y - \dim \mathcal{S}$.

- Since $X$ is non-degenerate, there exists $x \in X^{\text{sm}}(\mathbb{C})$ such that $\text{rank}_{\text{Betti}}(X, x) = 2 \dim X$. Denote $s = \pi(x) \in \mathcal{S}(\mathbb{C})$.

- Since the Betti map is a group isomorphism when restricted to $A_s = \pi^{-1}(s)$ (Prop 1(iii)), we have that $\text{rank}_{\text{Betti}}(Y, y) \geq 2 \dim Y_s = 2(\dim Y - \dim \mathcal{S})$ for a generic $y \in Y^{\text{sm}}_s(\mathbb{C})$. Thus $y \in Y^{\text{sm}}_s(\mathbb{C})$ as $\mathcal{S}$ is smooth and $Y^{\text{sm}} \to \mathcal{S}$ is a smooth morphism.
Product with a non-degenerate subvariety ctd.

- Now \((x, y) \in (X \times_S Y)^{sm}(\mathbb{C})\), and 
  \(\text{rank}_{Betti}(X \times_S Y, (x, y)) = 2(\dim X + \dim Y - \dim S)\), by definition (i) we are done. (end of proof Sketch of Lemma 4)

- The proof above has the following consequence. Let \(M \geq 1\) be an integer and \(X^{[M]} := X \times_S X \times_S \cdots \times_S X\) (\(M\)-copies) for any subvariety \(X\) of \(\mathcal{A}\), and \(\omega_M\) the Betti form on \(\mathcal{A}^{[M]} := \mathcal{A} \times_S \mathcal{A} \times_S \cdots \times_S \mathcal{A}\) (\(M\)-copies).

**Corollary 5**

Assume \(\pi|_{X^{sm}}\) is smooth and \(x \in X^{sm}(\mathbb{C})\) satisfies \((\omega|_X^\dim X)_x \neq 0\). Then we have \((\omega_M|_X^{\dim X^{[M]}})_{(x, \ldots, x)} \neq 0\).

**Proof Sketch:** By equation (*) of the definition, we have 
\(\text{rank}_{Betti}(X, x) = 2 \dim X\). Let \(s = \pi(x)\). By assumption 
\((x, \cdots, x) \in (X^{[m]})^{sm}(\mathbb{C})\). Thus 
\(\text{rank}_{Betti}(X^{[m]}, (x, \cdots, x)) = 2 \dim X + 2(m - 1) \dim X_s = 2 \dim X^{[m]}\), 
thus by equation (*) again we’re done.
The Faltings-Zhang map

- Motivation: The new gap principle is about the differences of points on each curve $C$ taken in its Jacobian $\text{Jac}(C)$; we can precisely view this operation by setting the subvariety $C - C$ of $\text{Jac}(C)$ to be the image of

$$C \times C \to \text{Jac}(C) = \text{Pic}^0(C)$$

$$(P, Q) \mapsto [Q - P]$$

Now we want to realize this difference in families.

- Let $\mathcal{C}_g \to \overline{\mathcal{M}}_g$ be the universal curve over the fine moduli space $\overline{\mathcal{M}}_g$ of smooth projective curves of genus $g$ with level-$N$-structure.

- Let $\text{Pic}(\mathcal{C}_g/\overline{\mathcal{M}}_g)$ be the relative Picard scheme; it is a group scheme over $\overline{\mathcal{M}}_g$ and can be decomposed as the union of open and closed subschemes $\text{Pic}^d(\mathcal{C}_g/\overline{\mathcal{M}}_g)$ for all $d \in \mathbb{Z}$ (here $d$ is the degree of a line bundle)
Consider the difference group law
\[
\text{Pic}(\mathcal{C}_g/\mathcal{M}_g) \times_{\mathcal{M}_g} \text{Pic}(\mathcal{C}_g/\mathcal{M}_g) \to \text{Pic}(\mathcal{C}_g/\mathcal{M}_g),
\]
when restricted to \(\text{Pic}^1 \times \text{Pic}^1\), we get an \(\mathcal{M}_g\)-morphism
\[
\text{Pic}^1(\mathcal{C}_g/\mathcal{M}_g) \times_{\mathcal{M}_g} \text{Pic}^1(\mathcal{C}_g/\mathcal{M}_g) \to \text{Pic}^0(\mathcal{C}_g/\mathcal{M}_g) = \text{Jac}(\mathcal{C}_g/\mathcal{M}_g).
\]

