Large genus bounds for the distribution of triangulated surfaces in moduli space

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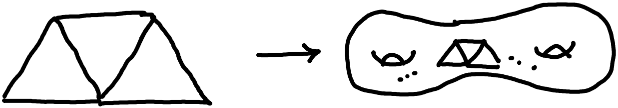
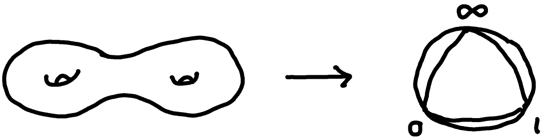
Triangulated surfaces

Glue T equilateral triangles to form a genus g compact Riemann surface:



Motivation:

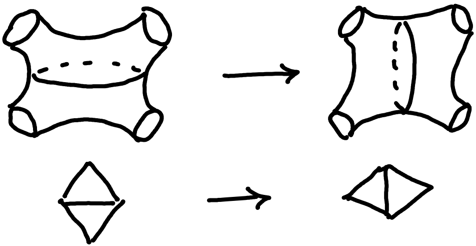
- Number Theory: Belyi maps are branched covers to $\mathbb{P}^1$ branched at 0, 1 and ∞ $\simeq$ Riemann surfaces defined over $\overline{\mathbb{Q}}$



Triangulated surfaces

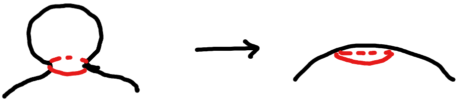
Motivation (cont.):

- Probability: distribution of surfaces as $T \to \infty$, g fixed conjecturally related to Liouville quantum gravity (Guillarmou-Rhodes-Vargas) triangulation locally looks like UIPT in the fixed genus limit in $g \to \infty$, $T \sim Cg$ range, triangulation locally looks like PSHT (Budzinski-Louf)
- Coarse geometry of Teichmuller space: use discrete models like pants graph to understand Teichmuller/moduli space (Brock, Cavendish-Parlier, Rafi-Tao)



Metric geometry of surfaces

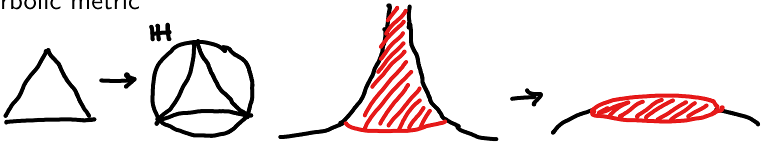
Arbitrary Riemannian surfaces roughly behave like hyperbolic surfaces after normalizing area (Gromov):



- systole grows like $\log g$ (Gromov, Buser-Sarnak)
- homological systole grows like $\log g$ (Gromov)
- pants decomposition with lengths $\lesssim g$ (Buser-Seppala)
- homologically independent loops (Balacheff-Parlier-Sabourau)

Random triangulated surfaces

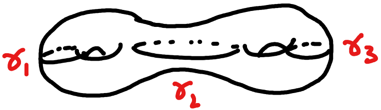
- glue T triangles to form a connected closed surface, expected genus is around $T/4$ (Gamburd)
- if $T \sim 4g$, then the flat metric on S is roughly similar to the hyperbolic metric



- Geometry of random triangulated surface?
  - shortest geodesic? $\geq C$ with probability 1 asymptotically
  - diameter? $\leq C \log g$ with probability 1 asymptotically
  - Cheeger constant? $\geq C$ with probability 1 asymptotically (Brooks-Makover)

Random hyperbolic surfaces

- Fenchel-Nielsen coordinates on $\mathcal{M}_g$:



- Weil-Petersson volume form: $d\ell_1 \wedge d\tau_1 \wedge \ldots \wedge d\ell_{3g-3} \wedge d\tau_{3g-3}$
- Geometry of random hyperbolic surface in $\mathcal{M}_g$?
  - shortest geodesic? $\geq C$ with high probability asymptotically
  - diameter? $\leq C \log g$ with probability 1 asymptotically
  - Cheeger constant? $\geq C$ with probability 1 asymptotically (Mirzakhani)

Triangulated surface vs. Hyperbolic surface

More results describing similarities:

- WP volume of $\mathcal{M}_g$ grows like g^{2g}, number of triangulated surfaces with $T \sim Cg$ also grows like g^{2g} (Mirzakhani, Budzinski-Louf)
- choose a random (hyperbolic, or triangulated with $T \sim Cg$) surface and a point on the surface (with respect to hyperbolic metric, or flat metric), probability of small injectivity radius at the point tends to 0 (Mirzakhani, Budzinski-Louf)
- random hyperbolic surfaces have total pants length $\geq g^{7/6-\epsilon}$, random triangulated surfaces also have total pants length $\geq g^{7/6-\epsilon}$ (Guth-Parlier-Young)

Conjecture: Triangulated surfaces are well-distributed in $\mathcal{M}_g$ as $g \to \infty$.

