

Framed mapping class groups and strata of Abelian differentials

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Translation surfaces

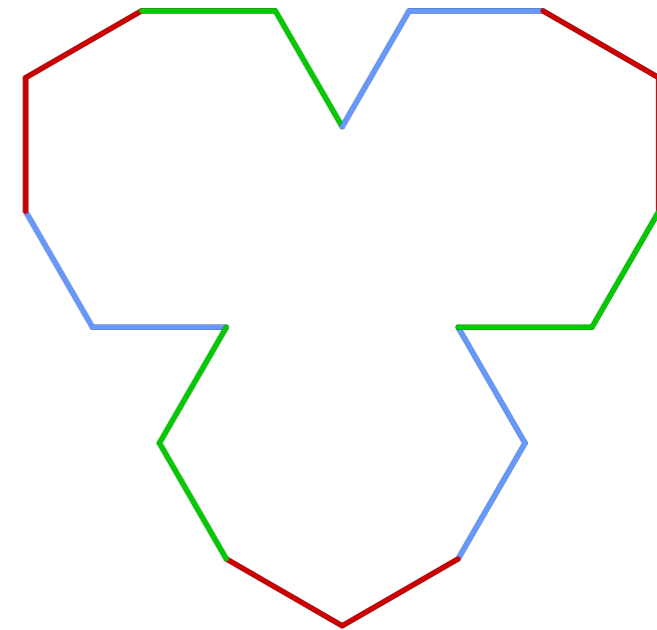
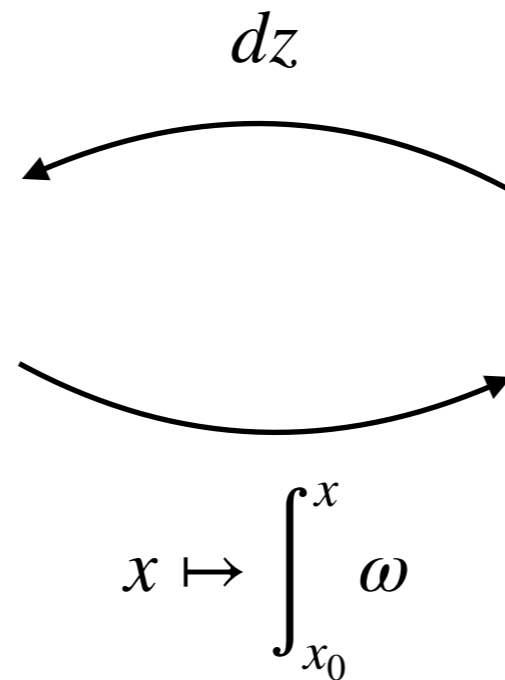
Algebraic geometry

(X, ω) : X Riemann surface,
 ω a holomorphic 1-form

$$\left(X^3 + Y^4 = 1, \frac{dX}{Y^3} \right)$$

Geometric geometry

Surface with atlas of charts to \mathbb{C} ,
transitions $z \mapsto z + c$ (translations)



These are the same thing!

Strata

A *stratum* parameterizes all translation surfaces of the same “geometric type”

Every differential has $2g-2$ zeroes (with multiplicity).
Geometry: cone points of flat metric

$\kappa = \{\kappa_1, \dots, \kappa_n\}$: partition of $2g-2$

Stratum $\mathcal{H}\Omega(\kappa)$: all translation surfaces
with cone angle set κ

Period coordinates: each $\mathcal{H}\Omega(\kappa)$ is a
complex orbifold of dimension $2g+n-1$

Strata

Strata host a fascinating dynamical system ($SL_2(\mathbb{R})$ action) with rich connections to algebraic geometry.

But there's lots of topology here too!

- Tautological family of translation surfaces lives over \mathcal{H}
- Topology of \mathcal{H} itself: $K(\pi, 1)$?

$\pi_0(\mathcal{H}\Omega(\kappa))$ understood (Kontsevich-Zorich).

Always ≤ 3 components.

Fix $\mathcal{H} \subseteq \mathcal{H}\Omega(\kappa)$. What is $\pi_1(\mathcal{H})$?

Strategy

Fix $\mathcal{H} \subseteq \mathcal{H}\Omega(\kappa)$. What is $\pi_1(\mathcal{H})$?

Approach: \mathcal{H} carries a “tautological family” of translation surfaces.