On the other hand, by GIT, we have an \(\mathcal{M}_g\)-morphism
\[
\mathcal{C}_g \to \text{Pic}^1(\mathcal{C}_g/\mathcal{M}_g).
\]

Thus combining the two morphisms we get an \(\mathcal{M}_g\)-morphism
\[
\mathcal{D}_1 : \mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g \to \text{Jac}(\mathcal{C}_g/\mathcal{M}_g)
\]

The restriction of \(\mathcal{D}_1\) to each fibre is precisely \((P, Q) \mapsto [Q - P]\).

We denote by \(\mathcal{C}_g - \mathcal{C}_g\) the image of \(\mathcal{D}_1\).
The Faltings-Zhang map and a construction of non-degenerate subvarieties

- We can generalize $D_1$ to $D_M$ for an arbitrary $M \geq 1$ integer.
- Let $C_g^{[M]}$ (resp. $\text{Jac}(C_g/M_g)^{[M]}$) be the $M$-th fibred power over $M_g$. Then we get an $M_g$-morphism

$$D_M : C_g^{[M+1]} \to \text{Jac}(C_g/M_g)^{[M]} ,$$

such that over each fibre it is
$$(P_0, P_1, \cdots, P_M) \mapsto (P_1 - P_0, \cdots, P_M - P_0).$$

- This Faltings-Zhang map $D_M$ can be used to construct non-degenerate subvarieties, i.e. we have the following theorem.

**Theorem 6**

Let $S \to M_g$ be a generically finite morphism. Then $D_M(C_g^{[M+1]} \times M_g) S$ is a non-degenerate subvariety of $\text{Jac}(C_g/M_g)^{[M]} \times M_g S$ for $M \geq \dim S + 1$. 
A more general theorem

Theorem 6 on the previous slide is a special case of a more general theorem of Gao’s. First we set some notations.

- Recall the notations $\pi : A \to S$ an abelian scheme and $\iota : A \to \mathcal{A}_g$ is the modular map.

- For each integer $M \geq 1$, set $A^{[M]} := A \times_S \cdots \times_S A$ ($M$-copies). Define the (more general) **Faltings-Zhang map**

$$D^A_M : A^{[M+1]} \to A^{[M]}$$

...to be the $S$-morphism fibrewise defined by

$$(P_0, P_1, \cdots, P_M) \mapsto (P_1 - P_0, \cdots, P_M - P_0).$$

- For each $M \geq 1$, let $\iota^{[M]} : A^{[M]} \to \mathcal{A}_{gM}$ be the modular map. (Now we’re ready to state the more general theorem)
A more general theorem ctd.

**General Theorem 7 about Non-degenerate Subvarieties**

Let $X$ be an irreducible subvariety of $\mathcal{A}$ such that $\pi|_X$ is dominant to $S$. Assume that the geometric generic fibre $X_{\eta}$ of $X \to S$ is irreducible. Assume moreover:

(a) $\dim X > \dim S$.
(b) $X_{\eta}$ is not contained in any proper subgroup of $\mathcal{A}_{\eta}$.
(c) On the geometric generic fibre $\mathcal{A}_{\eta}$ of $\mathcal{A} \to S$, the stabilizer $\text{Stab}_{\mathcal{A}_{\eta}}(X_{\eta})$ of $X_{\eta}$, is finite.

