THE MAGIC NUMBER CONJECTURE FOR THE $m = 2$ AMPLITUHEDRON AND PARKE-TAYLOR IDENTITIES

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Abstract. The *amplituhedron* $\mathcal{A}_{n,k,m}$ is a geometric object introduced in the context of scattering amplitudes in $\mathcal{N} = 4$ super Yang Mills. It generalizes the positive Grassmannian (when $n = k+m$), cyclic polytopes (when $k = 1$), and the bounded complex of the cyclic hyperplane arrangement (when $m = 1$). Of substantial interest are the *tilings* of the amplituhedron, which are analogous to triangulations of a polytope. In [KWZ20], it was observed that the known tilings of $\mathcal{A}_{n,k,2}$ have cardinality $\binom{n-2}{k}$ and the known tilings of $\mathcal{A}_{n,k,4}$ have cardinality the *Narayana number* $\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$; generalizing these observations, [KWZ20] conjectured that for even m the tilings of $\mathcal{A}_{n,k,m}$ have cardinality the *MacMahon number*, the number of plane partitions which fit inside a $k \times (n-k-m) \times \frac{m}{2}$ box. We refer to this prediction as the *Magic Number Conjecture*. In this paper we prove the Magic Number Conjecture for the $m = 2$ amplituhedron: that is, we show that each tiling of $\mathcal{A}_{n,k,2}$ has cardinality $\binom{n-2}{k}$. We prove this by showing that all positroid tilings of the hypersimplex $\Delta_{k+1,n}$ have cardinality $\binom{n-2}{k}$, then applying *T-duality*. In addition, we give combinatorial necessary conditions for tiles to form a tiling of $\mathcal{A}_{n,k,2}$; we give volume formulas for *Parke-Taylor polytopes* and certain positroid polytopes in terms of circular extensions of *cyclic partial orders*; and we prove new variants of the classical *Parke-Taylor identities*.

Contents

1. Introduction

The (tree) *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is the image of the positive Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ under the *amplituhedron map* $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$, a map induced by matrix multiplication by a positive matrix $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. The amplituhedron was introduced by Arkani-Hamed and Trnka [AHT14] in order to give a geometric interpretation of *scattering amplitudes* in $\mathcal{N} = 4$ super Yang–Mills theory (SYM); more specifically, they reformulated the BCFW recurrence for computing scattering amplitudes as giving a *tiling* of the $m = 4$ amplituhedron. Here, a *tiling* of $\mathcal{A}_{n,k,m}(Z)$ comes from a collection of km-dimensional cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ on which $\tilde{Z}$ is injective, such that the images of the cells are disjoint and cover a dense subset of the amplituhedron. The notion of tiling can be thought of as a generalization of the notion of triangulation of a polytope.[1]

While the case $m = 4$ is most directly relevant to physics, the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ makes sense for any positive n, k, m such that $k+m \leq n$, and has a very rich geometric and combinatorial structure. It generalizes cyclic polytopes (when $k = 1$), cyclic hyperplane arrangements [KW19] (when $m = 1$), and the positive Grassmannian (when $k = n - m$), and it is connected to the hypersimplex and the positive tropical Grassmanian [LPW23, PSBW23] (when $m = 2$). This paper will focus on the case $m = 2$. In this case, the amplituhedron $\mathcal{A}_{n,k,2}$ is also closely related to some scattering amplitudes, correlators of determinant operators and form factors in planar $\mathcal{N} = 4$ super Yang–Mills theory [KL20, CHCM23, BT23].

In [KWZ20] it was observed that the known tilings of $\mathcal{A}_{n,k,2}(Z)$ have cardinality $\binom{n-2}{k}$, the known tilings of $\mathcal{A}_{n,k,4}(Z)$ have cardinality the Narayana number $\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$, and all tilings of $\mathcal{A}_{n,1,m}$ for even m have cardinality $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$.[2] [KWZ20, Conjecture 8.1] generalized these observations by predicting that when m is even, the amplituhedron $\mathcal{A}_{n,k,m}$ has a tiling with cardinality

$$M_{n,k,m} := M\left(k, n-k-m, \frac{m}{2}\right), \quad \text{where} \quad M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{\ell=1}^{c} \frac{i+j+\ell-1}{i+j+\ell-2}$$

is the *MacMahon number*.[3] This number has many remarkable interpretations: $M(a, b, c)$ counts the number of *plane partitions* which fit inside an $a \times b \times c$ box, collections of c noncrossing lattice paths inside an $a \times b$ rectangle, rhombic tilings of a hexagon with side lengths (a, b, c, a, b, c), perfect matchings of a honeycomb lattice with parameters a, b, c, *Kekulé structures* of a hexagon-shaped benzenoid with parameters a, b, c, and the dimension of the degree c component of the homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}_{a,a+b}]$, see [Mac16, GD52, Cyv86, Kek58, CG88, BGCT88, Hod43]. (Plane partitions also naturally appear in the computation of the Euler characteristic of the Hilbert scheme of points in a 3-fold [Che96], and in the study of Donaldson-Thomas invariants [MNOP06a, MNOP06b].) After [KWZ20] appeared, [GL20] restated the conjecture but specified that *each* tiling should have this cardinality. We refer to the prediction that tilings of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ should have cardinality $M_{n,k,m}$ as the *Magic Number Conjecture*.

In this paper we prove the Magic Number Conjecture for the $m = 2$ amplituhedron; the following result appears as Theorem 5.10 and Corollary 5.11.

Theorem A (Magic Number Theorem for $\mathcal{A}_{n,k,2}(Z)$)**.** *Suppose a collection $\mathcal{C}$ of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ gives rise to a tiling of $\mathcal{A}_{n,k,2}(Z)$ for all $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. Then $\mathcal{C}$ has cardinality $M_{n,k,2} = \binom{n-2}{k}$.*

Recall that the *hypersimplex* $\Delta_{k+1,n}$ is the convex hull of all 0/1 vectors $e_I \in \mathbb{R}^n$, where I is a $(k + 1)$-element subset of $[n]$. It is the *uniform matroid polytope* and can also be viewed as the

[1]We don't require tiles to intersect in a common "face". For polytopes, this is weaker than a triangulation and it is sometime referred to as a *dissection*.

[2]This last statement comes from the fact that every triangulation of the cyclic polytope $C(n, m)$ contains exactly $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$ simplices when m is even [Bay93, Ram97] and that for $C(n, m)$ tilings and triangulations coincide [OT10].

[3][KWZ20, Remark 8.2] pointed out that the conjecture could be generalized to all m using $M(k, n-k-m, \lfloor \frac{m+1}{2} \rfloor)$.

image of the positive Grassmannian $\mathrm{Gr}_{k+1,n}^{\geq 0}$ under the *moment map*. To prove the Magic Number Conjecture for $m = 2$, we prove that each tiling of the hypersimplex $\Delta_{k+1,n}$ by positroid polytopes has cardinality $\binom{n-2}{k}$, then apply the T-duality theorem from [PSBW23, Theorem 11.6] (which appears here as Theorem 2.22).

Theorem B. *Every positroid tiling of the hypersimplex $\Delta_{k+1,n}$ consists of $\binom{n-2}{k}$ tiles.*

The above theorem generalizes the result from [SW21] that every finest regular positroid subdivision of $\Delta_{k+1,n}$ has cardinality $\binom{n-2}{k}$.

We also establish some necessary conditions for a collection of cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ to give rise to a tiling of $\mathcal{A}_{n,k,2}(Z)$ (equivalently, $\Delta_{k+1,n}$). These results appear in Theorem 6.11 and Proposition 6.15, and use the combinatorics of *bicolored subdivisions* (cf. Definition 2.9).

Our third main result is a formula for the volume of certain polytopes as the number of *circular extensions* of a *partial cyclic order*. Circular extensions and partial cyclic orders are analogues of linear extensions and partial orders. The following result appears as Corollary 4.9; see Proposition 7.8 for a generalization.

Theorem C. *Let σ be a* bicolored subdivision *(see Definition 2.9), let Γ_σ be the corresponding positroid polytope (see Theorem 2.12), and let C_σ be the corresponding partial cyclic order (see Definition 4.6). Then the normalized volume of Γ_σ equals the number of circular extensions of C_σ.*

We note that [AJVR20] also studied some polytopes contained in $[0,1]^n$ whose volumes can be computed in terms of circular extensions of partial cyclic orders; they were motivated in part by [Sta12, Exercise 4.56(d)]. In general our polytopes are different from the polytopes of [AJVR20] although in a very special case they agree, see Remark 7.10.

A key tool in our proof of the Magic Number Theorem is the *Parke-Taylor* function[4] associated to a permutation.

Definition 1.1. Let $w = w_1 \dots w_n \in S_n$ be a permutation on $[n]$. The *Parke-Taylor function* of w is

$$\mathrm{PT}(w) := \frac{1}{P_{w_1\, w_2} P_{w_2\, w_3} \cdots P_{w_n\, w_1}}, \tag{1}$$

where the P_{ij} are Plücker coordinates on $\mathrm{Gr}_{2,n}$, with the convention that Plücker coordinates are antisymmetric in their indices.

For example, if $w = 2143$ then

$$\mathrm{PT}(w) = \frac{1}{P_{21}P_{14}P_{43}P_{32}} = -\frac{1}{P_{12}P_{14}P_{34}P_{23}}.$$

We view the Parke-Taylor function $\mathrm{PT}(w)$ as a rational function on (the affine cone over) the Grassmannian $\mathrm{Gr}_{2,n}$. It is well-defined on (the cone over) the top-dimensional matroid stratum of $\mathrm{Gr}_{2,n}$, that is, the locus where no Plücker coordinates vanish. Parke-Taylor functions have numerous connections e.g. with the cohomology of the moduli space $\mathcal{M}_{0,n}$ of n points on the Riemann sphere in relation with *scattering equations* [CHY14] and *Lie polynomials* [FM21].

[4]From the seminal paper of the physicists Stephen J. Parke and T.R. Taylor [PT86].

As a corollary of our work, we obtain many relations for Parke-Taylor functions (cf. Theorem 7.11), one for each *tricolored subdivision* of an n-gon (cf. Definition 4.5). One such relation is as follows.

Corollary. *Let $S = \{s_1 < \cdots < s_r\} \subseteq [n]$, and let $\mathcal{D}_S$ be the set of permutations on $[n]$ with $w_n = n$ in which $s_1, \ldots, s_r$ appear in order. Then we have that*

$$\sum_{w \in \mathcal{D}_S} \mathrm{PT}(w) = 0.$$

The result above appears later as Corollary 7.14. In the special case that $r = n-1$, this result recovers the "classical" Parke-Taylor identity, which is also called the $U(1)$*-decoupling identity.* In Theorem 7.11 we give a generalization of Corollary 7.14, where the sum is over circular extensions of certain cyclic partial orders.

The structure of this paper is as follows. In Section 2 we provide background on the positive Grassmannian, the hypersimplex, and the amplituhedron, including a well-known triangulation of the hypersimplex into w-simplices. In Section 3 we give a simple combinatorial characterization of exactly which w-simplices lie in a given tile for the hypersimplex. In Section 4, we introduce partial cyclic orders and use them to rephrase the results of the previous section. In Section 5, we define the weight function of a positroid polytope using Parke-Taylor functions; prove that the weight function of each tile Γ_σ of $\Delta_{k+1,n}$ is constant; and deduce the Magic Number Theorem. In Section 6, we extend the notion of weight function to facets of tiles and prove additional necessary conditions for when a collection of bicolored subdivsions gives rise to a positroid tiling of $\Delta_{k+1,n}$. In Section 7, we introduce Parke–Taylor polytopes and prove new relations among Parke–Taylor functions. Finally in Appendix A we give an interpretation of our previous results in terms of the G-amplituhedron.

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2. Background: the hypersimplex, the $m = 2$ amplituhedron, and their tilings

In this section, we review background on the positive Grassmannian, the hypersimplex, and the amplituhedron. We introduce the notion of positroid tiles and tilings of a space X which is the surjective image of the positive Grassmannian. We review the characterization of positroid tiles for the hypersimplex [LPW23] and for the $m = 2$ amplituhedron [PSBW23] using the combinatorics of bicolored subdivisions; we also state the T-duality theorem of [PSBW23] which connects positroid tilings of the hypersimplex and $m = 2$ amplituhedron. Finally, we discuss a triangulation of the hypersimplex into w-simplices [Sta77, Stu96, LP07], which is the simultenous refinement of all positroid tilings and is essential to the proofs in later sections.

2.1. **The (positive) Grassmannian.** The *Grassmannian* $\mathrm{Gr}_{k,n} = \mathrm{Gr}_{k,n}(\mathbb{R})$ is the space of all k-dimensional subspaces of an n-dimensional vector space $\mathbb{R}^n$. Let $[n]$ denote $\{1, \ldots, n\}$, and $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. We can represent a point $V \in \mathrm{Gr}_{k,n}$ as the row-span of a full-rank $k \times n$ matrix C with entries in $\mathbb{R}$. Then for $I = \{i_1 < \cdots < i_k\} \in \binom{[n]}{k}$, the *Plücker coordinate* $P_I(C)$ be the $k \times k$ minor of C using the columns I. The Plücker coordinates of V are independent of the choice of matrix representative C (up to common rescaling). The *Plücker embedding* $V \mapsto \{P_I(C)\}_{I \in \binom{[n]}{k}}$ embeds $\mathrm{Gr}_{k,n}$ into projective space[5].

Definition 2.1 (Positive Grassmannian)**.** [Lus94, Pos06] We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if (up to a global change of sign) $P_I(C) \geq 0$ for all $I \in \binom{[n]}{k}$. Similarly, V is *totally positive* if $P_I(C) > 0$ for all $I \in \binom{[n]}{k}$. We let $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k,n}^{>0}$ denote the set of totally nonnegative and totally positive elements of $\mathrm{Gr}_{k,n}$, respectively. The set $\mathrm{Gr}_{k,n}^{\geq 0}$ is called the *totally nonnegative Grassmannian*, or sometimes just the *positive Grassmannian*.

If we partition $\mathrm{Gr}_{k,n}^{\geq 0}$ into strata based on which Plücker coordinates are strictly positive and which are 0, we obtain a cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ into *positroid cells* [Pos06] that glue together to form a CW complex [PSW09]. Each positroid cell S gives rise to a matroid $\mathcal{M}$, whose bases are precisely the k-element subsets I such that the Plücker coordinate P_I does not vanish on S; the matroid $\mathcal{M}$ is called a *positroid*.

2.2. **Positroid tiles and tilings.** Given any surjective map ϕ from a cell complex onto a topological space X, say of dimension d, it is natural to try to decompose X using images of d-dimensional cells under ϕ. This leads to the definition of *ϕ-induced tiling* [Wil23, Definition 2.19] which we apply in the case that our cell complex is the positive Grassmannian.[6] When ϕ is the amplituhedron map $\tilde{Z}$, tilings were first studied in [AHT14] (where they were referred to as *triangulations*).

Definition 2.2. Let $\phi : \mathrm{Gr}_{k,n}^{\geq 0} \to X$ be a continuous surjective map where $\dim X = d$. The image $\phi(S)$ of a positroid cell $S \subset \mathrm{Gr}_{k,n}^{\geq 0}$ is an *open tile* for X (with respect to ϕ) if ϕ is injective on S and $\dim S = d$. The closure $\overline{\phi(S)}$ of an open tile is called a *tile* for X.