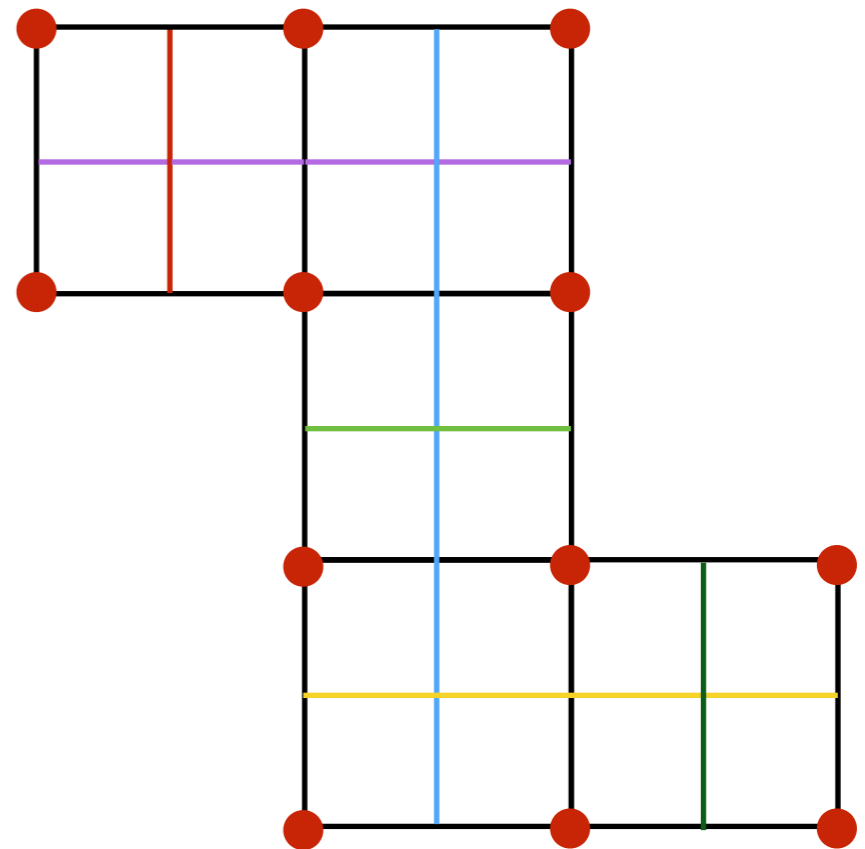
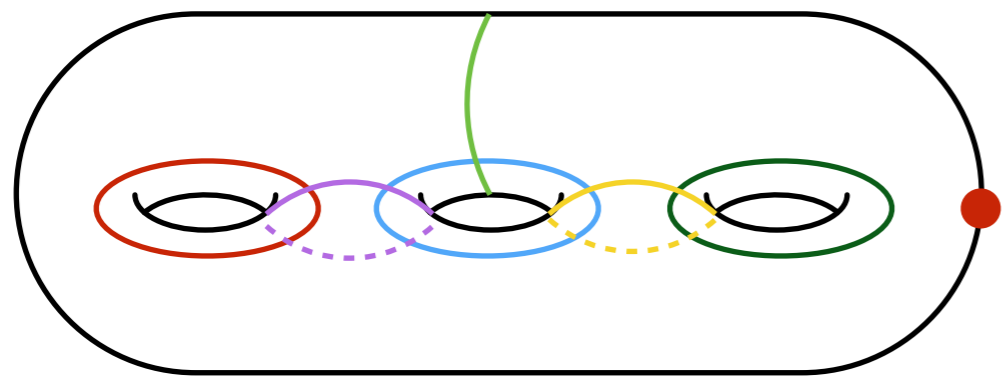
Given a family $p : E \rightarrow B$ of surfaces Σ , there is a *monodromy representation* $\rho : \pi_1(B) \rightarrow \mathbf{Mod}(\Sigma)$

Recall: $\mathbf{Mod}(\Sigma)$ is the *mapping class group* of diffeomorphisms up to isotopy.