Metrics on $\mathcal{T}_g$

Hyperbolic surfaces $\rightarrow$ bi-Lipschitz metric d_L:

$$d_L(X, Y) = \inf\{\log L | f : X \rightarrow Y \text{ is an } L\text{-bi-Lipschitz map}\}$$

Riemann surfaces $\rightarrow$ Teichmuller metric d_T:

$$d_T(X, Y) = \inf\left\{\frac{1}{2}\log K | f : X \rightarrow Y \text{ is a } K\text{-quasiconformal map}\right\}$$

- $d_T \leq d_L$
- Douady-Earle extension $\Rightarrow d_L \leq C_1 d_T + C_2$

Well-distribution results

Theorem (Upper Bound)

For $X \in \mathcal{M}_g$, there are at most $(\log g)^{Cg}$ T-triangulated surfaces in $B_{d_T}(X, R)$.

Theorem (Lower Bound)

For $X \in \mathcal{M}_g$ with $\operatorname{sys} X \geq 1$, there exists a Cg-triangulated surface in $B_{d_T}(X, R)$.

Triangulations vs. pants decompositions

Pants decompositions are not well-distributed!

- $\sim g^g$ different topological types pair of pants corresponding to trivalent graphs of degree $2g-2$
- for each topological type:
  $R_L = \{\ell_i \leq L, \tau_i \leq \ell_i \text{ for all } i \in \{1, ..., 3g-3\}\} \subset \mathcal{T}_g$
- $\text{vol}_{WP}(R_L) \sim L^{6g-6}$
- L needs to be at least $\sim g^{1/2}$ for every hyperbolic surface to have a $\leq L$-pants decomposition, but $(g^{1/2})^{6g-6} \cdot g^g$ is much larger than g^{2g} (which is $vol_{WP}(\mathcal{M}_g)$)
- there are hyperbolic surfaces with g^g different $\leq g^{1/2}$-pants decompositions!

Lower bounds

How can we approximate a hyperbolic surface X by a triangulated surface?

- cover X by balls $U_i(x_i, 1/2)$ such that $x_i \notin U_j$ for $i \neq j$
- choose $\{\psi_i\}$, a partition of unity subordinate to U_i with uniformly bounded derivatives
- choose charts $\chi_{i,j} : U_i \to B(1/2, 1/2) \subset \mathbb{R}^2$ with uniformly bounded derivatives
- get embedding $f : X \to \mathbb{R}^N$ given by coordinates $\psi_i \cdot \chi_{i,j}$
- all curvatures of $f(X)$ are bounded since derivatives of f are bounded
- $N \sim g$, but $f(X)$ is contained in the k-skeleton of a cubical lattice, where $k = O(1)$

Lower bounds

Pushing the surface from k-skeleton to $k-1$-skeleton:

Repeat till we have pushed surface to 2-skeleton to get triangulation (after a little more work).

Upper bounds

Step 1. Non-quantitative bound for translation surfaces.

- Translation surface: each triangle can be assigned a rotational orientation, such that all the gluings are translations.
- Given $X \in \mathcal{M}_g$, how many ways can we exhibit X as a triangulated translation surface?

Step 2. Quantitative bound for translation surfaces.

- Make proof of step 1 quantitative.

Step 3. Quantitative bound for triangulated surfaces.

- Use 6-degree branched covers to reduce the triangulated surface case to the translation surface case.

1-forms and Hodge norms

To each triangulated translation surface (X, ρ), we have a canonical holomorphic 1-form ϕ that is $\zeta^i dz$ on each triangle, where $\zeta^6 = 1$.

Then $\psi = 2\,\mathrm{Re}\,\phi$ is a real harmonic 1-form in $H^1(X, \mathbb{Z})$, which uniquely determines the triangulation.
Hodge norm:

$$\|\psi\|^2 = \int_X \psi \wedge *\psi = \int_X \langle \psi, \psi \rangle_\rho \sim g$$

So we want to count $H^1(X, \mathbb{Z})$ lattice points in a $\sim g^{1/2}$ ball.
Volume of a $g^{1/2}$ Euclidean ball is $\sim C^g$ and volume of the lattice is 1, but we still need to know more about that lattice to count lattice points inside the ball. What if the lattice is long and thin?