A *positroid tiling* of X (with respect to ϕ) is a collection $\{\overline{\phi(S)}\}_{S \in \mathcal{C}}$ of tiles satisfying:

- (disjointness): open tiles $\phi(S)$ and $\phi(S')$ for distinct positroid cells S and S' in $\mathcal{C}$ are disjoint
- (covering): $\cup_{S \in \mathcal{C}} \overline{\phi(S)} = X$.

We will focus here on ϕ-induced tilings for two different maps ϕ: the moment map μ and the amplituhedron map $\tilde{Z}$, which map $\mathrm{Gr}_{k,n}^{\geq 0}$ onto the hypersimplex and the amplituhedron, respectively.

[5] We will sometimes abuse notation and identify C with its row-span.

[6] There are many reasonable variations of this definition. One might want to relax the injectivity assumption, or to impose further restrictions on how boundaries of the images of cells should overlap. Note that in the literature, positroid tilings are sometimes called *(positroid) triangulations*. We avoid this terminology in order to avoid confusion with the notion of e.g. polytopal triangulations.

In the subsequent subsections, we will discuss the combinatorics of tiles and tilings for these choices of ϕ, which turn out to be closely related.

2.3. **The moment map and the hypersimplex.** We use $\{e_1, \dots, e_n\}$ to denote the standard basis of $\mathbb{R}$, and define $e_I := \sum_{i \in I} e_i \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ and $I \subset [n]$, we use the notation $x_I := \sum_{i \in I} x_i$. For $i \neq j \in [n]$, we use the notation $[i, j] := \{i, i+1, \dots, j-1, j\}$ for the cyclic interval from i to j.

The *moment map* from the Grassmannian $\mathrm{Gr}_{k+1,n}$ to $\mathbb{R}^n$ is defined as follows.

Definition 2.3 (The moment map)**.** Let A be a $(k+1) \times n$ matrix representing a point of $\mathrm{Gr}_{k+1,n}$. The *moment map* $\mu : \mathrm{Gr}_{k+1,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}.$$

It is well-known that the image of the Grassmannian $\mathrm{Gr}_{k+1,n}$ under the moment map is the *hypersimplex* $\Delta_{k+1,n}$, defined below. If one restricts the moment map to $\mathrm{Gr}_{k+1,n}^{\geq 0}$ then the image is again the hypersimplex $\Delta_{k+1,n}$ [TW15, Proposition 7.10].

Definition 2.4 (The hypersimplex)**.** The $(k+1, n)$-*hypersimplex* $\Delta_{k+1,n} \subset \mathbb{R}^n$ is the convex hull of the points e_I where I runs over $\binom{[n]}{k+1}$.

See Figure 1 for an example of the hypersimplex. The hypersimplex $\Delta_{k+1,n}$ is $(n-1)$-dimensional, as it is contained in the hyperplane $x_{[n]} = k + 1$.

Remark 2.5. Under the projection $\pi : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$, we see that the hypersimplex is linearly equivalent to the polytope in $\mathbb{R}^{n-1}$ defined by

$$\{(x_1, \dots, x_{n-1}) \mid 0 \leq x_i \leq 1; k \leq x_{[n-1]} \leq k + 1\}.$$

That is, the projected hypersimplex $\pi(\Delta_{k+1,n})$ is the slice of the unit hypercube $\square_{n-1}$ contained between the hyperplanes $x_{[n-1]} = k$ and $x_{[n-1]} = k + 1$.

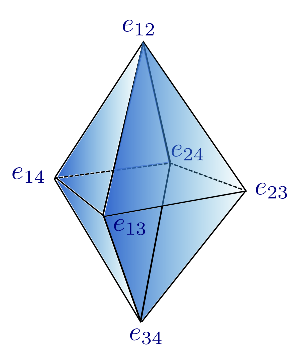

FIGURE 1. The hypersimplex $\Delta_{2,4}$ projected in $\mathbb{R}^3$.

The moment map images of positroid cells, and their closures, can easily be described using the combinatorics of positroids.

Definition 2.6. Let $\mathcal{M} \subset \binom{[n]}{k+1}$ be a positroid. The *positroid polytope* $\Gamma_{\mathcal{M}}$ is defined as

$$\Gamma_{\mathcal{M}} := \text{conv}\{e_I : I \in \mathcal{M}\}.$$

We denote by $\Gamma^\circ_{\mathcal{M}}$ the relative interior of $\Gamma_{\mathcal{M}}$.

Proposition 2.7 ([TW15, Proposition 7.10])**.** *Let $S \subset \text{Gr}^{\geq 0}_{k+1,n}$ be a positroid cell, and $\mathcal{M}$ its corresponding positroid. Then*

$$\mu(S) = \Gamma^\circ_{\mathcal{M}} \qquad \textit{and} \qquad \overline{\mu(S)} = \Gamma_{\mathcal{M}}.$$

For the reader's convenience, below we specialize Definition 2.16 with ϕ being the moment map μ and X being the hypersimplex $\Delta_{k+1,n}$.

Definition 2.8. Let $\mu : \text{Gr}^{\geq 0}_{k+1,n} \to \Delta_{k+1,n}$ be the moment map. The positroid polytope $\Gamma_{\mathcal{M}} = \overline{\mu(S)}$, which is the closure of the image of a positroid cell S, is a *tile* for $\Delta_{k+1,n}$ if μ is injective on S and $\dim S = n-1$. A *positroid tiling* of $\Delta_{k+1,n}$ is a collection of tiles with disjoint interiors whose union is $\Delta_{k+1,n}$.

2.4. **Tiles of $\Delta_{k+1,n}$ from bicolored subdivisions.** In this section we review the characterization of tiles of $\Delta_{k+1,n}$ from *bicolored subdivisions*. Here, we will describe the tiles directly as subsets of $\mathbb{R}^n$; see [LPW23, Section 3] for a discussion of the cells whose moment map images are the tiles. We first review the necessary combinatorics of bicolored subdivisions.

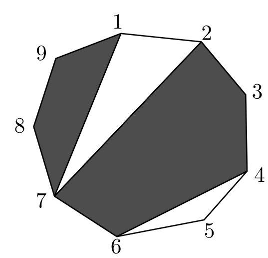

FIGURE 2. A bicolored subdivision of type $(5, 9)$.

Definition 2.9 (Bicolored subdivisions and arcs)**.** Let $\mathbf{P}_n$ be a convex n-gon with vertices labeled from 1 to n in clockwise order. A *bicolored subdivision* σ is a partition of $\mathbf{P}_n$ into black and white polygons such that two polygons sharing an edge have different colors. We say that σ has *type* (k, n) or that σ *is a* (k,n)*-bicolored subdivision* if any triangulation of the black polygons consists of exactly k black triangles. See Figure 2.

Given a pair of vertices i, j of $\mathbf{P}_n$, we say that the arc $i \to j$ is:

- *compatible* with σ if the arc either bounds a polygon or lies entirely inside a polygon of σ;
- *facet-defining* if it bounds a black polygon of σ to its left;
- *internal* if it is not a boundary edge of $\mathbf{P}_n$.

Suppose an arc $i \to j$ is compatible with σ. The *area to the left of* $i \to j$, denoted by $\text{area}_\sigma(i \to j)$ or $\text{area}(i \to j)$, is the number of black triangles to the left of $i \to j$ in any triangulation of the black polygons of σ.

Example 2.10. For the subdivision σ pictured in Figure 2, $3 \to 7$, $6 \to 4$, $1 \to 2$ are compatible, whereas $1 \to 4$ is not; $6 \to 4$, $2 \to 7$, $4 \to 3$ are facet-defining arcs, whereas $1 \to 2$ is not facet-defining. Moreover, $\text{area}(3 \to 7) = 2$, $\text{area}(6 \to 4) = 5$, and $\text{area}(1 \to 2) = 0$.

Remark 2.11. Bicolored subdivisions of type (k, n) are in bijection with *separable permutations* on $[n-1]$ with k descents [PSBW23, Corollary 12.6], which are enumerated by a refinement of the *large Schröder numbers* [S$^+$, A175124].

The following characterization of tiles of $\Delta_{k+1,n}$ in terms of bicolored subdivisions of type (k, n) appears in [PSBW23] as Proposition 9.5 and 9.6; the characterization of tiles of $\Delta_{k+1,n}$ in terms of tree plabic graphs (which are dual to bicolored subdivisions) had previously appeared in [LPW23, Propositions 3.15, 3.16], confirming conjectures of [ŁPSV19].

Theorem 2.12. *The positroid tiles for $\Delta_{k+1,n}$ are in bijection with bicolored subdivisions of type (k, n). Moreover, if σ is such a bicolored subdivision, then the corresponding tile Γ_σ is cut out of $\mathbb{R}^n$ by the equality $x_{[n]} = k + 1$ and by the following inequalities:*

$$\text{area}(i \to j) \leq x_{[i,j-1]} \leq \text{area}(i \to j) + 1 \qquad \textit{for any compatible arc } i \to j \textit{ of } \sigma,$$

or, alternatively, by the following facet inequalities:

(1) $x_i \geq 0$, if there is a white polygon of σ with vertex i;
(2) $x_{[i,j-1]} \geq \text{area}(i \to j)$, if $i \to j$ is a facet-defining arc of σ.

Example 2.13. For σ in Figure 2, if $x \in \Gamma_\sigma$ then $2 \leq x_{[3,6]} \leq 3$ ($3 \to 7$ is a compatible arc). Some facet inequalities of Γ_σ are: $x_5 \geq 0$ (5 is a vertex of a white polygon of σ), $x_{[6,3]} \geq 5$ ($6 \to 4$ is a facet-defining arc).

2.5. **The amplituhedron.** Building on [AHBC$^+$16, Hod13], Arkani-Hamed and Trnka [AHT14] introduced the *(tree) amplituhedron*, which they defined as the image of the positive Grassmannian under a positive linear map. Let $\text{Mat}_{n,p}^{>0}$ denote the set of $n \times p$ matrices whose maximal minors are positive.

Definition 2.14 (Amplituhedron)**.** Let $Z \in \text{Mat}_{n,k+m}^{>0}$, where $k + m \leq n$. The *amplituhedron map* $\tilde{Z} : \text{Gr}_{k,n}^{\geq 0} \to \text{Gr}_{k,k+m}$ is defined by $\tilde{Z}(C) := CZ$, where C is a $k \times n$ matrix representing an element of $\text{Gr}_{k,n}^{\geq 0}$, and CZ is a $k \times (k + m)$ matrix representing an element of $\text{Gr}_{k,k+m}$. The *amplituhedron* $\mathcal{A}_{n,k,m}^Z \subset \text{Gr}_{k,k+m}$ is the image $\tilde{Z}(\text{Gr}_{k,n}^{\geq 0})$.

It is convenient to use the following coordinates when discussing points in $\mathcal{A}_{n,k,m}(Z)$.

Definition 2.15 (Twistor coordinates)**.** Fix $Z \in \text{Mat}_{n,k+m}^{>0}$ with rows $Z_1, \dots, Z_n \in \mathbb{R}^{k+m}$. Given $Y \in \text{Gr}_{k,k+m}$ with rows $y_1, \dots, y_k$, and $\{i_1, \dots, i_m\} \subset [n]$, we define the *twistor coordinate* $\langle\langle Y i_1 i_2 \cdots i_m \rangle\rangle$ to be the determinant of the matrix with rows $y_1, \dots, y_k, Z_{i_1}, \dots, Z_{i_m}$.

Note that the twistor coordinates are defined only up to a common scalar multiple.

Specializing Definition 2.16, with ϕ being the amplituhedron map $\tilde{Z}$ and X being the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ we have the following.

Definition 2.16. Let $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ be the amplituhedron map. The image $Z_S^\circ := \tilde{Z}(S)$ of a positroid cell S is an *open tile* for $\mathcal{A}_{n,k,m}(Z)$ if $\tilde{Z}$ is injective on S and $\dim S = mk$. The closure $Z_S := \overline{\tilde{Z}(S)}$ of an open tile is a *tile* for $\mathcal{A}_{n,k,m}(Z)$. A *positroid tiling* of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{Z_S\}_{S\in\mathcal{C}}$ of tiles whose union is $\mathcal{A}_{n,k,m}(Z)$ and such that the open tiles $\{Z_S^\circ\}_{S\in\mathcal{C}}$ are pairwise disjoint.

In this paper it is convenient to work with "all-Z" tilings, as defined below.

Definition 2.17 (All-Z tilings)**.** We call $\{Z_S\}_{S\in\mathcal{C}}$ an *all-Z tiling* of the amplituhedron $\mathcal{A}_{n,k,m}$ if $\{Z_S\}_{S\in\mathcal{C}}$ is a tiling of $\mathcal{A}_{n,k,m}(Z)$ for all $Z \in \mathrm{Mat}_{n,k+m}^{>0}$.

Remark 2.18. In this paper we will restrict our attention to all-Z tilings[7]. However, for ease of notation, we often drop the phrase "all-Z." When we refer to tilings $\{Z_S\}$ we will always mean all-Z tilings.

There is a natural notion of *facet* of a tile, generalizing the notion of facet of a polytope.

Definition 2.19 (Facet of a cell and a tile)**.** Given two positroid cells S' and S, we say that S' is a *facet* of S if $S' \subset \partial S$ and S' has codimension 1 in $\overline{S}$. If S' is a facet of S and Z_S is a tile of $\mathcal{A}_{n,k,m}(Z)$, we say that $Z_{S'}$ is a *facet* of Z_S if $Z_{S'} \subset \partial Z_S$ and has codimension 1 in Z_S.

In this paper, we will be concerned with the amplituhedron when $m = 2$. All subsequent sections are in this setting.

2.6. **Tiles of $\mathcal{A}_{n,k,2}$ from bicolored subdivisions.** In [PSBW23, Theorem 4.25], a subset of the authors of the present work showed that positroid tiles for $\mathcal{A}_{n,k,2}(Z)$ are in bijection with the bicolored subdivisions of type (k, n), by associating a plabic graph to each bicolored subdivision and showing that these plabic graphs exactly correspond to the $2k$-dimensional cells on which the amplituhedron map is injective. Moreover, [PSBW23, Theorem 4.28] gives a concise description of these positroid tiles in terms of inequalities. These two results are summarized in the following theorem.

Theorem 2.20. *The positroid tiles for $\mathcal{A}_{n,k,2}(Z)$ are in bijection with the bicolored subdivisions of type (k, n). Moreover, if σ is such a bicolored subdivision, then the corresponding open tile is*

$$Z_\sigma^\circ = \{Y \in \mathrm{Gr}_{k,k+2} \mid (-1)^{\mathrm{area}(i\to j)}\langle\langle Yij\rangle\rangle > 0 \textit{ for all compatible arcs } i \to j \textit{ of } \sigma \textit{ with } i < j\}.$$

Alternatively, let $\tilde{\sigma}$ be a triangulation of the black polygons of σ. Then

(1) Z_σ° is cut out by the inequalities $(-1)^{\mathrm{area}(i\to j)}\langle\langle Yij\rangle\rangle > 0$ for all arcs $i \to j$ in $\tilde{\sigma}$ with $i < j$.
(2) the facets of Z_σ lie on the hypersurfaces $\langle\langle Yij\rangle\rangle = 0$, where $i \to j$ a facet-defining arc of σ.