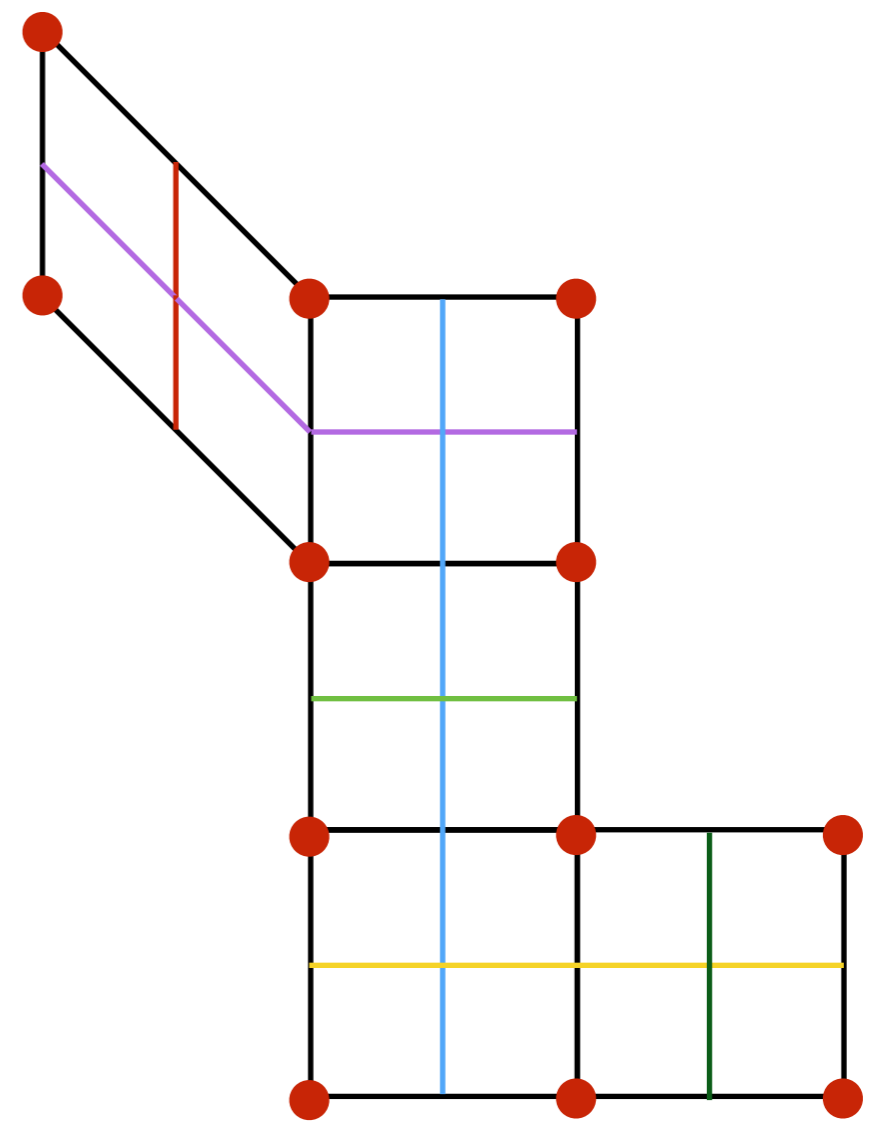
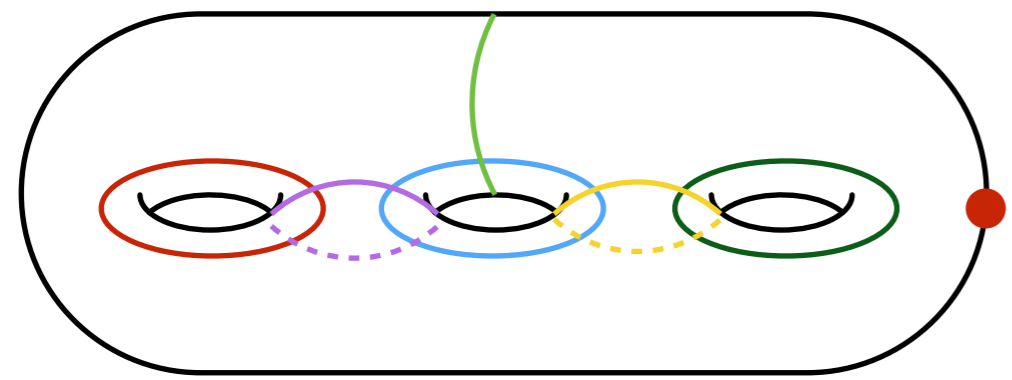
Idea: choose “marking” of reference fiber. Propagate marking along loop: see how it changes upon return.

Will assume all boundary components and punctures fixed *pointwise*.

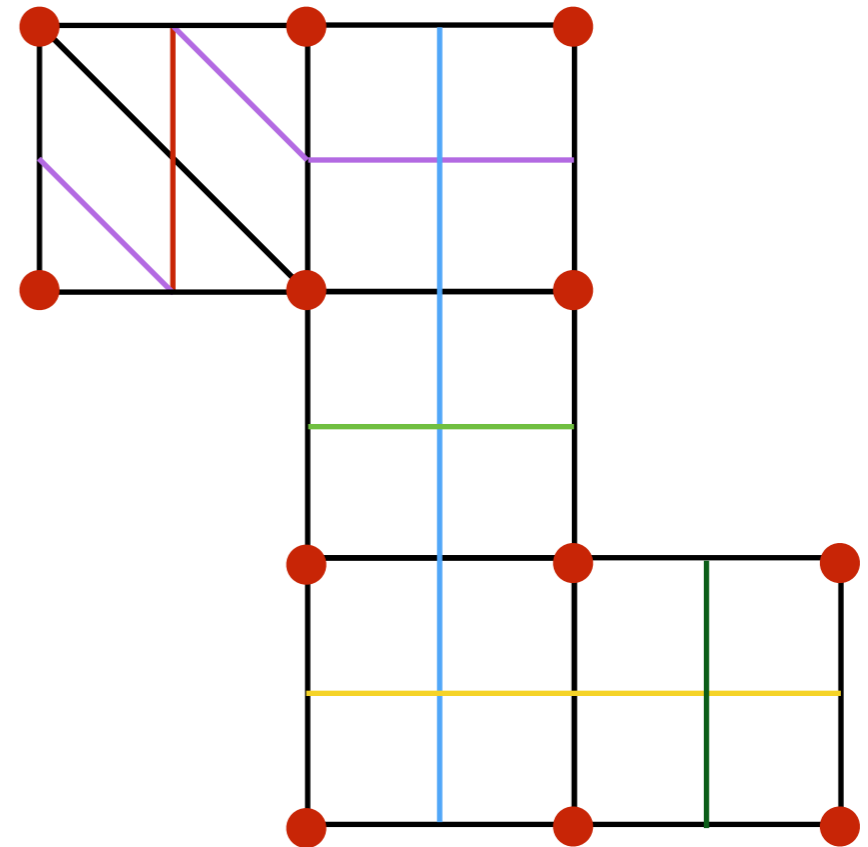
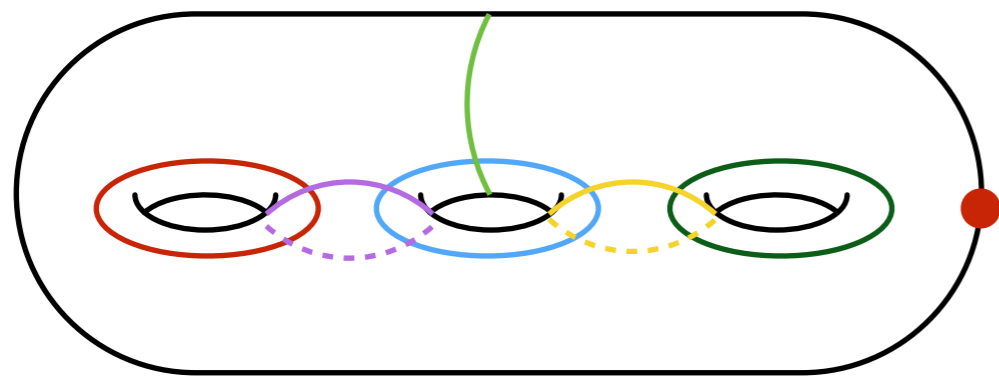
Monodromy