Then as subvarieties of $\mathcal{A}^{[M]}$, we have that

(i) $X^{[M]}$ is non-degenerate if $M \geq \dim S$ and $\iota^{[M]}|_{X^{[M]}}$ is generically finite.
   (Here $X^{[M]} = X \times S \times \cdots \times S X$ for each integer $M \geq 1$.)
(ii) $\mathcal{D}_M^A(X^{[M+1]})$ is non-degenerate if $M \geq \dim X$ and $\iota^{[M]}|_{\mathcal{D}_M^A(X^{[M+1]})}$ is generically finite.
A few remarks

- In practice, to verify the extra generic finiteness condition required in (i) and (ii), one often uses the following observations:
  - For (i), $\iota^M|_{X^M}$ is generically finite if $\iota|_X$ is generically finite.
  - For (ii), $\iota^M|_{\mathcal{D}_M^A(X^{M+1})}$ is generically finite if $\iota$ (not $\iota|_X$) is quasi-finite.

- Hypothesis (c) in the Theorem can be weakened into the following hypothesis:
  (c') On the geometric generic fibre $A_{\eta}$ of $A \to S$, the neutral component of stabilizer $\text{Stab}_{A_{\eta}}(X_{\eta})$ is contained in the $\overline{\mathbb{C}(\eta)}/\mathbb{C}$-trace of $A_{\eta}$, where $\overline{\mathbb{C}(\eta)}$ is an algebraic closure of the function field of $S$. 


Non-degeneracy is an algebraic property

**Theorem 8 on the degeneracy locus**

To each \( X \), one can associate an intrinsically defined Zariski closed subset \( X^{\text{deg}} \) of \( X \) such that the following property holds true:

\[ X \text{ is non-degenerate} \iff X \neq X^{\text{deg}}. \]

Moreover, if \( X \) is defined over an algebraically closed field \( F \), so is \( X^{\text{deg}} \).

Outline of the proofs of both theorems 7 and 8: The major step is to establish a criterion *in simple geometric terms* for an irreducible subvariety \( X \) of the universal abelian variety \( \mathbb{A}_g \) to be degenerate. Roughly speaking, the proof for this desired criterion consists of two steps:

- **Step 1** translates the degeneracy property to an unlikely intersection problem in \( \mathbb{A}_g \) by invoking the mixed Ax-Schanuel theorem for \( \mathbb{A}_g \). More precisely, we show that:

  \[ X \text{ is degenerate} \iff X \text{ is a union of subvarieties satisfying an appropriate unlikely intersection property}. \]

- **Step 2** solves this unlikely intersection problem, and prove that the union mentioned in Step 1 is a *finite* union. (This step uses the notion of *weakly optimal subvarieties* of Habegger-Pila.)
A bit more detail: the t-th degeneracy locus

Recall our abelian scheme $\pi_S : A \to S$.

**Definition**

A closed irreducible subvariety $Z$ of $A$ is called a *generically special subvariety of sg type* of $A$ if there exists a finite covering $S' \to S$ inducing a morphism $\rho : A' := A \times_S S' \to A$ such that

$$Z = \rho(\sigma' + \sigma_0' + B'),$$

where $B'$ is an abelian subscheme of $A'/S'$, $\sigma'$ is a torsion section of $A'/S'$ and $\sigma_0'$ is a constant section of $A'/S'$.

- By “constant section” we mean the following: let $C' \times S'$ be the largest constant abelian subscheme of $A'/S'$. We say that a section $\sigma_0' : S' \to A'$ is a constant section if there exists $c' \in C'(\mathbb{C})$ such that $\sigma_0$ is the composite of $S' \to C' \times S'$, $s' \mapsto (c', s')$ and the inclusion $C' \times S' \subseteq A'$. 
A bit more detail: the t-th degeneracy locus

For any locally closed irreducible subvariety \( Y \) of \( A \), denote by \( \langle Y \rangle_{sg} \) the smallest generically special subvariety of sg type of \( A|_{\pi_S(Y)} := \pi_S^{-1}(\pi_S(Y)) \) that contains \( Y \).