Example 2.21. For σ in Figure 2: if $Y \in Z_\sigma^\circ$ then $(-1)^2\langle\langle Y37\rangle\rangle > 0$ ($3 \to 7$ is a compatible arc). There is a facet of Z_σ lying on $\langle\langle Y64\rangle\rangle = 0$ ($6 \to 4$ is a facet-defining arc).

[7]Currently, there is no known example of tiling which is not an all-Z tiling.

2.7. **Correspondence between the hypersimplex and the amplituhedron.** From Theorems 2.12 and 2.20, we see that tiles of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ are both in bijection with bicolored subdivisions of type (k, n), and thus in bijection with each other. In fact, the bijection between tiles of the hypersimplex $\Delta_{k+1,n}$ and tiles of the amplituhedron $\mathcal{A}_{n,k,2}$ induces a bijection on tilings as well.

Theorem 2.22 ([PSBW23, Theorem 11.6]). *Let $\mathcal{S}$ be a collection of bicolored subdivisions of type (k, n). Then $\{\Gamma_\sigma\}_{\sigma\in\mathcal{S}}$ is a positroid tiling of $\Delta_{k+1,n}$ if and only if $\{Z_\sigma\}_{\sigma\in\mathcal{S}}$ is an all-Z tiling of $\mathcal{A}_{n,k,2}$.*

There is a particularly nice class of tilings which we can now describe in terms of bicolored subdivisions.

Definition 2.23 (Bicolored subdivisions of kermit type). Let $v \in [n]$, let $I = \{i_1, \dots, i_k\} \in \binom{[n]\setminus\{v\}}{k}$ and let σ_I be the bicolored subdivision of the n-gon obtained by drawing black triangles with vertices $\{v, i_\ell, i_\ell + 1\}$ for $\ell = 1, \dots, k$. (We then repeatedly merge black triangles sharing a common edge into a larger black polygon.) We say that σ_I^v is a *kermit bicolored subdivision based at vertex v*. We will often denote σ_I^1 by σ_I.

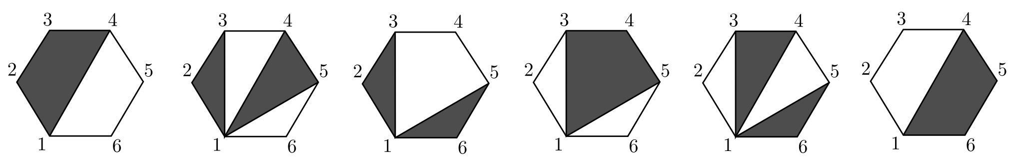

FIGURE 3. The bicolored subdivisions (based at vertex $v = 1$) giving rise to the kermit tilings of $\Delta_{3,6}$ and of $\mathcal{A}_{6,2,2}$.

Proposition 2.24. [PSBW23, Proposition 11.32] *Choose $0 \le k \le n-2$ and let I run over $\binom{[2,n-1]}{k}$. Then the collections $\{\Gamma_{\sigma_I}\}$ and $\{Z_{\sigma_I}\}$ are positroid tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$, respectively. In particular,*

$$\Delta_{k+1,n} = \bigcup_I \Gamma_{\sigma_I} \text{ and } \mathcal{A}_{n,k,2}(Z) = \bigcup_I Z_{\sigma_I}.$$

Moreover if we apply a cyclic shift to our definitions (i.e. rotate the labels of the n-gon), we obtain positroid tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$.

For any $v \in [n]$, we refer to the tiling $\{\Gamma_{\sigma_I^v}\}$ of $\Delta_{k+1,n}$, and the tiling $\{Z_{\sigma_I^v}\}$ of $\mathcal{A}_{n,k,2}$, as *kermit tilings (based at v)*, see Figure 3.

See [PSBW23, Table 1] for many other correspondences between the hypersimplex and the $m = 2$ amplituhedron.

2.8. **The triangulation of the hypersimplex into w-simplices.** In this section, we review a unimodular triangulation of the hypersimplex, originally due to Stanley [Sta77], and also studied by Sturmfels [Stu96] and Lam–Postnikov [LP07]. The maximal simplices of this triangulation are indexed by permutations with a fixed number of descents, which are counted by the Eulerian numbers. We follow the presentation of [LP07].

Definition 2.25. Let $w \in S_n$. A letter $i < n$ is a *left descent* (or a *left descent bottom*) of w if i occurs to the right of $i+1$ in w. In other words, $w^{-1}(i) > w^{-1}(i+1)$. And we say that $i \in [n]$ is a *cyclic left descent* of w if either $i < n$ is a left descent of w or if $i = n$ and 1 occurs to the left of n in w, that is, $w^{-1}(1) < w^{-1}(n)$. We let $\mathrm{cDes_L}(w)$ denote the set of cyclic left descents of w. We frequently refer to cyclic left descents as simply *cyclic descents*.

Remark 2.26. Left and right descents and descent sets are discussed extensively in [BB05, Chapter 1]. Left descents are sometimes called *recoils* in the literature.

Definition 2.27. Choose $0 \le k \le n-2$. We define $D_{k+1,n}$ to be the set of permutations $w \in S_n$ with $k+1$ cyclic descents and $w_n = n$. We let D_n be the set of permutations $w \in S_n$ with $w_n = n$.

Note that $|D_{k+1,n}|$ equals the *Eulerian number* $E_{k,n-1} := \sum_{\ell=0}^{k+1}(-1)^\ell\binom{n}{\ell}(k+1-\ell)^{n-1}$.

Example 2.28. The set $D_{2,4}$ contains the following $E_{2,3} = 4$ permutations.

w	1324	3124	2134	2314
$\mathrm{cDes_L}(w)$	$\{2,4\}$	$\{2,4\}$	$\{1,4\}$	$\{1,4\}$

It will often be convenient for us to identify elements of D_n with n-cycles in S_n. In Section 4, we will also identify elements of D_n and n-cycles with total cyclic orders.

Notation 2.29. For $w = w_1w_2\dots w_n$ in S_n, we write $(w) = (w_1\ w_2 \cdots w_n)$ for the cycle which sends $w_1 \mapsto w_2 \mapsto \cdots \mapsto w_n \mapsto w_1$. Note that $w \mapsto (w)$ gives a bijection between D_n and the set of n-cycles in S_n.

Definition 2.30 (w-simplices)**.** For $w = w_1w_2\dots w_n \in D_{k+1,n}$, let $w^{(a)}$ denote the cyclic rotation of w ending at a. We define

$$I_r = I_r(w) := \mathrm{cDes_L}(w^{(r)}).$$

Note that I_r in fact depends only on (w) rather than w itself. We define the *w-simplex* $\Delta_{(w)} \subseteq \Delta_{k+1,n}$ to be the convex hull of the points $e_{I_1}, \dots, e_{I_n}$; this is an $(n-1)$-dimensional simplex. We call

$$I_{w_1} \to I_{w_2} \to \cdots \to I_{w_n} \to I_{w_1}$$

the *circuit* of $\Delta_{(w)}$ and sometimes abuse notation by equating $\Delta_{(w)}$ and its circuit.

Example 2.31. From Example 2.28, there are four w-simplices in $\Delta_{2,4}$, see Figure 4. For $w = 1324$, to determine the vertices of $\Delta_{(w)}$, we compute the cyclic descents of the rotations of w.

r	1	2	3	4
$w^{(r)}$	3241	4132	2413	1324
$I_r = \mathrm{cDes_L}(w^{(r)})$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$	$\{2,4\}$

Therefore, Δ_{1324} is the convex hull of $e_{12}, e_{23}, e_{13}, e_{24}$, i.e. it is the red simplex in Figure 4.

Remark 2.32. [PSBW23, Section 10] uses left cyclic descent tops rather than left cyclic descent bottoms. It also uses the notation Δ_w for w-simplices. The bijection $w \mapsto \Delta_{(w)}$ from $D_{k+1,n}$ to w-simplices in Definition 2.30 is slightly different from the bijection $u \mapsto \Delta_u$ of [PSBW23, Definition 10.3]. In particular, let u be the permutation obtained from w by subtracting 1 from each number and rotating so n is at the end. Then $\Delta_{(w)} = \Delta_u$. The left cyclic descent tops of $u^{(r-1)}$ are the

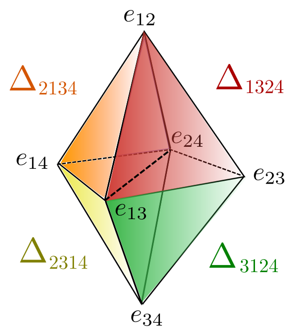

FIGURE 4. w-simplices for $\Delta_{2,4}$.

same as the left cyclic descent bottoms of $w^{(r)}$, so the sets $I_r(w)$ defined here are the same as the sets $I_r(u)$ in [PSBW23, Definition 10.3].

Remark 2.33. For a w-simplex $\Delta_{(w)}$, the consecutive elements $I_{w_{i-1}} \to I_{w_i}$ of the circuit are related by

$$I_{w_i} = I_{w_{i-1}} \setminus \{w_i - 1\} \cup \{w_i\}.$$

(In particular, r is always in I_r and $r-1$ is never in I_r.) From this, one can see that, letting u be as in Remark 2.32, $\Delta_{(w)}$ agrees with the simplex denoted $\Delta_{(u)}$ in [LP07, Section 2.4]. In particular, the circuit $I_{w_1} \to I_{w_2} \to \cdots \to I_{w_n} \to I_{w_1}$ is the circuit used to define $\Delta_{(u)}$.

The following triangulation of the hypersimplex first appeared in [Sta77], though the description there was slightly different.

Proposition 2.34 ($\Delta_{k+1,n}$ is the union of w-simplices [Sta77])**.** *The w-simplices $\{\Delta_{(w)} : w \in D_{k+1,n}\}$ are the maximal simplices of a triangulation of the hypersimplex $\Delta_{k+1,n}$. Moreover, projecting $\{\Delta_{(w)} : w \in S_n\}$ into $\mathbb{R}^{n-1}$ (see Remark 2.5), we obtain the maximal simplices in a triangulation of the hypercube $\square_{n-1}$ which refines the subdivision of the hypercube into hypersimplices.*

It follows from [LP07, Theorem 2.7] that the triangulation

$$\Delta_{k+1,n} = \bigcup_{w \in D_{k+1,n}} \Delta_{(w)}$$

is obtained by cutting the hypersimplex with all facet hyperplanes of all positroid tiles. Another way to say this is that the triangulation into w-simplices is the simultaneous refinement of all positroid tilings. From this, one can conclude the following.

Proposition 2.35 ([LP07])**.** *Every positroid tile[8] for $\Delta_{k+1,n}$ has a triangulation into w-simplices.*

We will later need to discuss facet-sharing of w-simplices. The characterization of facet-sharing was given in [LP07, Theorem 2.9]. The hyperplane on the shared facet lies can be deduced readily from [LP07] (we also prove it in Corollary 3.4).

[8]In fact, this statement hold for any full-dimensional positroid polytope.

Proposition 2.36. *Let $u, w \in D_{k+1,n}$. Then $\Delta_{(w)}$ and $\Delta_{(u)}$ share a facet if and only if (w) and (u) are related by swapping an adjacent pair of numbers, i.e. if $(w) = (A\ i\ j\ B)$ and $(u) = (A\ j\ i\ B)$, where $j \neq i \pm 1$. The shared facet, which we denote by $\Delta^{(ij)}_{(w)}$ or $\Delta^{(ji)}_{(u)}$, lies on a hyperplane $x_{[i,j-1]} = c$.*

Finally, the results from this section have analogues in the $m = 2$ amplituhedron.

Remark 2.37. Cutting $\Delta_{k+1,n}$ with the facet hyperplanes of all positroid tiles decomposes the hypersimplex into w-simplices $\Delta_{(w)}$. Analogously, cutting $\mathcal{A}_{n,k,2}(Z)$ with the facet hypersurfaces of all positroid tiles decomposes the amplituhedron into *w-chambers* $\Delta^Z_{(w)}$ (cf. Definition A.1), which are also indexed by $w \in D_{k+1,n}$ [PSBW23, Corollary 11.20]. Each chamber consists of points whose twistor coordinates have specified signs, which can be read off of w. Some w-chambers may be empty for a particular choice of Z, but all are nonempty for some $Z \in \mathrm{Mat}^{>0}_{n,k+2}$.

Remark 2.38. The bijection between tiles of the hypersimplex and the amplituhedron can be further refined using w-simplices and w-chambers. Each tile Z_σ of the amplituhedron is a union of w-chambers [PSBW23, Corollary 10.17]. If $\Delta^Z_{(w)}$ is nonempty, then $\Delta^Z_{(w)} \subset Z_\sigma$ if and only if $\Delta_{(w)} \subset \Gamma_\sigma$ [PSBW23, Proposition 11.1].

3. Combinatorial results on tiles

In this section, we characterize which w-simplices are contained in a tile Γ_σ.

Theorem 3.1. *Let σ be a bicolored subdivision of type (k, n), and Γ_σ the corresponding tile of $\Delta_{k+1,n}$. Then*

$$\Gamma_\sigma = \bigcup \Delta_{(w)}$$

where the union is over $w \in D_{k+1,n}$ which satisfy

- *for every white polygon of σ with vertices $v_1, \ldots, v_r$ in **clockwise** order, we see $v_1, \ldots, v_r$ in order in (w).*
- *for every black polygon of σ with vertices $v_1, \ldots, v_r$ in **counterclockwise** order, we see $v_1, \ldots, v_r$ in order in (w).*

Example 3.2. For the bicolored subdivision σ in Figure 5, Γ_σ is the union of all w-simplices $\Delta_{(w)}$ such that $w \in D_{3,6}$ satisfies:

- we see $1, 2, 3$ and $1, 4, 6$ in order in (w) (from the white polygons);
- we see $1, 4, 3$ and $4, 6, 5$ in order in (w) (from the black polygons).

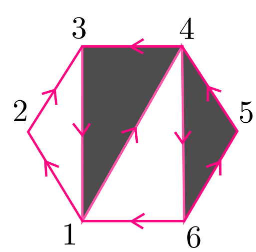

FIGURE 5. A bicolored subdivision σ, together with the orientation of the boundary of the white (resp. black) polygons clockwise (resp. counterclockwise).

These permutations w are:

512436	351246	531246	125436	312546	152436	315246	154236
315426	231546	514236	351426	531426	235146	253146	523146

For example, 512436 satisfies the conditions above: $5\boxed{12}4\boxed{3}6$, $5\boxed{1}2\boxed{4}3\boxed{6}$, $5\boxed{1}2\boxed{43}6$, $\boxed{5}12\boxed{4}3\boxed{6}$.

Before proving Theorem 3.1, we need a straightforward lemma.

Lemma 3.3. *Consider $w \in D_n$ with $(w) = (\cdots q\ a \cdots r\ b \cdots)$ and corresponding w-simplex*

$$\Delta_{(w)} = \cdots I_q \to I_a \to \cdots \to I_r \to I_b \to \cdots .$$

If j appears weakly between a and r in (w), then

$$|I_a \cap [a, b-1]| = |I_j \cap [a, b-1]|.$$

If j appears weakly between b and q in (w), then

$$|I_a \cap [a, b-1]| - 1 = |I_b \cap [a, b-1]| = |I_j \cap [a, b-1]|.$$

Proof. First, suppose j appears weakly between a and r. If $j = a$, the lemma clearly holds. If $j \neq a$, either $j-1, j$ are both in $[a, b-1]$ or neither $j-1, j$ is in $[a, b-1]$. This is because $j \neq b$ by assumption. Now, the number x preceding j in (w) is also weakly between a and r. By definition, $I_j = I_x \setminus \{j-1\} \cup \{j\}$, so we have

$$|I_x \cap [a, b-1]| = |I_j \cap [a, b-1]|.$$

This shows that $|I_j \cap [a, b-1]|$ is constant as j ranges over all numbers appearing weakly between a and r in (w).