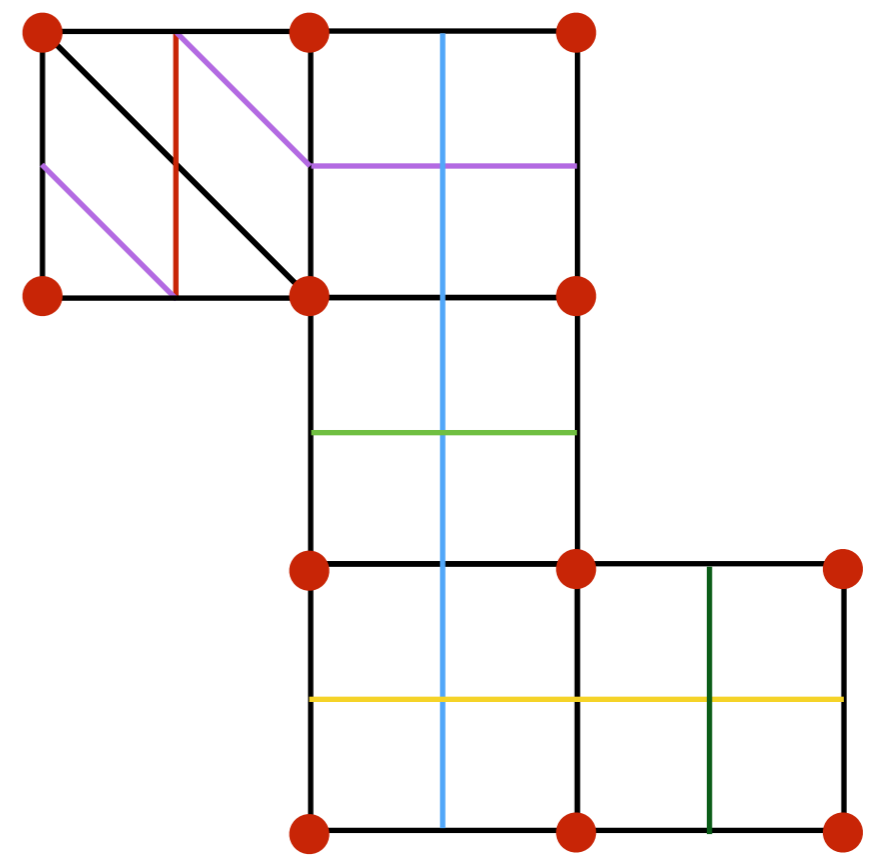
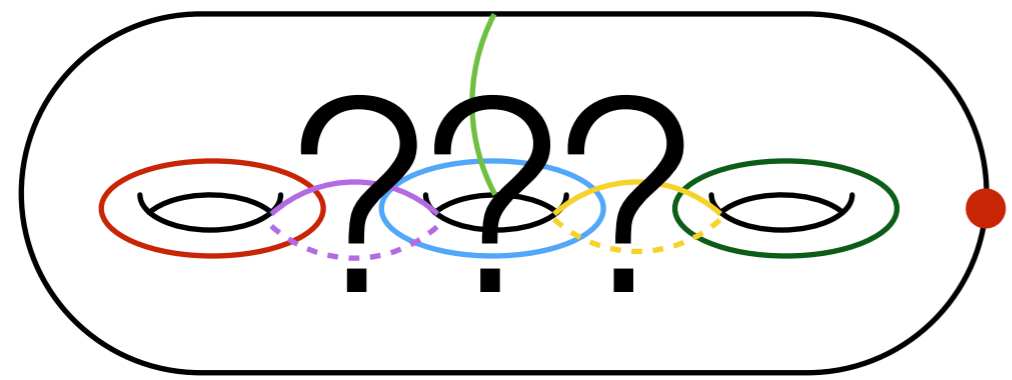
Monodromy