**Definition**

Let \( X \) be a closed irreducible subvariety of \( A \). For any \( t \in \mathbb{Z} \), define the *t-th degeneracy locus* of \( X \), denoted by \( X^{\text{deg}}(t) \), to be the union of positive dimensional closed irreducible subvarieties \( Y \subseteq X \) such that

\[
\dim \langle Y \rangle_{sg} - \dim \pi_S(Y) < \dim Y + t.
\]

When \( t = 0 \), we abbreviate \( X^{\text{deg}}(0) := X^{\text{deg}} \).

We say that \( X \) is *degenerate* if \( X^{\text{deg}} \) is Zariski dense in \( X \).

- Note that \( X^{\text{deg}} = X \) clearly holds if \( X \) is a multi-section and \( g < \dim S \).
- The locus on which \( \text{rank}_\mathbb{R}(db_\Delta|_X)_x \) is smaller than expected is precisely \( X^{\text{deg}}(t) \) for some \( t \leq 0 \).
The t-th degeneracy locus ctd.

More precisely, we have the following result. Recall the modular map $\iota : \mathcal{A} \to \mathcal{A}_g$ and the naive bound $\operatorname{rank}_\mathbb{R}(db_\Delta|_X) \leq 2 \dim \iota(X)$.

**Theorem 9**

Let $x \in X^{\text{sm}}(\mathbb{C}) \cap A_\Delta$. Then for each integer $\ell \leq \dim \iota(X)$, we have

$$\operatorname{rank}_\mathbb{R}(db_\Delta|_X)_x < 2\ell \iff \iota(x) \in \iota(X)^{\text{deg}}(\ell - \dim \iota(X)) \iff x \in X^{\text{deg}}(\ell - \dim X).$$

This is not yet satisfactory, as the $X^{\text{deg}}(t)$ thus defined is a priori a complicated subset of $X$. However, we show that they are all Zariski closed in $X$.

**Theorem 10**

The set $X^{\text{deg}}(t)$ is Zariski closed in $X$ for each $t \in \mathbb{Z}$.

Proof of Theorem 10 uses ideas from (mixed) Shimura varieties. (unfortunately we don’t have time to go into detail on this beautiful proof)
Some ingredients used in the proof of Theorem 9

- For simplicity we consider the case $X \subseteq \mathcal{A}_g$. We want to translate the question of the generic rank of the Betti map to the $t$-th degeneracy locus of $X$ for some particular $t$.

- Recall the uniformizations defined earlier

$$\mathcal{X}_{2g,a}^+ \xrightarrow{\tilde{\pi}} \mathcal{H}_g^+$$

$$\downarrow u \quad \quad \quad \quad \quad \downarrow u_G$$

$$\mathcal{A}_g \xrightarrow{\pi} \mathbb{A}_g$$

and the uniformized universal Betti map $\tilde{b} : \mathcal{X}_{2g,a}^+ \to \mathbb{R}^{2g}$ defined earlier. Let $X$ be an irreducible subvariety of $\mathcal{A}_g$.

- Fix a complex analytic irreducible component $\tilde{X}$ of $u^{-1}(X)$. 

slides by Yujie Xu
Weak Ax-Schanuel for $\mathcal{A}_g$

One of the most important tools we use to study the Betti map is the following weak Ax-Schanuel theorem for $\mathcal{A}_g$ due to Gao.

- Note $\mathcal{X}^+_{2g,a}$ can be embedded as an open, in the usual topology, semi-algebraic subset of a complex flag variety (thus algebraic) $\mathcal{X}^\lor_{2g,a}$.

**Definition**

(i) A subset $\tilde{Y}$ of $\mathcal{X}^+_{2g,a}$ is said to be **irreducible algebraic** if it is a complex analytic irreducible component of $\mathcal{X}^+_{2g,a} \cap W$ for some algebraic subvariety $W$ of $\mathcal{X}^\lor_{2g,a}$.

(ii) An irreducible subvariety $Y$ of $\mathcal{A}_g$ is said to be **bi-algebraic** if one (and hence any) complex analytic irreducible component $\tilde{Y}$ of $u^{-1}(Y)$ is algebraic.