We now turn to the second case. An identical argument shows that $|I_j \cap [a, b-1]|$ is constant as j ranges over numbers weakly between b and q in (w). Since $I_b = I_r \setminus \{b-1\} \cup \{b\}$, we conclude that

$$|I_b \cap [a, b-1]| = |I_r \cap [a, b-1]| - 1 = |I_a \cap [a, b-1]| - 1.$$

This shows the desired equalities. □

Lemma 3.3 implies that $\Delta_{(w)}$ satisfies a number of inequalities.

Corollary 3.4. *Consider a w-simplex $\Delta_{(w)} \subset \Delta_{k+1,n}$ and let I_a be as in Definition 2.30. Then*

(1) For all $a, b \in [n]$, all points in $\Delta_{(w)}$ satisfy

$$|I_b \cap [a, b-1]| \leq x_{[a,b-1]} \leq |I_a \cap [a, b-1]|.$$

(2) A minimal inequality description of $\Delta_{(w)}$ as a subset of $\mathbb{R}^n$ is given by the equation $x_{[n]} = k+1$ together with the n facet inequalities:

$$|I_b \cap [a, b-1]| \leq x_{[a,b-1]}$$

where a immediately precedes b in (w).

Proof. For (1): Say $\Delta_{(w)} = \cdots \to I_q \to I_a \to \cdots \to I_r \to I_b \to \cdots$. Lemma 3.3 shows that all vertices e_{I_j} of $\Delta_{(w)}$ lie on one of the hyperplanes

$$x_{[a,b-1]} = |I_a \cap [a, b-1]| \quad \text{or} \quad x_{[a,b-1]} = |I_a \cap [a, b-1]| - 1 = |I_b \cap [a, b-1]|.$$

Thus, all points in $\Delta_{(w)}$ satisfy

$$|I_b \cap [a, b-1]| \leq x_{[a,b-1]} \leq |I_a \cap [a, b-1]|.$$

For (2): The w-simplex $\Delta_{(w)}$ is codimension 1 in $\mathbb{R}^n$, and its affine span is the hyperplane $x_{[n]} = k+1$.

If a immediately precedes b in w, then by Lemma 3.3, $n-1$ vertices of $\Delta_{(w)}$ lie on the hyperplane

$$x_{[a,b-1]} = |I_b \cap [a, b-1]|.$$

These $n-1$ vertices form a facet. This gives rise to n facet hyperplanes of $\Delta_{(w)}$, which is the complete list of facet hyperplanes as $\Delta_{(w)}$ is a simplex. □

Example 3.5. The w-simplex Δ_{1324} from Example 2.31 is the subset of $\mathbb{R}^4$ given by the equation $x_{[4]} = 3$ together with the 4 facet inequalities

$$1 \leq x_{[1,2]}, \quad 1 \leq x_{[3,1]}, \quad 1 \leq x_{[2,3]} \quad \text{and} \quad 0 \leq x_4.$$

Proof of Theorem 3.1. Because Γ_σ is triangulated by w-simplices (cf. Proposition 2.35), $\Delta_{(w)}$ is either contained in Γ_σ or it does not intersect the interior of Γ_σ. So it suffices to determine which $\Delta_{(w)}$ intersect the interior of Γ_σ.

By Theorem 2.12, a point in $\mathbb{R}^n$ is in the interior of Γ_σ if and only if $x_{[n]} = k+1$ and for each arc $a \to b$ compatible with σ, the point satisfies

$$\text{area}(a \to b) < x_{[a,b-1]} < \text{area}(a \to b) + 1.$$

Points in $\Delta_{(w)}$ satisfy the equality $x_{[n]} = k+1$ for all $w \in D_{k+1,n}$. By Corollary 3.4, the above inequalities hold for a point in $\Delta_{(w)}$ if and only if

$$\text{area}(a \to b) = |I_b \cap [a, b-1]| \quad \text{for all arcs } a \to b \text{ compatible with } \sigma. \tag{2}$$

We now show that (2) holds if and only if w satisfies the conditions of the theorem.

($\Longrightarrow$): Suppose that w satisfies the conditions of the theorem. Choose a compatible arc $a \to b$. We proceed by induction on the size of the polygon to the left of the arc $a \to b$. The base case is $b = a+1$, and in this case,

$$\text{area}(a \to a+1) = 0 = |I_b \cap [a, a]|.$$

The last equality holds because I_b is obtained from some other subset I_x by removing $b-1 = a$ and adding b, and so does not contain a.

Now, suppose $\text{area}(i \to j) = |I_j \cap [i, j-1]|$ for all arcs which are compatible with σ and have an $(r-1)$-gon to the left. Suppose the arc $a \to b$ has an r-gon to its left. Choose some vertex c of this r-gon so that $a \to c$ and $c \to b$ are also compatible with σ. Suppose the triangle with vertices a, b, c is white; the argument when the triangle is black is very similar. Then

$$\text{area}(a \to b) = \text{area}(a \to c) + \text{area}(c \to b) = |I_c \cap [a, c-1]| + |I_b \cap [c, b-1]|$$

where the second equality is by the inductive hypothesis. Note that we see a, c, b clockwise around the boundary of some white polygon in σ. So by assumption, we have

$$(w) = (\cdots a \cdots c \cdots b \cdots).$$

Lemma 3.3 implies that $|I_b \cap [a, c-1]| = |I_c \cap [a, c-1]|$, so we conclude that

$$\text{area}(a \to b) = |I_c \cap [a, c-1]| + |I_b \cap [c, b-1]| = |I_b \cap [a, b-1]|$$

as desired.
($\Longleftarrow$): We show the contrapositive. That is, we show that if w does not satisfy the conditions of the theorem, then (2) does not hold.

Suppose the conditions of the theorem fail for some white polygon of σ (the argument for a black polygon is similar). Then there are vertices a, c, b which appear in clockwise order around this white polygon, but we see $(w) = (\cdots a \cdots b \cdots c \cdots)$.

We have $\text{area}(a \to b) = \text{area}(a \to c) + \text{area}(c \to b)$ because the triangle on a, b, c is white. If $\text{area}(i \to j) = |I_j \cap [i, j-1]|$ for all three arcs $a \to b, a \to c$ and $c \to b$, then we would have

$$|I_b \cap [a, b-1]| = |I_c \cap [a, c-1]| + |I_b \cap [c, b-1]|.$$

But by Lemma 3.3, $|I_c \cap [a, c-1]| = |I_b \cap [a, c-1]| - 1$ and thus

$$|I_b \cap [a, b-1]| = |I_c \cap [a, c-1]| + |I_b \cap [c, b-1]| = |I_b \cap [a, b-1]| - 1.$$

This is impossible, so we must have $\text{area}(i \to j) \neq |I_j \cap [i, j-1]|$ for one of $a \to b, a \to c$ or $c \to b$. Since all three arcs are compatible with σ, this means (2) does not hold. $\square$

In light of Remark 2.38, Theorem 3.1 has the following corollary for the amplituhedron.

Corollary 3.6. *Let σ be a (k, n)-bicolored subdivision and Z_σ the corresponding tile of $\mathcal{A}_{n,k,2}(Z)$. Then*

$$Z_\sigma = \bigcup \Delta^Z_{(w)}$$

where the union is over $w \in D_{k+1,n}$ which satisfy

- *for every white polygon of σ with vertices $v_1, \ldots, v_r$ in* ***clockwise*** *order, we see $v_1, \ldots, v_r$ in order in (w).*
- *for every black polygon of σ with vertices $v_1, \ldots, v_r$ in* ***counterclockwise*** *order, we see $v_1, \ldots, v_r$ in order in (w).*

4. Cyclic orders and their circular extensions

We now interpret some of the previous results in terms of *(partial) cyclic orders* and *circular extensions.* Cyclic orders and circular extensions are circular analogues of partial orders and linear extensions.

Definition 4.1. A *(partial) cyclic order* on a finite set X is a ternary relation $C \subset X^3$ such that for all $a, b, c, d \in X$:

(cyclicity) $$(a, b, c) \in C \implies (c, a, b) \in C$$

(asymmetry) $$(a, b, c) \in C \implies (c, b, a) \notin C$$

(transitivity) $$(a, b, c) \in C \text{ and } (a, c, d) \in C \implies (a, b, d) \in C$$

A cyclic order C is *total* if for all $a, b, c \in X$, either $(a, b, c) \in C$ or $(a, c, b) \in C$.

Informally, a total cyclic order C on $[n]$ is a way of placing $1, \ldots, n$ on a circle, just as a total order is a way of placing $1, \ldots, n$ on a line.

Definition 4.2. Let $w = w_1 \dots w_n \in S_n$. The w-order C_w is the total cyclic order obtained by placing $w_1, w_2, \dots, w_n$ on the circle clockwise. We identify this total cyclic order with the n-cycle (w) and so may write (w) for C_w or write $C_w = (w_1\ w_2 \dots w_n)$.

Note that each total cyclic order on $[n]$ is of the form C_w for a unique permutation $w \in D_n$ (cf. Definition 2.27 for the definition of D_n). We move interchangeably between $w \in D_n$, the n-cycle (w) and the total cyclic order C_w.

Definition 4.3. A total cyclic order C is a *circular extension* of a cyclic order C' if $C' \subset C$. We let $\mathrm{Ext}(C)$ denote the set of all circular extensions of a cyclic order C. In an abuse of notation, if C is a partial cyclic order on $[n]$, we sometimes write $(w) \in \mathrm{Ext}(C)$ if $C_w \in \mathrm{Ext}(C)$.

Not all cyclic orders have a circular extension [Meg76], that is, $\mathrm{Ext}(C)$ could be empty. Moreover, the problem of determining whether a cyclic order has a circular extension is NP-complete [GM78].

Definition 4.4. Let $x_1, \dots, x_m$ be a sequence of m distinct elements of $[n]$ (for $3 \le m \le n$). We let $C = C_{(x_1,x_2\dots,x_m)}$ denote the partial cyclic order on $[n]$ in which for each triple $1 \le i < j < \ell \le m$ we have $(x_i, x_j, x_\ell) \in C$ (which implies by cyclicity that also (x_j, x_ℓ, x_i) and (x_ℓ, x_i, x_j) lie in C). We call this partial cyclic order a *chain*.

We now generalize the notion of bicolored subdivision to *tricolored* (or "partially bicolored") subdivision, which will be useful in Section 7. We will then associate a partial cyclic order to every tricolored subdivision.

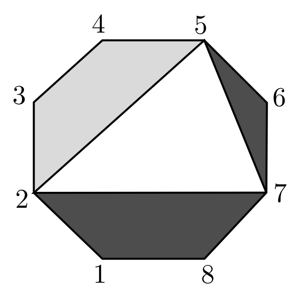

FIGURE 6. A tricolored subdivision σ of type $(3, 2, 8)$. It gives rise to the cyclic order C_σ which is the union of the chains $C_{(2,5,7)}$, $C_{(5,7,6)}$, and $C_{(1,8,7,2)}$.

Definition 4.5 (Tricolored subdivisions)**.** Let $\mathbf{P}_n$ be a convex n-gon with vertices labeled from 1 to n in clockwise order. A *tricolored subdivision* τ is a partition of $\mathbf{P}_n$ into black, white, and grey polygons such that two polygons sharing an edge have different colors. We say that τ has *type* (k, ℓ, n) if any triangulation of the black (respectively, grey) polygons consists of exactly k black (respectively, ℓ grey) triangles. See Figure 6.

Definition 4.6 (τ-order)**.** Let τ be a tricolored subdivision of $\mathbf{P}_n$ with q polygons $P_1, \dots, P_q$ which are black or white (we ignore the grey polygons). If P_a is white (respectively, black), we let $v_1, \dots, v_r$ denote its list of vertices read in clockwise (respectively, counterclockwise) order. We then associate the chain $C_a = C_{(v_1,\dots,v_r)}$ to P_a. Finally we define the *τ-order* to be the partial cyclic order which is the union of the partial cyclic orders associated to the black and white polygons:

$$C_\tau := C_1 \cup \dots \cup C_q.$$

We leave it as an exercise to verify that C_τ is a partial cyclic order. See Figure 6 for an example.

To rephrase Theorem 3.1 and Corollary 3.6 in terms of cyclic orders, we need one straightforward lemma. Recall the definition of $D_{k+1,n}$ and D_n from Definition 2.27.

Lemma 4.7. *Let σ be a bicolored subdivision of type (k,n). Suppose $w \in D_n$ and C_w is a circular extension of C_σ. Then $w \in D_{k+1,n}$.*

Proof. We proceed by induction on n. The base case is $n = 3$. For this n, k is either 0 or 1. In either case, C_σ is a total cyclic order, and so $C_w = C_\sigma$. One can check that $w \in D_{k+1,3}$.

Now, suppose $n > 3$. Without loss of generality, we may assume a polygon P of σ contains the edges of $\mathbf{P}_n$ between $n-1$ and n and between n and 1. Let σ' be the subdivision of $\mathbf{P}_{n-1}$ obtained by removing the triangle on vertices $n-1, n, 1$ from σ, and let $w' \in S_{n-1}$ be the permutation obtained from $w^{(n-1)}$ by removing n (recall that $w^{(n-1)}$ denotes the rotation of w ending at $n-1$). Notice that $C_{w'}$ is a circular extension of $C_{\sigma'}$.

If the polygon P is black, then by the inductive hypothesis, w' has k cyclic left descents. Because P is black and C_w is a circular extension of C_σ, we must have $C_w = (1 \cdots n \cdots n-1 \cdots)$. So to obtain $w^{(n-1)}$ from w', we put in n somewhere to the right of 1 and to the left of $n-1$. This means n is a cyclic descent of $w^{(n-1)}$. So $w^{(n-1)}$ has $k+1$ cyclic descents, and so does w.

The argument is very similar if P is white. Then w' has $k+1$ cyclic descents, and to obtain $w^{(n-1)}$ from w', we put n somewhere (cyclically) to the right of $n-1$ and to the left of 1. This means $w^{(n-1)}$ and w both have $k+1$ cyclic left descents. □

Theorem 3.1, Corollary 3.6 and Lemma 4.7 imply the following. Recall from Definition 4.3 that we identify total cyclic orders C_w with n-cycles (w).

Corollary 4.8. *Let σ be a bicolored subdivision of type (k,n). Then*

$$\Gamma_\sigma = \bigcup_{(w)\in \mathrm{Ext}(C_\sigma)} \Delta_{(w)} \qquad \text{and} \qquad Z_\sigma = \bigcup_{(w)\in \mathrm{Ext}(C_\sigma)} \Delta^Z_{(w)}.$$

That is, Γ_σ (resp. Z_σ) is the union of w-simplices $\Delta_{(w)}$ (resp. w-chambers $\Delta^Z_{(w)}$) where the w-order C_w is a circular extension of the σ-order C_σ.