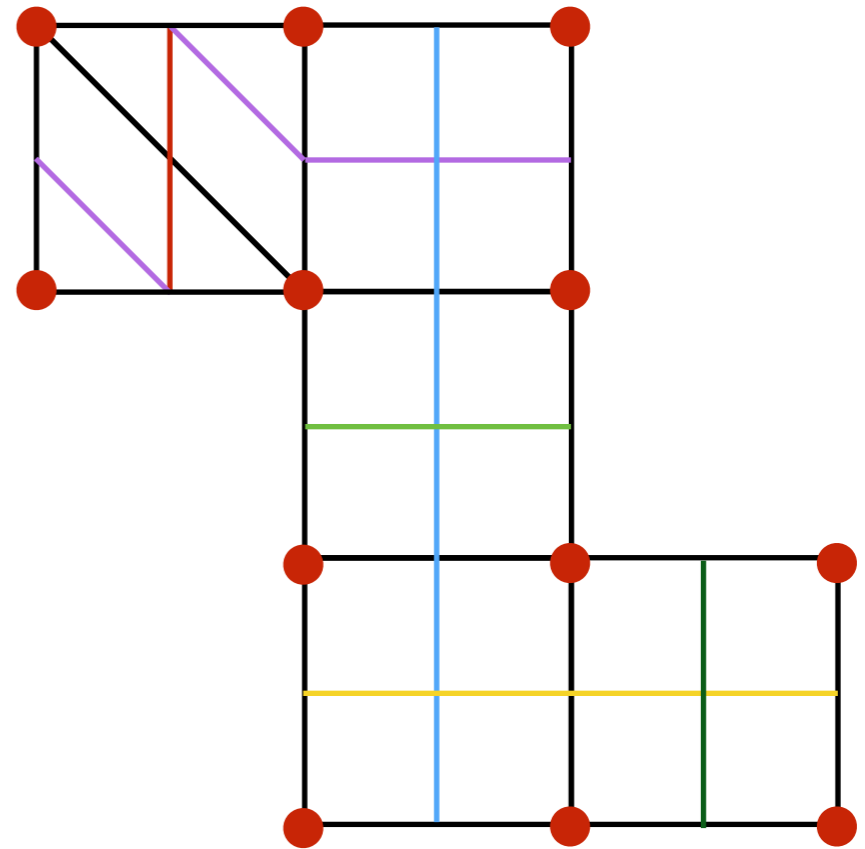
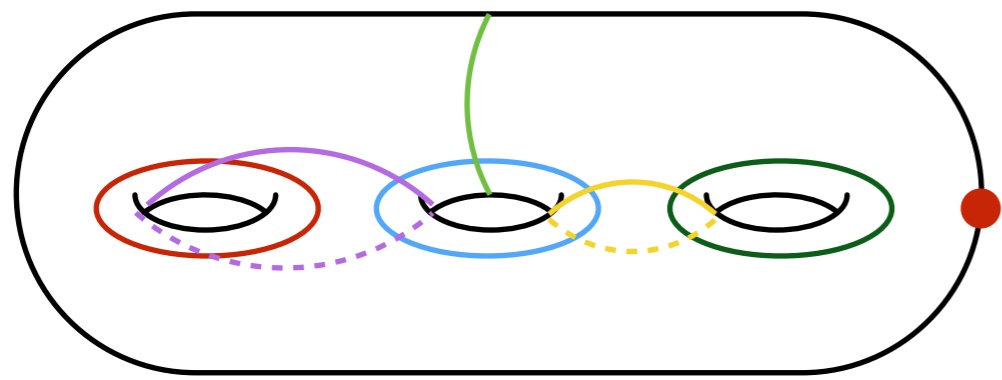
Monodromy