- The intersection of two bi-algebraic subvarieties of $\mathcal{A}_g$ is a finite union of irreducible bi-algebraic subvarieties of $\mathcal{A}_g$.

- For any subset $Z$ of $\mathcal{A}_g$, there exists a smallest bi-algebraic subvariety $\mathcal{A}_g$ that contains $Z$. We denote it by $Z^{\text{biZar}}$. 
Weak Ax-Schanuel for $\mathbb{A}_g$ ctd.

Weak Ax-Schanuel Theorem

Let $\tilde{Z}$ be a complex analytic irreducible subset of $\mathcal{X}_{2g,a}$. Then

$$\dim \tilde{Z}^{\text{Zar}} + \dim(u(\tilde{Z}))^{\text{Zar}} \geq \dim \tilde{Z} + \dim(u(\tilde{Z}))^{\text{biZar}},$$

where $\tilde{Z}^{\text{Zar}}$ means the smallest irreducible algebraic subset of $\mathcal{X}_{2g,a}$ which contains $\tilde{Z}$. 
Proof ingredients for Theorem 9 ctd.

The proof of Theorem 9 basically boils down to proving the following Lemma.

**Lemma 11**

Let $d = \dim X$. For any integer $\ell \in \{1, \cdots, d\}$, let

$$\tilde{X}_{< 2\ell} := \{\tilde{x} \in \tilde{X} : \text{rank}_\mathbb{R}(\tilde{b}|_{\tilde{X}})_{\tilde{x}} < 2\ell, u(\tilde{x}) \in X^{\text{sm}}(\mathbb{C})\}.$$  

Then $X^{\text{deg}}(\ell - d) \cap X^{\text{sm}}(\mathbb{C}) \subseteq u(\tilde{X}_{< 2\ell}) \subseteq X^{\text{deg}}(\ell - d)$.

In particular, if $\text{rank}_\mathbb{R}(\tilde{b}|_{\tilde{X}})_{\tilde{x}} < 2\ell$ for all $\tilde{x} \in \tilde{X}$ with $u(\tilde{x}) \in X^{\text{sm}}(\mathbb{C})$, then $X^{\text{deg}}(\ell - d)$ is Zariski dense in $X$.

**Proof Ingredients of Lemma:**

- To prove the inclusion $u(\tilde{X}_{< 2\ell}) \subseteq X^{\text{deg}}(t)$: take any $\tilde{x} \in \tilde{X}_{< 2\ell}$, set $r := \tilde{b}(\tilde{x}) \in \mathbb{R}^{2g}$.

- We identify $X^{+}_{2g,a}$ as the semi-algebraic space $\mathbb{R}^{2g} \times \mathcal{N}^{+}_g$ with the complex structure specified earlier. In particular, $\tilde{b}^{-1}(r) = \{r\} \times \mathcal{N}^{+}_g$. 

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Proof ingredients for Theorem 9 ctd.

- Proof Ingredients of Lemma 11 ctd.:
  - Property (ii) of the Betti map implies that \( \tilde{b}^{-1}(r) \cap \tilde{X} \) is complex analytic.
  - The assumption of this lemma implies that
    \[
    \dim_{\mathbb{R}}(\tilde{b}^{-1}(r) \cap \tilde{X}) > 2(d - \ell)
    \]
  - Thus there exists a complex analytic irreducible subset \( \tilde{W} \) in \( \mathcal{H}_g^+ \) of dimension \( \geq d - \ell + 1 \) such that
    \[
    \tilde{x} \in \{r\} \times \tilde{W} \subseteq \tilde{X}.
    \]
  - Thus it suffices to prove the following assertion:
    \[
    Y := u((\{r\} \times \tilde{W}))^{\text{Zar}} \text{ satisfies }
    \dim \langle Y \rangle_{\text{sg}} - \dim \pi(Y) < \dim Y + (\ell - d). \quad (^\dagger)
    \]
  - Apply weak Ax-Schanuel for \( \mathfrak{H}_g \) to \( \{r\} \times \tilde{W} \), and we get
    \[
    \dim(\{r\} \times \tilde{W})^{\text{Zar}} + \dim Y \geq \dim(\{r\} \times \tilde{W}) + \dim Y^{\text{biZar}}.
    \]
Proof ingredients for Theorem 9 ctd.