Proof. Theorem 3.1 and Corollary 3.6 imply that

$$\Gamma_\sigma = \bigcup \Delta_{(w)} \qquad \text{and} \qquad Z_\sigma = \bigcup \Delta^Z_{(w)}$$

where the union is over $\{w \in D_{k+1,n} \colon C_w \text{ a circular extension of } C_\sigma\}$. Lemma 4.7 implies that this set is equal to $\{v \in D_n \colon C_w \text{ a circular extension of } C_\sigma\}$, which we identify with the set of (n-cycles which are) circular extensions of C_σ. □

Because the w-simplices are unimodular, Corollary 4.8 gives rise to an expression for the normalized volume of each hypersimplex tile.

Corollary 4.9. *Let σ be a bicolored subdivision. The normalized volume of Γ_σ is the number of circular extensions of C_σ. That is, $\mathrm{Vol}(\Gamma_\sigma) = |\,\mathrm{Ext}(C_\sigma)|$.*

We can also interpret Proposition 2.24 in terms of cyclic orders, which will be useful in Section 7.

Corollary 4.10. *Let $w \in D_{k+1,n}$. Then C_w is a cyclic extension of exactly one of the "kermit" cyclic orders C_{σ_I}, where I runs over $\binom{[2,n-1]}{k}$. The same statement holds if we replace σ_I by σ_I^v.*

Proof. We know from Proposition 2.34 and Proposition 2.24 that

$$\Delta_{k+1,n} = \bigcup_{w \in D_{k+1,n}} \Delta_{(w)} = \bigcup_{I \in \binom{[2,n-1]}{k}} \Gamma_{\sigma_I}$$

where in both unions, the pieces have pairwise disjoint interiors. Now the result follows from Corollary 4.8. □

Now if we use Corollary 4.10 and let k range over the interval $0 \le k \le n-2$, we obtain the following.

Corollary 4.11. *Let $w \in D_n$, and choose $v \in [n]$. Then C_w is a circular extension of exactly one of the "kermit" cyclic orders $C_{\sigma_I^v}$, where I runs over all subsets of $[n] \setminus \{v\}$.*

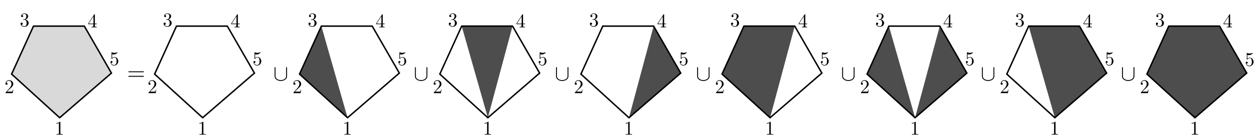

FIGURE 7. The all-grey tricolored subdivision of the n-gon corresponds to the union of all kermit subdivisions for fixed n and $0 \le k \le n-2$. Here $v = 1$. We will see in Example 7.4 and Proposition 7.6 that this equation can be interpreted as a tiling of the unit hypercube $\square_4 \subset \mathbb{R}^4$.

We represent Corollary 4.11 by the diagram in Figure 7. The left-hand side, which is the all-grey tricolored subdivision of the n-gon, represents the trivial cyclic order $C_\emptyset$ containing no triples. Each of the eight bicolored subdivisions at the right is a kermit subdivision σ_I and represents the partial cyclic order C_{σ_I}. Corollary 4.11 then says that each circular extension of $C_\emptyset$ is a circular extension of precisely one of the partial cyclic orders C_{σ_I} from the right-hand side.

Remark 4.12. There is a natural notion of *sub-cyclic order* of a cyclic order. If C is a cyclic order on a set X, and $Y \subset X$ is a subset of Y, then C restricts to a *sub-cyclic order* $C|_Y$ on Y^3: namely,

$$C|_Y = \{(a, b, c) \mid a, b, c \in Y \text{ and } (a, b, c) \in C\}.$$

Note that Corollary 4.11 also applies to sub-cyclic orders of a cyclic order.

5. MAGIC NUMBER THEOREM FOR THE $m = 2$ AMPLITUHEDRON

In this section, we show that every tiling of the hypersimplex $\Delta_{k+1,n}$ and every all-Z tiling of $\mathcal{A}_{n,k,2}$ consists of $\binom{n-2}{k}$ tiles.

Recall the definition of the Parke-Taylor function from Definition 1.1. Let $\mathbf{I}_n = 1 \dots n$ denote the identity permutation on $[n]$.

Definition 5.1. Let $\Delta_{(w)}$ be a w-simplex for $\Delta_{k+1,n}$. The *weight function* of $\Delta_{(w)}$ is

$$\Omega(\Delta_{(w)}) := \mathrm{PT}(w)$$

viewed as a rational function on $\widehat{\mathrm{Gr}}_{2,n}$, the affine cone over the Grassmannian $\mathrm{Gr}_{2,n}$.

We note that the weight function $\Omega(\Delta_{(w)})$ is well-defined on $\widehat{\mathrm{Gr}}^{\circ}_{2,n}$, the open locus in $\widehat{\mathrm{Gr}}_{2,n}$ where all Plücker coordinates are nonvanishing. We also remark that $\mathrm{PT}(w)$ depends only on the n-cycle (w); that is, if $(u) = (w)$, then $\mathrm{PT}(w) = \mathrm{PT}(u)$.

Remark 5.2. A point in $\widehat{\mathrm{Gr}}^{\circ}_{2,n}/T$ where $T = (\mathbb{R}^*)^n$ can be represented with a matrix C whose top row is $(1,\dots,1)$ and the bottom row is $(z_1,\dots,z_n)$. Then the Plücker coordinates P_{ij} of C are simply $z_j - z_i$ and Definition 1.1 reads:

$$\mathrm{PT}(w) = \frac{1}{(z_{w_2} - z_{w_1})(z_{w_3} - z_{w_2})\dots(z_{w_n} - z_{w_{n-1}})(z_{w_1} - z_{w_n})}. \tag{3}$$

Some of our proofs about Parke-Taylor functions will use the formulation of (3).

Definition 5.3. Let $\Gamma_S \subset \Delta_{k+1,n}$ be a full-dimensional positroid polytope. We define the *weight function* of Γ_S to be the sum of the weight functions of w-simplices included in Γ_S:

$$\Omega(\Gamma_S) := \sum_{\Delta_{(w)} \subset \Gamma_S} \Omega(\Delta_{(w)})$$

The first step in our argument is to show that all tiles have the same weight function. We contrast this with the normalized volume of tiles, which is far from constant (see Figure 8 for an example).

Proposition 5.4. *Let Γ_σ be a tile of $\Delta_{k+1,n}$. Then the weight function of Γ_σ is*

$$\Omega(\Gamma_\sigma) = (-1)^k\,\mathrm{PT}(\mathbf{I}_n). \tag{4}$$

In particular, all tiles of $\Delta_{k+1,n}$ have the same weight function.

We need some preliminaries before proceeding to the proof of Proposition 5.4.

Definition 5.5. For a function $F \in \mathbb{C}(\widehat{\mathrm{Gr}}_{2,n})$ all of whose poles are simple, we define its *residue* at $P_{ij} = 0$ by

$$\mathrm{Res}_{P_{ij}=0}\, F := \lim_{P_{ij}\to 0} P_{ij} F.$$

This definition resembles, of course, the usual definition of a residue, only that we define it for functions rather than forms.

Definition 5.6. Let C be a cyclic order on $[n]$, and let $i, j \in [n]$. We let

$$\mathrm{Ext}_{(ij)}(C) := \{C' \in \mathrm{Ext}(C) : C' = (\cdots i\ j \cdots)\}$$

be the set of circular extensions of C in which i immediately precedes j.

The next lemma constrains the poles of $\Omega(\Gamma_\sigma)$.

Lemma 5.7. *Let σ be a bicolored subdivision of $\mathbf{P}_n$. Then*

$$\mathrm{Res}_{P_{ij}=0}\,\Omega(\Gamma_\sigma) = 0,$$

for all i, j such that (i, j) is not an edge of a white or black polygon of σ.

Proof. First, suppose (i,j) is a diagonal (but not an edge) of a (black or white) polygon in σ. Consider $\Delta_{(w)} \subset \Gamma_\sigma$. By Theorem 3.1, i,j are not adjacent in C_w, hence $P_{ij}=0$ is not a pole of $\Omega(\Delta_{(w)})$. It follows that also $\Omega(\Gamma_\sigma)$ does not have a pole at $P_{ij}=0$.

Now suppose i,j are not vertices of the same polygon in σ. By Corollary 4.8, the w-simplices $\Delta_{(w)} \subset \Gamma_\sigma$ such that $\mathrm{PT}(w)$ has poles at $P_{ij}=0$ are exactly those for which $(w) \in \mathrm{Ext}_{(ij)}(C_\sigma) \cup \mathrm{Ext}_{(ji)}(C_\sigma)$. We define the map

$$\begin{aligned} \phi\colon\ \mathrm{Ext}_{(ij)}(C_\sigma) &\to \mathrm{Ext}_{(ji)}(C_\sigma)\\ C_w = (\cdots i\ j \cdots) &\mapsto (\cdots j\ i \cdots) =: C_{\phi(w)} \end{aligned}$$

which swaps the order of i and j. The map ϕ is well-defined and a bijection: w satisfies Theorem 3.1 if and only if $\phi(w)$ does, since we are not changing the order of numbers which are vertices of the same polygon. (The geometric interpretation of ϕ—see Corollary 4.8 and Proposition 2.36—is that $\Delta_{(w)}$ and $\Delta_{(\phi(w))}$ are w-simplices in Γ_σ which intersect in a common facet on a hyperplane $x_{[i,j-1]}=d$.) We claim that for all $(w) \in \mathrm{Ext}_{(ij)}(C_\sigma)$

$$\mathrm{Res}_{P_{ij}=0}\left(\Omega(\Delta_{(w)}) + \Omega(\Delta_{(\phi(w))})\right) = 0. \tag{5}$$

It then will follow immediately from (5) that $\Omega(\Gamma_\sigma)$ does not have a pole at $P_{ij}=0$.

To show (5), suppose $C_w = (\cdots h\ i\ j\ l\ \cdots)$ and $C_{\phi(w)} = (\cdots h\ j\ i\ l\ \cdots)$. Let Q be the product of common factors in $\Omega(\Delta_{(w)})$ and $\Omega(\Delta_{(\phi(w))})$, which are of the type $1/P_{rs}$, with $i,j \notin \{r,s\}$. Then

$$\begin{aligned} \Omega(\Delta_{(w)}) + \Omega(\Delta_{(\phi(w))}) &= Q\left(\frac{1}{P_{h,i}P_{i,j}P_{j,l}} + \frac{1}{P_{h,j}P_{j,i}P_{i,l}}\right) = \frac{Q}{P_{i,j}}\frac{P_{h,j}P_{i,l}-P_{h,i}P_{j,l}}{P_{h,i}P_{j,l}P_{h,j}P_{i,l}} = \\ &= \frac{Q}{P_{i,j}}\frac{P_{h,l}P_{i,j}}{P_{h,i}P_{j,l}P_{h,j}P_{i,l}} = \frac{QP_{h,l}}{P_{h,i}P_{j,l}P_{h,j}P_{i,l}}, \end{aligned}$$

where in the third equality we used the Plücker relation $P_{h,j}P_{i,l} - P_{h,i}P_{j,l} + P_{h,l}P_{i,j} = 0$. Therefore $\Omega(\Delta_{(w)}) + \Omega(\Delta_{(\phi(w))})$ does not have a pole at $P_{ij}=0$, which shows (5). □

The next proof will use the formulation from (3), so we introduce the following notation.

Notation 5.8. We define the notation $\omega(\Delta_{(w)}) := \Omega(\Delta_{(w)})|_{P_{ij}\mapsto(z_j-z_i)}$, $\omega(\Gamma_\sigma) := \Omega(\Gamma_\sigma)|_{P_{ij}\mapsto(z_j-z_i)}$, and $\mathrm{pt}(w) := \mathrm{PT}(w)|_{P_{ij}\mapsto(z_j-z_i)}$.

Proof of Proposition 5.4. Consider the function

$$F = \Omega(\Gamma_\sigma) - (-1)^k\,\mathrm{PT}(\mathbf{I}_n),$$

which is a rational function on $\widehat{\mathrm{Gr}}_{2,n}$. We would like to show that F is the zero function. We first perform a series of reductions.

First, it suffices to show that $F(V)=0$ for all $V \in \widehat{\mathrm{Gr}}^{\circ}_{2,n}$, the dense subset of $\widehat{\mathrm{Gr}}_{2,n}$ where all Plücker coordinates are nonvanishing. Note that F is well-defined on this subset.

Choose $V \in \widehat{\mathrm{Gr}}^{\circ}_{2,n}$ and let A be a matrix representative. Recall that the torus $(\mathbb{R}^*)^n$ acts on $\widehat{\mathrm{Gr}}_{2,n}$ by rescaling columns. Scaling a column of A by $t \in \mathbb{R}^*$ will multiply $F(A)$ by t^{-2}. This gives the second reduction: to show $F(V)$ is zero, it suffices to show that F is zero on a single point in the torus orbit of V.

The torus orbit of V contains a point of the form

$$B = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ z_1 & z_2 & \cdots & z_{n-1} & z_n \end{bmatrix}$$

where $z_1, \dots, z_n \in \mathbb{R}$. Note that the Plücker coordinate $P_{ij}(B)$ is equal to $(z_j - z_i)$. This is the third reduction: to show that $F(B)$ is zero, it suffices to show that

$$f(z_1, \dots, z_n) := F|_{P_{ij} \mapsto (z_j - z_i)}$$

is zero as a function on $\mathbb{R}^n$.

To show that $f(z_1, \dots, z_n) = 0$, we proceed by induction on n. The base case $n = 3$ is simple and can be checked by hand. Now suppose we have proven $f = 0$ for all $n' < n$. In what follows, we will deduce $f = 0$ for n by carefully analyzing the poles of f.

Because PT functions are homogeneous of degree $-n$ in Plücker coordinates, f is either identically zero or of degree $-n$ in the z_i. Because PT functions have (only) simple poles of the form $P_{ij} = 0$, any poles of f are simple and lie at $z_j - z_i = 0$ for some i, j. Lemma 5.7 implies that all poles of $\Omega(\Gamma_\sigma)$ are of the form $P_{ij} = 0$ where (i, j) is an edge of some polygon in σ. The poles of $\mathrm{PT}(\mathbf{I}_n)$ are at $P_{i-1,i} = 0$. So the only possible poles of f are $z_j - z_i = 0$ for (i, j) an edge of a polygon in σ (this includes the case $z_i - z_{i-1} = 0$).

We claim that f does not have poles of the form $z_i - z_{i-1} = 0$. That is, we have the following.

Claim 5.9. Let $i \in [n]$. Then, using Notation 5.8, we have

$$\lim_{z_i \to z_{i-1}} (z_i - z_{i-1})\ \omega(\Gamma_\sigma) = \lim_{z_i \to z_{i-1}} (z_i - z_{i-1})(-1)^k\ \mathrm{pt}(\mathbf{I}_n)$$

and thus $\lim_{z_i \to z_{i-1}} (z_i - z_{i-1}) f = 0$.