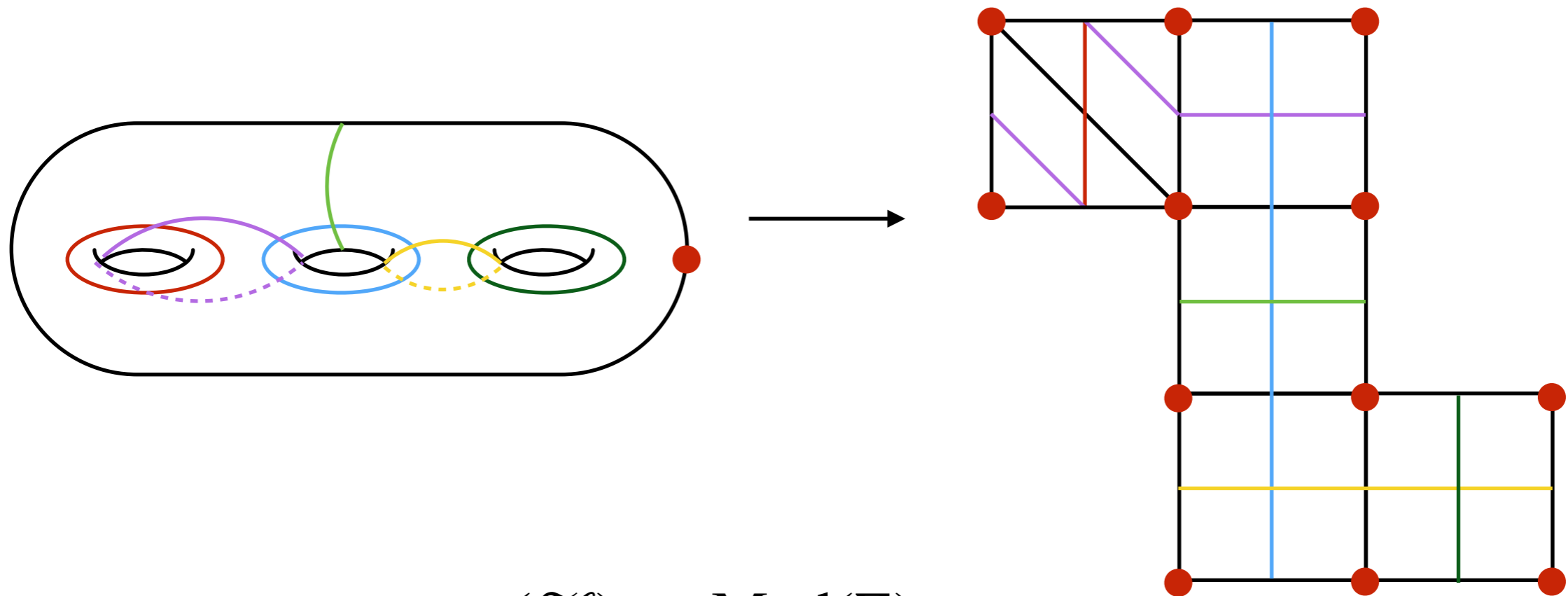
Monodromy



Monodromy



Monodromy



Gives us map $\rho_{\mathcal{H}} : \pi_1(\mathcal{H}) \rightarrow \text{Mod}(\Sigma)$

- A method to study this mysterious group
- Tells us about translation surfaces

How do we describe the image?

Framed mapping class groups

$\text{Mod}(\Sigma)$ acts on set of isotopy classes of framings

$\text{Mod}(\Sigma)[\phi]$: stabilizer of ϕ

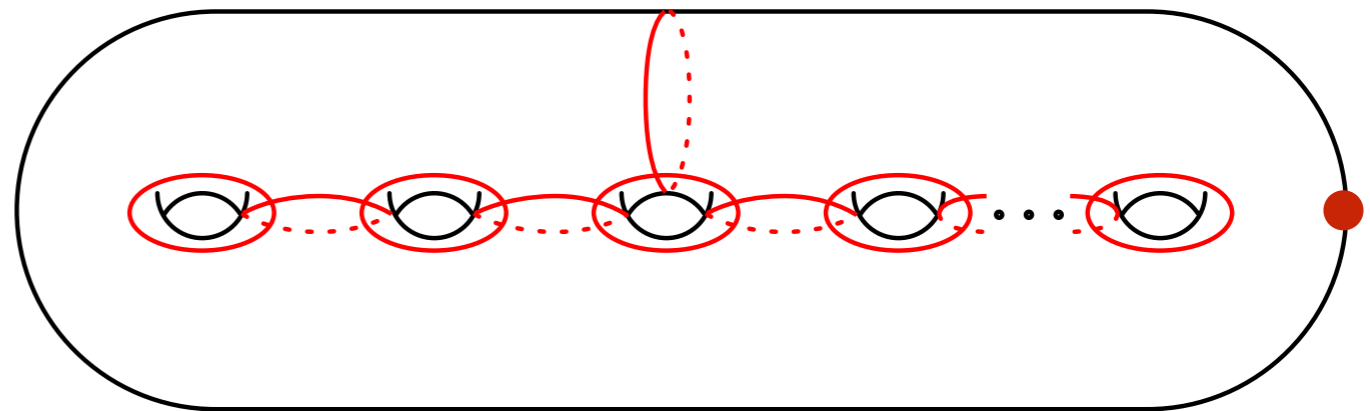
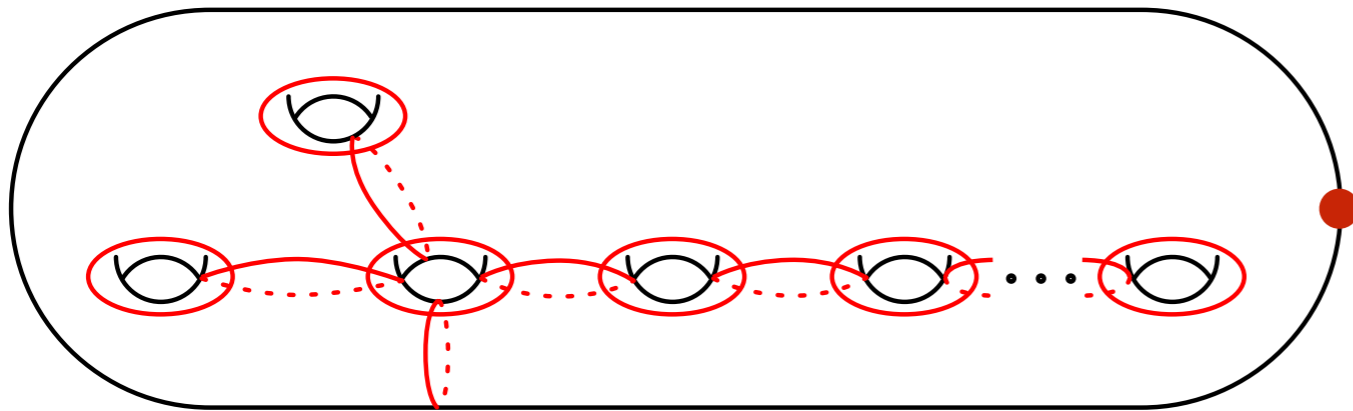
Invariant horizontal vector field \longrightarrow invariant framing ϕ

$$\rho_{\mathcal{H}} : \pi_1(\mathcal{H}) \rightarrow \text{Mod}(\Sigma)[\phi]$$

To study $\rho_{\mathcal{H}}$, must understand $\text{Mod}(\Sigma)[\phi]$!