Geometry of surface $\to$ geometry of lattice

Find short independent lattice vectors to make sure lattice is not long and thin.

non-separating annular regions on $X \to$ lattice vectors

Theorem (Balacheff-Parlier-Sabourau)

If X is a hyperbolic surface with $\operatorname{sys} X \geq 1$, *there exist* $\gamma_1, \ldots, \gamma_{2g}$ *homologically independent loops such that* $\ell(\gamma_i) \leq C \dfrac{\log g}{2g - i + 1} g$.

This gives homologically independent annular regions.

Quantitative?

- if $X, Y \in \mathcal{T}_g$ and $d_T(X, Y) < R$, there is an R'-quasiconformal and C-bi-Lipschitz map $f : X \to Y$
- consider $f^*(\psi)$, which gives element of $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{R})$ with bounded Hodge norm
- use same method to count lattice points

Two issues:

- Cohomology class does not uniquely determine the triangulation anymore since $f^*(\psi)$ is not harmonic.
  Solution: $f^*(\psi)$ is still close to the harmonic representative in the cohomology class, so use this to prove a quantitative version of this statement.
- Need to understand growth of d_T.
  Solution: use Bers embedding to bound d_T (locally) in a genus-independent way.

Bers embedding

- fix $X = \mathbb{H}/\Gamma \in \mathcal{T}_g$, for $\Gamma \subset PSL_2(\mathbb{R})$, and let $X^* = \mathbb{L}/\Gamma$
- any $Y \in \mathcal{T}_X$ can be described by a Beltrami differential $\mu \in M(X)$, equivalently Γ-invariant Beltrami differential on $\mathbb{H}$ (for $\gamma \in \Gamma$, $\mu \circ \gamma = \mu$)
- extend μ to $\widehat{\mu}$ on $\mathbb{P}^1$ by 0 on $\mathbb{P}^1 \setminus \mathbb{H}$ and let $f^{\widehat{\mu}}$ be the solution that preserves 0, 1, and ∞
- have homomorphism $\eta : \Gamma \to \Gamma_Y \subset PSL_2(\mathbb{C})$ such that $f^{\widehat{\mu}} \circ \gamma = \eta(\gamma) \circ f^{\widehat{\mu}}$
- $f^{\widehat{\mu}}(\mathbb{H})/\Gamma_Y = Y$ by construction, and $f^{\widehat{\mu}}(\mathbb{L})/\Gamma_Y = X^*$ since $f^{\widehat{\mu}}$ is conformal on $\mathbb{L}$
- Bers embedding $\beta_X : \mathcal{T}_g \to Q^\infty(X^*)$, by

$$\beta_X(Y) = \mathcal{S}(f^{\widehat{\mu}})$$

Here $\mathcal{S}$ is the Schwarzian derivative

Asymptotic geometry of Teichmuller metric

Nehari's bound:

$$\|\mathcal{S}f\|_\infty \le \frac{1}{2} \Rightarrow f : \mathbb{H} \to \mathbb{C} \text{ is a holomorphic injection} \Rightarrow \|\mathcal{S}f\|_\infty \le \frac{3}{2}$$

d_T is the Kobayashi metric on the region above. So it is locally bounded by Kobayashi metrics on the norm balls.

Translation surfaces $\to$ triangulated surfaces

Triangulated surface on X $\to$ meromorphic 6-differential ω that is dz^6 on each triangle, poles up to order 5
Canonical degree 6 branched cover:

- cover $X - \{\text{zeros}, \text{poles}\}$ by open balls $\{U_i\}$
- for each U_i, associate $U_{i,1}, ..., U_{i,6}$, each consisting of $(U_i, \phi_{i,j} = \zeta^j \omega^{1/6})$
- glue $U_{i,j}$ and $U_{i',j'}$ if the $U_i \cap U_{i'} \neq \emptyset$, and $f_{i,i'}^* \phi_{i,j} = \phi_{i',j'}$ for transition function $f_{i,i'}$
- compactify to obtain branched cover $F : \widetilde{X} \to X$ such that $F^*(\omega) = \phi^6$ for holomorphic 1-form ϕ

Reduces counting triangulated surfaces to counting triangulated translation surfaces on a higher dimensional moduli space.

Thank you for listening!