Before proving Claim 5.9, we explain why this claim implies that f is zero. Indeed, we can write $f = \frac{P}{Q}$, where P is a homogenous polynomial in $z_1, \dots, z_n$ of degree $h \geq 0$, and Q is a product of factors of the form $(z_j - z_i)$, where each factor appears at most once. Suppose for the sake of contradiction that f is not zero, and thus is of degree $-n$. Then the degree of Q is $h + n \geq n$. By Claim 5.9, the factors of Q are $(z_j - z_i)$ where (i, j) is an edge of a polygon in σ but is not an edge of $\mathbf{P}_n$. There are at most $n - 3$ such arcs, so $\deg(Q) \leq n - 3$. But this contradicts the fact that $\deg(Q) \geq n$.

We now prove Claim 5.9, which will conclude the proof. We would like to compute the sum over $\Delta_{(w)} \subset \Gamma_\sigma$ of terms

$$\lim_{z_i \to z_{i-1}} (z_i - z_{i-1}) \omega(\Delta_{(w)}).$$

Fix $\Delta_{(w)} \subset \Gamma_\sigma$. If $i - 1, i$ are not adjacent in the cyclic order C_w, then

$$\lim_{z_i \to z_{i-1}} (z_i - z_{i-1}) \omega(\Delta_{(w)}) = 0.$$

If $i-1, i$ are adjacent in C_w, their order is dictated by the color of the polygon p in σ containing the edge $(i-1, i)$. If p is white, then $C_w = (\cdots i-1\ i \cdots) \in \mathrm{Ext}_{(i-1,i)}(C_\sigma)$; if p is black, then $C_w = (\cdots i\ i-1 \cdots) \in \mathrm{Ext}_{(i,i-1)}(C_\sigma)$. Let $c = 0$ if p is white and let $c = 1$ if p is black.

Consider now the limit $z_i \to z_{i-1}$. Write $w = w_1 \dots w_n$ with the i and $i-1$ in positions $\{j, j+1\}$. It is immediate that

(6)

$$\lim_{z_i \to z_{i-1}} (z_i - z_{i-1})\omega(\Delta_{(w)}) = \lim_{z_{w_j} \to z_{w_{j+1}}} \frac{(-1)^c}{(z_{w_2} - z_{w_1})(z_{w_3} - z_{w_2}) \dots (z_{w_j} - z_{w_{j-1}})(z_{w_{j+2}} - z_{w_{j+1}}) \dots (z_{w_1} - z_{w_n})}$$
$$= \frac{(-1)^c}{(z_{w_2} - z_{w_1})(z_{w_3} - z_{w_2}) \dots (z_{w_{j+1}} - z_{w_{j-1}})(z_{w_{j+2}} - z_{w_{j+1}}) \dots (z_{w_1} - z_{w_n})}.$$

Let σ' be the bicolored subdivision of the $(n-1)$-gon with vertices $N' = \{1, 2, \dots, i-2, \star, i+1, \dots, n\}$ obtained from σ by contracting the boundary edge $(i-1, i)$ to the vertex $\star$. Let ρ be the map

$$\rho : \{C_w \in C_\sigma : i-1, i \text{ adjacent in } C_w\} \to \mathrm{Ext}(C_{\sigma'})$$

where $\rho(C_w) := C_{w'}$ is the cyclic order on N' obtained from C_w by identifying $i-1$ and i with $\star$. It is easy to see that ρ is a bijection. Then (6) can be rewritten as

$$\lim_{z_i \to z_{i-1}} (z_i - z_{i-1})\omega(\Delta_{(w)}) = (-1)^c \omega(\Delta_{(w')}). \tag{7}$$

By the induction hypothesis

$$\omega(\Gamma_{\sigma'}) = \sum_{(u) \in \mathrm{Ext}(C_{\sigma'})} \omega(\Delta_{(u)}) = (-1)^{k-c}\, \mathrm{pt}(\mathbf{I}_{N'}). \tag{8}$$

Putting everything together, we have

$$\begin{aligned}
\lim_{z_i \to z_{i-1}} (z_i - z_{i-1})\omega(\Gamma_\sigma) &= \sum_{\substack{(w) \in \mathrm{Ext}(C_\sigma): \\ i-1, i \text{ adj in } C_w}} \lim_{z_i \to z_{i-1}} (z_i - z_{i-1})\omega(\Delta_{(w)}) \\
&= \sum_{\substack{(w) \in \mathrm{Ext}(C_\sigma): \\ i-1, i \text{ adj in } C_w}} (-1)^c \omega(\Delta_{(w')}) \\
&= (-1)^c \sum_{(u) \in \mathrm{Ext}(C_{\sigma'})} \omega(\Delta_{(u)}) \\
&= (-1)^k\, \mathrm{pt}(\mathbf{I}_{N'}).
\end{aligned}$$

In the second equality we use (7), in the third the fact that ρ is a bijection, and in the fourth we use (8). By inspection, the last line is equal to

$$\lim_{z_i \to z_{i-1}} (z_i - z_{i-1})(-1)^k\, \mathrm{pt}(\mathbf{I}_n),$$

which proves Claim 5.9.

□

Theorem 5.10. *Every positroid tiling of $\Delta_{k+1,n}$ consists of $\binom{n-2}{k}$ tiles.*

Proof. Let $\{\Gamma_\sigma\}_{\sigma \in \mathcal{S}}$ be a positroid tiling for $\Delta_{k+1,n}$. Then we have

$$\Omega(\Delta_{k+1,n}) = \sum_{w \in D_{k+1,n}} \Omega(\Delta_{(w)}) = \sum_{\sigma \in \mathcal{S}} \sum_{\Delta_{(w)} \subset \Gamma_\sigma} \Omega(\Delta_{(w)}) = \sum_{\sigma \in \mathcal{S}} \Omega(\Gamma_\sigma) = |\mathcal{C}|(-1)^k\, \mathrm{PT}(\mathbf{I}_n),$$

where for the second and third equality we used that $\Gamma_\sigma = \cup_{\Delta_{(w)} \subset \Gamma_\sigma} \Delta_{(w)}$ and that the tiles $\{\Gamma_\sigma\}_{\sigma \in \mathcal{S}}$ have disjoint interiors and cover $\Delta_{k+1,n}$. For the last equality we used Proposition 5.4. Since $\Omega(\Delta_{k+1,n})$ and $\mathrm{PT}(\mathbf{I}_n)$ do not depend on $\mathcal{S}$, then $|\mathcal{S}|$ will only depend on k and n. Considering e.g. the kermit tiling in Proposition 2.24 whose size is $\binom{n-2}{k}$, we conclude that $|\mathcal{S}| = \binom{n-2}{k}$. □

Using Theorem 2.22, we have the following corollary of Theorem 5.10, which proves the Magic Number Conjecture for the $m = 2$ amplituhedron.

Corollary 5.11. *Every all-Z positroid tiling of $\mathcal{A}_{n,k,2}$ consists of $M_{n,k,2} = \binom{n-2}{k}$ tiles.*

Example 5.12. Consider the collection $\mathcal{S}$ of bicolored subdivisions in Figure 8. Then $\{\Gamma_\sigma\}_{\sigma \in \mathcal{S}}$ is a tiling for $\Delta_{4,7}$. This can be verified by checking that each w-simplex in $\Delta_{4,7}$ lies in a unique tile Γ_σ with $\sigma \in \mathcal{S}$. By T-duality (cf. Theorem 2.22), $\{Z_\sigma\}_{\sigma \in \mathcal{S}}$ is an all-Z tiling of $\mathcal{A}_{7,3,2}$. The tiling contains $10 = \binom{7-2}{3} = M_{7,3,2}$ tiles (cf. Corollary 5.11).

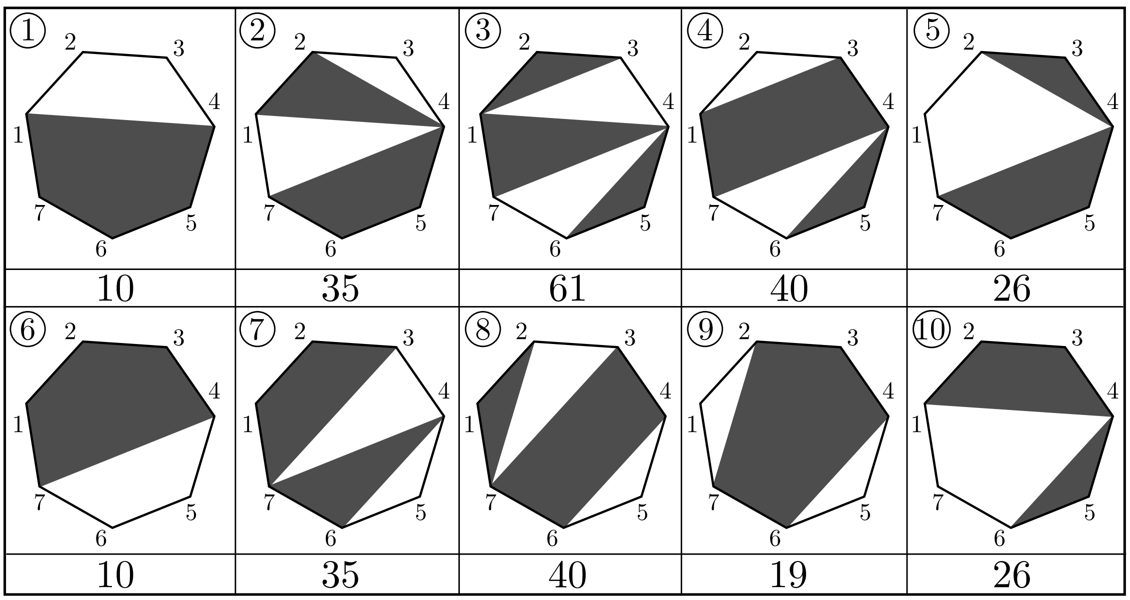

Figure 8. A collection of $10 = \binom{7-2}{3} = M_{7,3,2}$ bicolored subdivisions of type $(3, 7)$ (labelled from ① to ⑩) which gives a tiling for $\Delta_{4,7}$ and $\mathcal{A}_{7,3,2}$. The number in the box below each bicolored subdivision is the volume of the corresponding positroid polytope in $\Delta_{4,7}$. The sum of their volumes is the Eulerian number $E_{3,6} = 302$.

Corollary 5.13. *The weight function of $\Delta_{k+1,n}$ is given by*

$$\Omega(\Delta_{k+1,n}) = (-1)^k M_{n,k,2}\, \mathrm{PT}(\mathbf{I}_n) = (-1)^k \binom{n-2}{k} \mathrm{PT}(\mathbf{I}_n).$$

6. Combinatorial results on tilings

In the last section, we showed that any tiling of $\Delta_{k+1,n}$ or $\mathcal{A}_{n,k,2}$ consists of $\binom{n-2}{k}$ tiles. In this section, we prove another enumerative necessary condition for a collection of tiles to comprise a tiling. The strategy is again to use Parke-Taylor functions, but now for facets of tiles.

We start by defining a weight function for common facets of w-simplices. Recall the description of these facets from Proposition 2.36.

Notation 6.1. Choose $w \in S_n$ and let i and j be letters located in adjacent positions in w. Let $N_L := [i+1, j-1] \cup \{\star\}$ and $N_R := [j+1, i-1] \cup \{\star\}$ be subsets of $[n]/\sim$, where $\sim$ is the equivalence relation on $[n]$ obtained by identifying i and j, and $\star := i \sim j$. We define a total cyclic order C_L on the set N_L by taking the subword of w consisting of letters in $[i, j]$, and then identifying $i \sim j \sim \star$. Similarly, we define a total cyclic order C_R on N_R by taking the subword of w consisting of letters in $[j, i]$, and then identifying $i \sim j \sim \star$. We also define (w_L) to be the $|N_L|$-cycle on N_L such that $C_L = (w_L)$. We define (w_R) analogously.

Example 6.2. Let $n = 6$, $i = 1$, and $j = 4$. Let $w = 514236$. We have $N_L = \{2, 3, \star\}$, $N_R = \{5, 6, \star\}$ and $C_L = (\star\ 2\ 3) = (w_L)$ and $C_R = (5\ \star\ 6) = (w_R)$.

In the next definition, we use the notation $\widehat{\mathrm{Gr}}_{2,N_L}$ and $\widehat{\mathrm{Gr}}_{2,N_R}$ to denote Grassmannians whose matrix representatives have columns labeled by N_L, respectively N_R, rather than $[n]$.

Definition 6.3. Let $\Delta^{(ij)}_{(w)} = \Delta^{(ji)}_{(u)}$ be a common facet of $\Delta_{(w)}$ and $\Delta_{(u)}$. Then we define the *weight function* of $\Delta^{(ij)}_{(w)}$ to be the rational function on $\widehat{\mathrm{Gr}}_{2,N_L} \times \widehat{\mathrm{Gr}}_{2,N_R}$ given by

$$\Omega(\Delta^{(ij)}_{(w)}) := \mathrm{PT}(w_L)\,\mathrm{PT}(w_R) = \mathrm{PT}(u_L)\,\mathrm{PT}(u_R),$$

where we used Notation 6.1.

Because $\mathrm{PT}(u)$ depends only on the cycle (u), $\mathrm{PT}(w_L)$ and $\mathrm{PT}(w_R)$ are well-defined.

Example 6.4. We continue Example 6.2. Consider the facet $\Delta^{(14)}_{(w)}$ common to $\Delta_{(w)}$ and $\Delta_{(u)}$, where $w = 514236$ and $u = 541236$. We have $\Delta_{(w)} \subset \Gamma_\sigma$ where σ is as in Example 3.2. Since $(w_L) = (\star\ 2\ 3)$ and $(w_R) = (5\ \star\ 6)$,

$$\Omega(\Delta^{(14)}_{(w)}) = \mathrm{PT}(w_L)\,\mathrm{PT}(w_R) = \frac{1}{P_{\star 2}P_{23}P_{3\star}}\frac{1}{P_{5\star}P_{\star 6}P_{65}}.$$

Recall from Theorem 2.12 that the tile Γ_σ has a facet on the hyperplane $x_{[i,j-1]} = \mathrm{area}(i \to j)$ for each facet-defining arc $i \to j$. Denote this facet by $\Gamma^{(ij)}_\sigma$. Before defining the weight function for $\Gamma^{(ij)}_\sigma$, we characterize which w-simplices have a facet contained in $\Gamma^{(ij)}_\sigma$. Recall the notation $\mathrm{Ext}_{(ij)}(C_\sigma)$ from Definition 5.6.

Lemma 6.5. *Let σ be a bicolored subdivision and $i \to j$ an internal facet-defining arc. Then $\Delta_{(w)} \subset \Gamma_\sigma$ has a facet contained in $\Gamma^{(ij)}_\sigma$ if and only if $C_w \in \mathrm{Ext}_{(ij)}(C_\sigma)$.*

Proof. Suppose $\Delta_{(w)} \subset \Gamma_\sigma$ has a facet on the hyperplane $x_{[i,j-1]} = \mathrm{area}(i \to j)$. This implies by Corollary 3.4 that i, j are adjacent in C_w, so we have either $C_w = (\cdots i\ j \cdots)$ or $C_w = (\cdots j\ i \cdots)$. In the former case, Corollary 3.4 implies $\Delta_{(w)}$ has a facet inequality $x_{[i,j-1]} \geq \mathrm{area}(i \to j)$; in the latter, $\Delta_{(w)}$ has a facet inequality of the form $x_{[i,j-1]} \leq \mathrm{area}(i \to j)$. Since $\Delta_{(w)}$ satisfies the facet inequalities of Γ_σ, we have $x_{[i,j-1]} \geq \mathrm{area}(i \to j)$ and $C_w = (\cdots i\ j \cdots)$ as desired.