Simple generating sets

Theorem (Calderon - S.): For $g \geq 5$, any framing ϕ , $\text{Mod}(\Sigma)[\phi]$ is generated by finitely many Dehn twists.

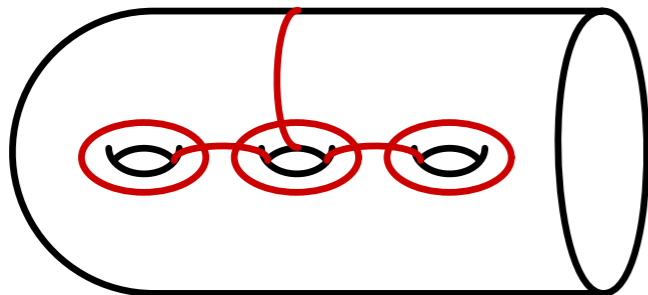


Even though $[\text{Mod}(\Sigma) : \text{Mod}(\Sigma)[\phi]] = \infty!$

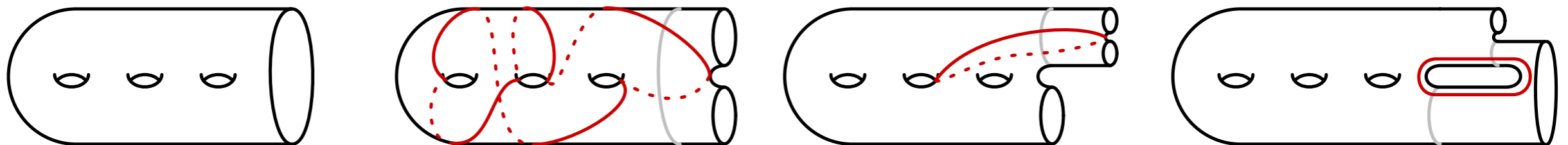
Simple generating sets

These come in a vast array of possibilities:

Start with the E_6 configuration



Now perform *any sequence* of “stabilizations”



The result generates **the** associated framed mapping class group!

Which one? The one uniquely specified by the condition that each distinguished curve has “zero holonomy” for the framing.

Monodromy of strata

Theorem (Calderon - S.):

Fix $g \geq 5$, κ partition of $2g-2$, and $\mathcal{H} \subseteq \mathcal{H}\Omega(\kappa)$
“non-hyperelliptic”*. Then

$$\rho_{\mathcal{H}} : \pi_1(\mathcal{H}) \rightarrow \text{Mod}(\Sigma)[\phi]$$

is surjective.

*: this is the generic case (hyperelliptic is classically understood)

Comments/corollaries (I)

There are various versions of the theorem, depending on how much data you track at the cone points. In each case we show that the monodromy surjects onto the stabilizer of the tangential structure.

Data	Domain	Invariant tangential structure
(Labeled) zeroes	(marked) stratum component	Framing on punctured surface (rel isotopy)
Prong structure (collection of <i>all</i> horizontals)	—//—	Framing on <i>pronged surface</i> (rel. isotopy)
Prong (choice of <i>specific</i> horizontal)	<i>Prong-marked stratum</i>	Framing on surface with boundary (rel <i>relative</i> isotopy)
Nothing	Stratum component	“mod- r framing” (<i>r-spin structure</i>) on closed surface

Comments/corollaries (II)

With a moderate amount of extra work, we can understand the monodromy action on *relative* homology $H_1(X, \text{Zeroes}(\omega); \mathbb{Z})$

This extends work of Gutierrez-Romo, who studied the case $\kappa = \{g - 1, g - 1\}$

We formulate the answer via a *crossed homomorphism*

$$\Theta_\phi : \text{PAut}(H_1(X, \mathbb{Z}; \mathbb{Z})) \rightarrow H^1(X; \mathbb{Z}/2\mathbb{Z})$$

measuring change in “mod-2 winding number” rel. the horizontal framing.

Corollary (C-S): Let \mathcal{H} be a non-hyperelliptic stratum component for $g \geq 5$. The relative homological monodromy is

$$\text{Ker}(\Theta_\phi) \leq \text{PAut}(H_1(X, \mathbb{Z}; \mathbb{Z}))$$

Comments/corollaries (III)

We can use our result to give a complete description of which curves can appear as the core of a cylinder in some marking.

Let (Σ, ϕ) be a framed surface and let c be a simple closed curve. We say that c is *admissible* if the winding number of c rel ϕ is zero.

The core curve of every cylinder is admissible. Conversely,

Corollary (C-S): There exists a marking $f : (\Sigma, \phi) \rightarrow (X, \omega)$ such that $f(c)$ is the core of a cylinder if and only if c is admissible.

Comments/corollaries (IV)

We have a similar result describing saddle connections. Interestingly, there are no obstructions here.

Corollary (C-S): For any $(X, \omega) \in \mathcal{H}$ and any arc α connecting distinct zeroes of ω , there is some path $\gamma(t)$ of differentials in \mathcal{H} such that α is realized as a saddle connection on $\gamma(1)$.

Similar corollaries are obtainable for any other configuration of geometric objects. One must analyze the $\mathbf{Mod}(\Sigma)[\phi]$ orbits of the underlying topological structure.

Comments/corollaries (V)

We now understand the *image* of $\rho_{\mathcal{H}}$

What about the kernel?

I know of *exactly one* element of any kernel!