- Proof Ingredients of Lemma 11 ctd.:
  - On the other hand we have
    \[ \widetilde{W}_{\text{Zar}} \subseteq \widetilde{W}_{\text{biZar}} \subseteq (u_G(\widetilde{W}))^{\text{biZar}} \subseteq \pi(Y)^{\text{biZar}}, \]
    where the last inclusion holds because \( u_G(\widetilde{W}) \subseteq \pi(Y) \) by definition of \( Y \).
  - Using the algebraic structures on \( \mathcal{X}^+_g, a \), we can prove that
    \((\{r\} \times \widetilde{W})^{\text{Zar}} = \{r\} \times \widetilde{W}^{\text{Zar}}: \) this is because \( \{r\} \times \widetilde{W}^{\text{Zar}} \) is semi-algebraic and complex analytic; and it is a general fact that
    \( \{r\} \times \widetilde{W}^{\text{Zar}} \) is algebraic in \( \mathcal{X}^+_g, a \), thus \( (\{r\} \times \widetilde{W})^{\text{Zar}} \subseteq \{r\} \times \widetilde{W}^{\text{Zar}} \);
    and since the algebraic structures on \( \mathcal{X}^+_g, a \) and \( \mathcal{H}^+_g \) are compatible under \( \tilde{\pi} \), we have that \( \tilde{\pi}|_{(\{r\} \times \widetilde{W})^{\text{Zar}}}: (\{r\} \times \widetilde{W})^{\text{Zar}} \to \widetilde{W}^{\text{Zar}} \) is dominant, and the reverse inclusion follows.
  - Combining the previous inequality and inclusions we get
    \( \dim \pi(Y)^{\text{biZar}} + \dim Y \geq \dim \widetilde{W} + \dim Y^{\text{biZar}}. \)
  - Thus we have
    \[ \dim \langle Y \rangle_{\text{sg}} - \dim \pi(Y) = \dim Y^{\text{biZar}} - \dim \pi(Y)^{\text{biZar}} \leq \dim Y - \dim \widetilde{W}, \]
    and thus \((\dagger)\) holds since \( \dim \widetilde{W} \geq d - \ell + 1. \)
1-parameter case

When \( \text{dim } S = 1 \), the criterion of non-degeneracy and the degeneracy locus are easier to describe.

**Definition**

An irreducible closed subvariety \( Y \) of \( \mathcal{A} \) is called a **generically special** subvariety of \( \mathcal{A} \) if it dominates \( S \) and if its geometric generic fibre \( Y \times_S \text{Spec} \mathbb{C}(S) \) is a finite union of \((Z \otimes_{\mathbb{C}} \mathbb{C}(S)) + B\), where \( Z \) is a closed irreducible subvariety of \( \mathcal{A}_{\mathbb{C}(S)/\mathbb{C}} \) (the \( \mathbb{C}(S)/\mathbb{C} \)-trace of \( \mathcal{A} \)) and \( B \) is a torsion coset in \( \mathcal{A} \otimes_{\mathbb{C}(S)} \mathbb{C}(S) \).

**Theorem 12 (Gao-Habegger ’19)**

Assume \( \text{dim } S = 1 \). Let \( X \) be an irreducible closed subvariety of \( \mathcal{A} \) which is dominant to \( S \). Then

(i) \( X \) is degenerate \( \iff \) \( X \) is generically special;

(ii) We have \( X^{\text{deg}} = \bigcup_{Y \subseteq X} Y \). The union of a finite union.

\( Y \) is a generically special subvariety of \( \mathcal{A} \).