Suppose $\Delta_{(w)} \subset \Gamma_\sigma$ and $C_w = (\cdots i\ j \cdots)$. Then $\Delta_{(w)}$ satisfies the inequalities

$$\mathrm{area}(i \to j) \leq x_{[i,j-1]} \leq \mathrm{area}(i \to j) + 1$$

which hold for all points in Γ_σ, and also has a facet on a hyperplane $x_{[i,j-1]} = c$ by Corollary 3.4. Since c is an integer, it is either equal to $\mathrm{area}(i \to j)$ or $\mathrm{area}(i \to j) + 1$. The hyperplane $x_{[i,j-1]} = \mathrm{area}(i \to j) + 1$ intersects the tile Γ_σ in codimension at least 2, as it is a bounding hyperplane but not a facet hyperplane. Therefore, $c = \mathrm{area}(i \to j)$ and $\Delta_{(w)}$ has a facet on the desired hyperplane. □

Definition 6.6. Let σ be a bicolored subdivision and $i \to j$ an internal facet-defining arc. Let $\Gamma_\sigma^{(ij)}$ denote the facet of Γ_σ on the hyperplane $x_{[i,j-1]} = \mathrm{area}(i \to j)$. Then we define the weight function of the facet $\Gamma_\sigma^{(ij)}$ as the rational function on $\widehat{\mathrm{Gr}}_{2,N_L} \times \widehat{\mathrm{Gr}}_{2,N_R}$ given by the sum

$$\Omega(\Gamma_\sigma^{(ij)}) := \sum_{\Delta_{(w)}^{(ij)} \subset \Gamma_\sigma^{(ij)}} \Omega(\Delta_{(w)}^{(ij)}) = \sum_{(w) \in \mathrm{Ext}_{(ij)}(C_\sigma)} \Omega(\Delta_{(w)}^{(ij)})$$

where the equality follows from Lemma 6.5.

We now show the facet analogue of Proposition 5.4.

Notation 6.7. Using Notation 6.1, we define $\mathrm{sh}_n^{(ij)} := \binom{n-2}{|N_L|-1} = \binom{n-2}{|N_R|-1}$. We also denote by $\mathbf{I}_{N_L}, \mathbf{I}_{N_R}$ the identity permutations on N_L, N_R respectively.

Proposition 6.8. *Let σ be a bicolored subdivision and $i \to j$ an internal facet-defining arc. Let Γ_σ be the corresponding tile for $\Delta_{k+1,n}$. Then*

$$\Omega(\Gamma_\sigma^{(ij)}) = (-1)^{k-1} \mathrm{sh}_n^{(ij)} \, \mathrm{PT}(\mathbf{I}_{N_L}) \, \mathrm{PT}(\mathbf{I}_{N_R}).$$

In particular, the weight function $\Omega(\Gamma_\sigma^{(ij)})$ does not depend on σ.

Proof. By the definition of $\Omega(\Gamma_\sigma^{(ij)})$ and $\Omega(\Delta_{(w)}^{(ij)})$, we have

$$\Omega(\Gamma_\sigma^{(ij)}) = \sum_{(w) \in \mathrm{Ext}_{(ij)}(C_\sigma)} \mathrm{PT}(w_L) \, \mathrm{PT}(w_R).$$

As the first step in our argument, we would like to rewrite this as a product of weight functions for tiles with smaller k and n.

Let σ_L (resp. σ_R) be the bicolored subdivision on a polygon with vertices N_L (resp. N_R) obtained from σ by taking all polygons to the left (resp. to the right) of the arc $i \to j$ and then contracting the edge (i, j) to a vertex $\star$, see Figure 9. We write k_L (resp. k_R) for the area of σ_L (resp. σ_R).

We will show that

$$\Omega(\Gamma_\sigma^{(ij)}) = \mathrm{sh}_n^{(ij)} \, \Omega(\Gamma_{\sigma_L}) \, \Omega(\Gamma_{\sigma_R}). \tag{9}$$

The terms of the left hand side are labeled by $(w) \in \mathrm{Ext}_{(ij)}(C_\sigma)$, while terms on the right hand side are labeled by elements of $\mathrm{Ext}(C_{\sigma_L}) \times \mathrm{Ext}(C_{\sigma_R})$. We define a map

$$\psi : \mathrm{Ext}_{(ij)}(C_\sigma) \to \mathrm{Ext}(C_{\sigma_L}) \times \mathrm{Ext}(C_{\sigma_R})$$

which sends C_w to (C_{w_L}, C_{w_R}). This map is well-defined and surjective by Theorem 3.1. Each pair (C_u, C_v) in the image has exactly $\mathrm{sh}_n^{(ij)} = \binom{n-2}{|N_L|-1}$ pre-images. Indeed, if $C_u = (u_1 \cdots u_p \ \star)$ and $C_v = (v_1 \cdots v_q \ \star)$, then any shuffle S of the tuples $(u_1, \ldots, u_p)$ and $(v_1, \ldots, v_q)$ will give rise to a circular order $C_w = (S \ i \ j)$ which is a pre-image of (C_u, C_v). Moreover, all preimages of (C_u, C_v) arise in this way.

Using this, we rewrite

$$\Omega(\Gamma_\sigma^{(ij)}) = \sum_{(w)\in \mathrm{Ext}_{(ij)}(C_\sigma)} \mathrm{PT}(w_L)\,\mathrm{PT}(w_R) = \mathrm{sh}_n^{(ij)} \sum_{(C_u,C_v)} \mathrm{PT}(u)\,\mathrm{PT}(v)$$

where the second sum is over $\mathrm{Ext}(C_{\sigma_L}) \times \mathrm{Ext}(C_{\sigma_R})$. The second sum can also be written as

$$\mathrm{sh}_n^{(ij)} \left(\sum_{(u)\in \mathrm{Ext}(C_{\sigma_L})} \mathrm{PT}(u) \right) \left(\sum_{(v)\in \mathrm{Ext}(C_{\sigma_R})} \mathrm{PT}(v) \right) = \mathrm{sh}_n^{(ij)} \Omega(\Gamma_{\sigma_L}) \Omega(\Gamma_{\sigma_R})$$

where the equality uses Corollary 4.8. This shows that (9) holds.

Now we apply Proposition 5.4 to (9) to obtain

$$\Omega(\Gamma_\sigma^{(ij)}) = \mathrm{sh}_n^{(ij)}\,(-1)^{k_L+k_R}\,\mathrm{PT}(\mathbf{I}_{N_L})\,\mathrm{PT}(\mathbf{I}_{N_R}).$$

Now, $k_L + k_R = k - 1$ because contracting $i \to j$ removes the triangles on either side of $i \to j$ in σ, and one of these triangles is black and the other is white. This completes the proof of the proposition. □

Remark 6.9. Equation (9) exposes the 'factorization' properties of the Parke-Taylor functions in connection with the geometry of tiles, see also Figure 9.

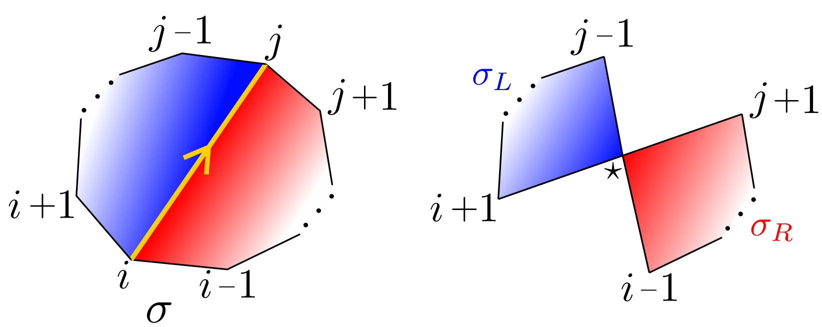

FIGURE 9. Left: a bicolored subdivision σ with the facet-defining arc $i \to j$ highlighted. Right: the bicolored subdivisions σ_L on $N_L = [i+1, j-1] \cup \{\star\}$ and σ_R on $N_R = [j+1, i-1] \cup \{\star\}$ obtained from σ by contracting the arc $i \to j$ and identifying i and j with $\star$. The blue and red colors highlight the respective regions before and after the contraction.

Example 6.10. Let σ be the bicolored subdivision from Example 3.2. Recall that there are 16 w-simplices contained in Γ_σ, i.e. $|\,\mathrm{Ext}(C_\sigma)| = 16$. The w-simplices that have $C_w = (\cdots 1\ 4 \cdots)$ are the following:

$\mathrm{Ext}_{(14)}(C_\sigma)$						
w	514236	351426	531426	235146	253146	523146

We have $N_L = \{2, 3, \star\}, N_R = \{5, 6, \star\}$. By Lemma 6.5, to compute $\Omega(\Gamma_\sigma^{(14)})$, we sum over the weight functions of the w-simplices with $(w) \in \mathrm{Ext}_{(14)}(C_\sigma)$. Performing the sum gives:

$$\Omega(\Gamma_\sigma^{(14)}) = \sum_{(w)\in \mathrm{Ext}_{(14)}(C_\sigma)} \Omega(\Delta_{(w)}^{(14)}) = 6 \frac{1}{P_{23}P_{3\star}P_{\star 2}} \frac{1}{P_{56}P_{6\star}P_{\star 5}} = \binom{4}{2} \mathrm{PT}(2, 3, \star)\,\mathrm{PT}(5, 6, \star),$$

which agrees with Proposition 6.8, where $\binom{n-2}{|N_L|-1} = \binom{4}{2} = 6$ is the number of shuffles between the sets $\{2,3\}, \{5,6\}$. Moreover, $\mathrm{PT}(2,3,\star) = \Omega(\Gamma_{\sigma_L})$ and $\mathrm{PT}(5,6,\star) = \Omega(\Gamma_{\sigma_R})$, where σ_L, σ_R are as in Figure 10.

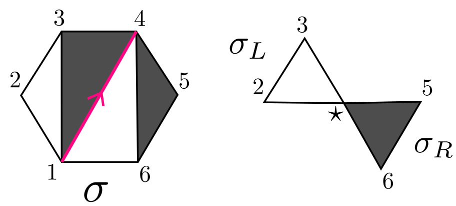

FIGURE 10. Left: a bicolored subdivision σ with the facet-defining arc $1 \to 4$ highlighted. Right: the bicolored subdivisions σ_L, σ_R obtained from σ by contracting the arc $1 \to 4$ and identifying 1 and 4 with $\star$.

Using Proposition 6.8 and a similar strategy to the proof of Theorem 5.10, we can prove the following necessary condition for a collection of tiles to give a tiling.

Theorem 6.11. *Let $\mathcal{T} = \{\Gamma_\sigma\}_{\sigma \in \mathcal{S}}$ be a tiling for $\Delta_{k+1,n}$ and $c \in [1,k]$. Let $i \to j$ be an internal arc of $\mathbf{P}_n$. Then the number of tiles in $\mathcal{T}$ that have $x_{[i,j-1]} \geq c$ as a facet inequality equals the number of tiles that have $x_{[i,j-1]} \leq c$ as a facet inequality. Equivalently, we have*

$$\#\{\sigma \in \mathcal{S} : i \to j \textit{ facet-defining, } \operatorname{area}(i \to j) = c\} = \#\{\sigma \in \mathcal{S} : j \to i \textit{ facet-defining, } \operatorname{area}(i \to j) = c - 1\}.$$

Proof. Let H denote the hyperplane $\{x_{[i,j-1]} = c\}$ and let $\mathcal{T}_A$, resp. $\mathcal{T}_B$, be the set of tiles in $\mathcal{T}$ with $x_{[i,j-1]} \geq c$, resp. $x_{[i,j-1]} \leq c$, as facet inequality (tiles in $\mathcal{T}_A$ are above H, while those in $\mathcal{T}_B$ are below). By Theorem 2.12, a tile $\Gamma_\sigma \in \mathcal{T}$ is in $\mathcal{T}_A$ precisely when $i \to j$ is a facet-defining arc and $c = \operatorname{area}_\sigma(i \to j)$; using Theorem 2.12 and the fact that $x_{[n]} = k+1$ for all points in the hypersimplex, a tile Γ_σ is in $\mathcal{T}_B$ precisely when $j \to i$ is a facet-defining arc and $c = \operatorname{area}_\sigma(i \to j)+1$. This shows that the two statements in the theorem are equivalent.

We would like to show that $|\mathcal{T}_A| = |\mathcal{T}_B|$. Note that if $\Gamma_\sigma \in \mathcal{T}_A$, then the facet $\Gamma_\sigma^{(ij)}$ lies on H; if instead $\Gamma_{\sigma'} \in \mathcal{T}_B$, then the facet $\Gamma_{\sigma'}^{(ji)}$ lies on H.

Consider a facet $\Delta_{(w)}^{(ij)}$ of a w-simplex, and suppose this facet is contained in the facet $\Gamma_\sigma^{(ij)}$ of a tile $\Gamma_\sigma \in \mathcal{T}_A$. This means $\Delta_{(w)}^{(ij)}$ is also contained in H. Because i precedes j in w, Corollary 3.4 implies $\Delta_{(w)}$ in fact lies above H. The w-simplex $\Delta_{(u)}$ which shares the facet $\Delta_{(u)}^{(ji)} = \Delta_{(w)}^{(ij)}$ with $\Delta_{(w)}$ must lie below H and must be contained in a tile $\Gamma_{\sigma'}$ in the tiling. In particular, $\Gamma_{\sigma'}$ must be in $\mathcal{T}_B$: $\Gamma_{\sigma'}$ cannot intersect $\Delta_{(w)}$ in its interior, so since $\Gamma_{\sigma'}$ contains $\Delta_{(u)}$, $\Gamma_{\sigma'}$ must have a facet on H and also be below H. The argument exchanging A and B is identical. In summary, $\Delta_{(w)}^{(ij)} = \Delta_{(u)}^{(ji)}$ is contained in $\Gamma_\sigma^{(ij)}$ for a tile $\Gamma_\sigma \in \mathcal{T}_A$ if and only if it is also contained in $\Gamma_{\sigma'}^{(ji)}$ for a tile $\Gamma_{\sigma'} \in \mathcal{T}_B$

This implies

$$\sum_{\substack{\Delta_{(w)}^{(ij)} \subset \Gamma_\sigma^{(ij)} \\ \text{and } \Gamma_\sigma \in \mathcal{T}_A}} \Omega(\Delta_{(w)}^{(ij)}) = \sum_{\substack{\Delta_{(u)}^{(ji)} \subset \Gamma_{\sigma'}^{(ji)} \\ \text{and } \Gamma_{\sigma'} \in \mathcal{T}_B}} \Omega(\Delta_{(u)}^{(ji)}).$$

The left hand side can be expressed as

$$\sum_{\Gamma_\sigma \in \mathcal{T}_A} \sum_{\Delta^{(ij)}_{(w)} \subset \Gamma^{(ij)}_\sigma} \Omega(\Delta^{(ij)}_{(w)}) = \sum_{\Gamma_\sigma \in \mathcal{T}_A} \Omega(\Gamma^{(ij)}_\sigma) = |\mathcal{T}_B| \cdot (-1)^{k-1} \mathrm{sh}^{(ij)}_n \, \mathrm{PT}(\mathbf{I}_{N_L}) \, \mathrm{PT}(\mathbf{I}_{N_R})$$

where the final equality uses Proposition 6.8. Analogously, the right hand side is

$$\sum_{\Gamma_{\sigma'} \in \mathcal{T}_B} \Omega(\Gamma^{(ji)}_{\sigma'}) = |\mathcal{T}_A| \cdot (-1)^{k-1} \mathrm{sh}^{(ij)}_n \, \mathrm{PT}(\mathbf{I}_{N_L}) \, \mathrm{PT}(\mathbf{I}_{N_R})$$

again using Proposition 6.8. (For the tiles in $\mathcal{T}_B$, $j \to i$ is a facet-defining arc rather than $i \to j$, but the expression in Proposition 6.8 is symmetric with respect to i and j, so we obtain the same expression.) This implies $|\mathcal{T}_A| = |\mathcal{T}_B|$. □

Example 6.12. Consider the collection $\mathcal{S}$ of bicolored subdivisions in Figure 8. There are two bicolored subdivisions, ② and ⑤ , that have $4 \to 7$ as facet-defining arc and area$(4 \to 7) = 2$. Correspondingly, ③ and ④ have $7 \to 4$ as facet-defining arc and area$(4 \to 7) = 2 - 1 = 1$. There is one bicolored subdivision, ⑦ , that has $4 \to 7$ as facet-defining arc and area$(4 \to 7) = 1$. Correspondingly, there is one bicolored subdivision, ⑥ , that has $7 \to 4$ as facet-defining arc and area$(4 \to 7) = 1 - 1 = 0$. This verifies $\mathcal{S}$ satisfies Theorem 6.11 for the arc $4 \to 7$.