Looijenga-Mondello ('14): $\pi_1(\mathcal{H}(4)^{odd}) \cong A(E_6)/\text{center}$

Wajnryb ('99): There is a non-central element $w \in A(E_6)$ such that $\mu(w) = 0$ under the map $\mu : A(E_6) \rightarrow \text{Mod}(\Sigma_{3,1})$

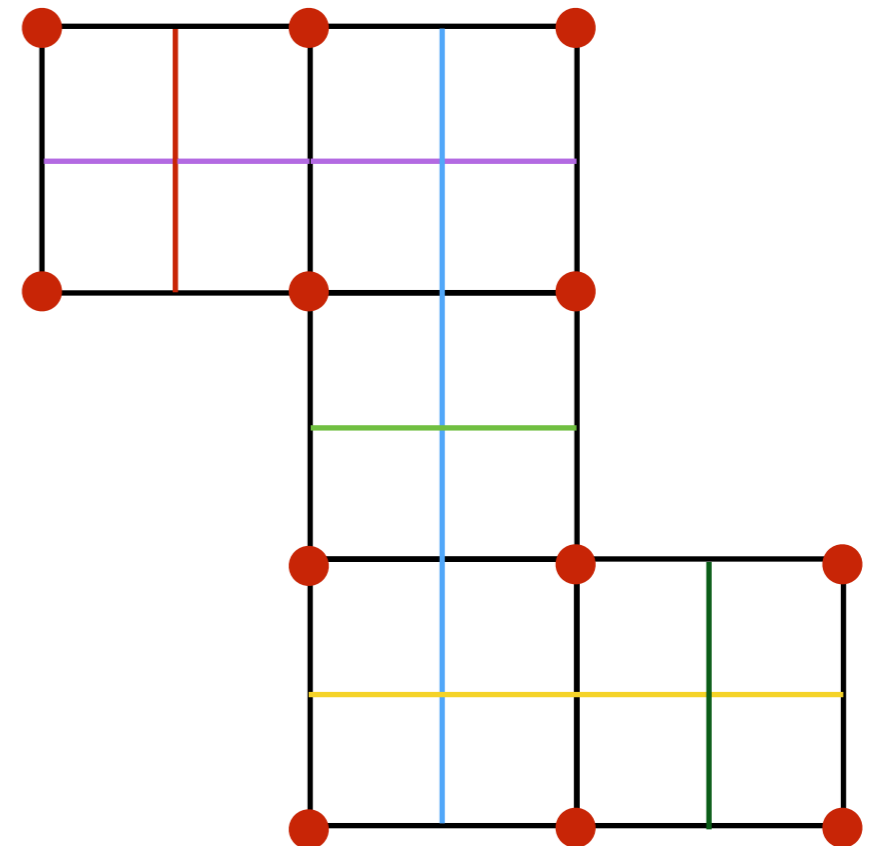
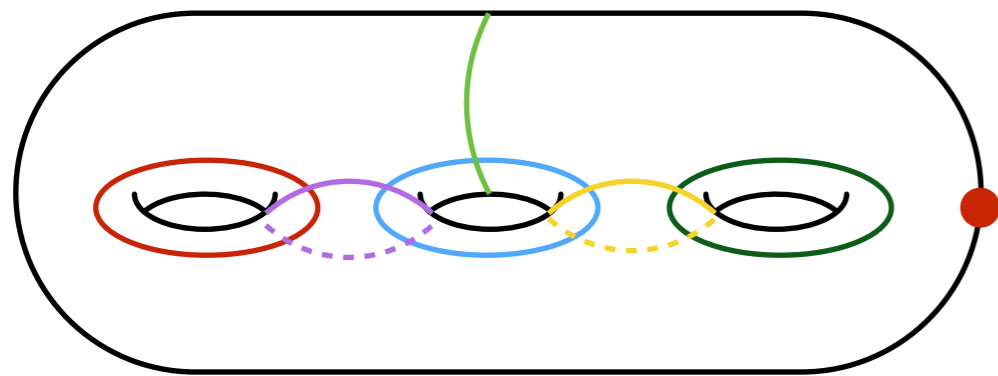
Challenge: give an “intrinsic” proof of this. Find some invariant capable of detecting nontriviality of elements $x \in \ker(\rho_{\mathcal{H}})$

Proof idea: Thurston-Veech

The *Thurston-Veech construction* allows one to build translation surfaces in a desired stratum while controlling configurations of cylinders.

We saw that *cylinder shears* map to Dehn twists under $\rho_{\mathcal{H}}$

So we find a configuration of curves which generate the framed mapping class group, and build a surface realizing these as cylinders.



(yes, I know that $3 < 5\dots$)

Finite generation

A few words on how we find our generating sets for $\mathbf{Mod}(\Sigma)[\phi]$

Admissible subgroup: \mathcal{T}_ϕ generated by all admissible Dehn twists.

Step 1: Find *one specific* finite generating set for \mathcal{T}_ϕ .

Method: “complex of admissible curves”

Step 2: Show that $\mathbf{Mod}(\Sigma)[\phi] = \mathcal{T}_\phi$.

Method: induct on number of boundary components by acting on “complex of framed arcs”. Atrocious connectivity argument 😓

Step 3: Use this to prove the “stabilization lemma”: one new twist suffices

Method: “framed change-of-coordinates principle”
(orbits of (configurations of) framed curves)