Corollary 6.13. *Suppose that a collection $\mathcal{S}$ of bicolored subdivisions indexes a positroid tiling of $\Delta_{k+1,n}$ (or an all-Z tiling of $\mathcal{A}_{n,k,2}$). Then for every diagonal (i, j) of the n-gon,*

$$\#\{\sigma \in \mathcal{S} : i \to j \text{ is a facet-defining arc}\} = \#\{\sigma \in \mathcal{S} : j \to i \text{ is a facet-defining arc}\}.$$

Proof. Fix a diagonal (i, j) of the n-gon. Let $\mathcal{S}^{(r)}_A$ (resp. $\mathcal{S}^{(r)}_B$) be the subdivisions in $\mathcal{S}$ where $i \to j$ (resp. $j \to i$) is a facet-defining arc and area$(i \to j) = r$ (resp. area$(i \to j) = r$). By Theorem 6.11, we have $|\mathcal{S}^{(r)}_A| = |\mathcal{S}^{(r-1)}_B|$. The number of subdivisions in $\mathcal{S}$ with $i \to j$ as a facet-defining arc is

$$\sum_{r=1}^{k} |\mathcal{S}^{(r)}_A| = \sum_{r=0}^{k-1} |\mathcal{S}^{(r)}_B|.$$

The right hand side is the number of $\sigma \in \mathcal{S}$ with $j \to i$ as a facet-defining arc.

Example 6.14. Consider the collection $\mathcal{S}$ of bicolored subdivisions in Figure 8. There are 3 bicolored subdivisions (②, ⑤, ⑦) that have $4 \to 7$ as facet-defining arc. Correspondingly, there are 3 bicolored subdivisions (③, ④, ⑥) that have $7 \to 4$ as facet-defining arc. Hence, $\mathcal{S}$ satisfies Corollary 6.13.

□

Using Corollary 6.13, we obtain another combinatorial necessary condition for a collection of bicolored subdivisions to index a tiling.

Proposition 6.15. *Suppose that a collection $\mathcal{S}$ of bicolored subdivisions indexes a positroid tiling of $\Delta_{k+1,n}$ (or an all-Z tiling of $\mathcal{A}_{n,k,2}$). Superimposing all bicolored subdivisions in $\mathcal{S}$, we obtain a $\binom{n-3}{k-1}$-fold covering of the n-gon by black polygons.*

Example 6.16. Consider the collection $\mathcal{S}$ of bicolored subdivisions in Figure 8. By drawing all the facet-defining arcs of the bicolored subdivisions in $\mathcal{S}$, we subdivide $\mathbf{P}_7$ in 13 regions, see Figure 11.

One can check that each region is covered by a black polygon in $\binom{7-3}{3-1} = \binom{4}{2} = 6$ bicolored subdivisions in $\mathcal{S}$ (cf. Proposition 6.15). For example, the red pentagonal region is covered by a black polygon in the bicolored subdivisions ②, ④, ⑥, ⑦, ⑨, ⑩. The blue triangular region is covered by a black polygon in the bicolored subdivisions ①, ②, ⑤, ⑦, ⑧, ⑨.

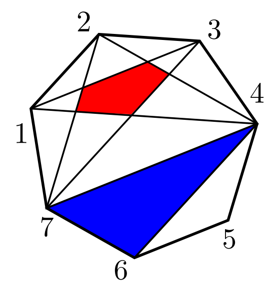

FIGURE 11. Regions in $\mathbf{P}_7$ obtained by refining all the bicolored subdivisions in Figure 8.

Proof of Proposition 6.15. Triangulate the black polygons in each subdivision in $\mathcal{S}$. Let $\mathcal{B}$ be the multi-set of black triangles appearing in the subdivisions in $\mathcal{S}$. Since each subdivision in $\mathcal{S}$ has area k and by Theorem 5.10 we have $|\mathcal{S}| = \binom{n-2}{k}$, $\mathcal{B}$ consists of $k\binom{n-2}{k}$ triangles.

Similarly to bicolored subdivisions, we call an arc $i \to j$ a facet-defining arc of a triangle t in $\mathcal{B}$ if (i, j) is an edge of t and the other vertex of t is in the cyclic interval $[i + 1, j - 1]$. The multiset $\mathcal{B}$ has the property that for all diagonals (i, j) of the n-gon,

$$\#\{t \in \mathcal{B} : i \to j \text{ a facet-defining arc}\} = \#\{t \in \mathcal{B} : j \to i \text{ a facet-defining arc}\}. \tag{10}$$

Indeed, if t is in the left hand multiset, then either $i \to j$ is a facet-defining arc of the black polygon p that t lies inside or $i \to j$ is an internal arc of p. In the former case, Corollary 6.13 guarantees another black polygon p' with $j \to i$ a facet-defining arc and the triangle t' in p' which has $j \to i$ as an edge is a triangle in the right hand set; in the latter, the neighboring triangle t' to t in p has $j \to i$ as a facet defining arc and is a triangle in the right hand multiset.

Now, choose some $t \in \mathcal{B}$. For each internal facet-defining arc $i \to j$ of t, (10) implies there is another triangle $t_{ij} \in \mathcal{B}$ for which $j \to i$ is a facet defining arc–meaning that t_{ij} lies on the opposite side of the diagonal (i, j). The triangles t_{ij} have disjoint interiors. For each t_{ij}, consider each "exposed" edge; that is, each internal facet-defining arc which is not an edge of t. Again, (10) implies that we can find a triangle in $\mathcal{B}$ which covers the other side of this edge. Again, all of the triangles we choose to cover the exposed edges of the t_{ij} have disjoint interiors. We continue this process, successively covering the "exposed" facet-defining arcs of the triangles we have obtained, until we have a collection of triangles in $\mathcal{B}$ which give a triangulation of the n-gon. Let $\mathcal{B}'$ be the multiset obtained from $\mathcal{B}$ by removing this collection of triangles. Note that $\mathcal{B}'$ still satisfies (10). So we may iterate this procedure of choosing a triangle in $\mathcal{B}'$, then successively choosing triangles to cover the exposed edges until we obtain a triangulation, then removing the triangluation from $\mathcal{B}'$ to obtain a new multiset. This procedure will terminate when our multi-set becomes empty; that is, when we have chosen $k\binom{n-2}{k}$ triangles in total.

Each time we iterate, we remove $n-2$ triangles from the multiset, since we are removing a triangulation of an n-gon. So the procedure terminates after $\binom{n-3}{k-1}$ steps, since

$$(n-2)\binom{n-3}{k-1} = k\binom{n-2}{k}.$$

In summary, the black polygons from the subdivisions in $\mathcal{S}$ cover the n-gon $\binom{n-3}{k-1}$ times, as desired.

□

7. Combinatorics of Parke-Taylor polytopes and Parke-Taylor identities

In this section we use tricolored subdivisions (cf. Definition 4.5) to introduce and study Parke-Taylor polytopes. We also use them to derive new Parke-Taylor identities.

7.1. **Parke-Taylor polytopes.** Our combinatorial results on polytopes associated to bicolored subdivisions in Section 3 have natural extensions to tricolored subdivisions as well. However, in order to make sense of this, we need to work with the projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ from Remark 2.5.

Definition 7.1. Given a bicolored subdivision σ, let $\tilde{\Gamma}_\sigma$ denote the projected polytope $\pi(\Gamma_\sigma)$. Given a w-simplex Δ_w, let $\tilde{\Delta}_w$ denote the projected simplex $\pi(\Delta_w)$.

We also need to extend Definition 2.9 to the setting of tricolored subdivisions.

Definition 7.2. Let τ be a tricolored subdivision. Given a pair of vertices i, j of $\mathbf{P}_n$, we say that the arc $i \to j$ is *compatible* with τ if the arc either is an edge of a black, white, or grey polygon, or lies entirely inside a black or white polygon of τ. If $i \to j$ is compatible with τ, the *area to the left of* $i \to j$ (respectively, *grey area to the left of* $i \to j$), denoted by $\text{area}(i \to j)$ (respectively, $\text{gr-area}(i \to j)$), is the number of black triangles (respectively, grey triangles) to the left of $i \to j$ in any triangulation of the black (respectively, grey) polygons of τ.

Definition 7.3. Let τ be a tricolored subdivision of $\mathbf{P}_n$ of type (k, ℓ, n). We define the *Parke-Taylor* polytope $\tilde{\Gamma}_\tau \subset \mathbb{R}^{n-1}$ by the following inequalities: for any compatible arc $i \to j$ with $i < j$,

$$\text{area}(i \to j) \leq x_{[i,j-1]} \leq \text{area}(i \to j) + \text{gr-area}(i \to j) + 1.$$

Example 7.4. If τ is the tricolored subdivision of $\mathbf{P}_n$ which is just a grey polygon on n vertices, then the Parke-Taylor polytope $\tilde{\Gamma}_\tau$ is the unit hypercube $\square_{n-1} \subset \mathbb{R}^{n-1}$.

Example 7.5. For the tricolored subdivision τ from Figure 6 of type $(3, 2, 8)$, the Parke-Taylor polytope $\tilde{\Gamma}_\tau \subset \mathbb{R}^7$ is defined by the inequalities $0 \leq x_i \leq 1$ for $1 \leq i \leq 7$, $3 \leq x_{[1,7]} \leq 6$, and

$$\begin{aligned} 1 \leq x_{[5,6]} \leq 2, \qquad 0 \leq x_{[2,4]} \leq 2, \qquad 1 \leq x_{[2,6]} \leq 4 \\ 2 \leq x_{[1,6]} \leq 5, \qquad 2 \leq x_{[2,7]} \leq 5. \end{aligned}$$

Note that this is not a minimal description of $\tilde{\Gamma}_\tau$, as the inequality $1 \leq x_{[2,6]} \leq 4$ is implied by $1 \leq x_{[5,6]} \leq 2$ and $0 \leq x_{[2,4]} \leq 2$.

The Parke-Taylor polytope $\tilde{\Gamma}_\tau$ is related to the (projected) positroid tiles as follows.

Proposition 7.6. *Let τ be a tricolored subdivision. Let $\mathcal{S}$ denote the set of bicolored subdivisions obtained from τ by replacing each grey polygon p by any bicolored subdivision σ_p of p. Then we have a covering of $\tilde{\Gamma}_\tau$ by projected positroid tiles:*

$$\tilde{\Gamma}_\tau = \bigcup_{\sigma \in \mathcal{S}} \tilde{\Gamma}_\sigma. \tag{11}$$

Moreover, there is a collection $\mathcal{K}$ of bicolored subdivisions obtained from τ by choosing a distinguished vertex v_p of each grey polygon p of τ and replacing p with all possible kermit subdivisions of p (based at v_p), such that we have a covering of $\tilde{\Gamma}_\tau$ by projected positroid tiles with disjoint interiors:

$$\tilde{\Gamma}_\tau = \bigcup_{\sigma \in \mathcal{K}} \tilde{\Gamma}_\sigma. \tag{12}$$

Remark 7.7. Equation (12) should remain true if we replace $\mathcal{K}$ by any collection $\mathcal{K}'$ of bicolored subdivisions obtained as follows: for each grey r-gon p of τ, choose a tiling $\mathcal{T}_a = \{\Gamma_{\sigma'}\}_{\sigma' \in \mathcal{S}_a}$ of $\Delta_{a+1,r}$ for $a = 0, \dots, r-2$. A bicolored subdivision σ is in $\mathcal{K}'$ if it is obtained from τ by replacing each p with some $\sigma' \in \mathcal{S}_a$. In this language, to obtain the collection $\mathcal{K}$ in Proposition 7.6, choose each $\mathcal{T}_a$ to be a kermit tiling based at v_p.

See Figure 12 for an example of the set $\mathcal{K}$.

Proof of Proposition 7.6. We first show $\bigcup_{\sigma \in \mathcal{S}} \tilde{\Gamma}_\sigma \subset \tilde{\Gamma}_\tau$.

Let $\sigma \in \mathcal{S}$ be a bicolored subdivision, and consider a point $x \in \tilde{\Gamma}_\sigma$. Choose an arc $i \to j$ (with $i < j$) that is compatible with τ. Clearly the arc $i \to j$ is also compatible with σ. We have that x satisfies

$$\text{area}_\sigma(i \to j) \leq x_{[i,j-1]} \leq \text{area}_\sigma(i \to j) + 1.$$

Note that $\text{area}_\tau(i \to j) \leq \text{area}_\sigma(i \to j)$, since σ can only have more black polygons than τ, and

$$\text{area}_\tau(i \to j) + \text{gr-area}_\tau(i \to j) \geq \text{area}_\sigma(i \to j),$$

since at worst all grey regions to the left of $i \to j$ in τ have been colored black in σ. This shows that $\bigcup_{\sigma \in \mathcal{S}} \tilde{\Gamma}_\sigma \subset \tilde{\Gamma}_\tau$.

Now we will construct the collection $\mathcal{K}$ and show $\tilde{\Gamma}_\tau \subset \bigcup_{\sigma \in \mathcal{K}} \tilde{\Gamma}_\sigma$. To construct $\mathcal{K}$, we will choose distinguished vertices of the grey polygons of τ oen by one.

Pick a grey polygon p of τ with no grey polygons to its left; that is, some boundary arc $i \to j$ of p (with $i < j$) has p to its left but no other grey polygon. Choose i as the distinguished vertex v_p of p. Repeat this process, choosing a grey polygon p' of τ such that for some boundary arc $a \to b$ of p' (with $a < b$), the grey polygons to the left of $a \to b$ are p' and grey polygons whose distinguished vertices have already been chosen. Choose a as the distinguished vertex $v_{p'}$.

Now, consider a point $x \in \tilde{\Gamma}_\tau$ whose coordinate sums $x_{[i,j]}$ are all non-integral. Let $r := \lfloor x_{[n-1]} \rfloor$. Lift x to the point $y = (x, r + 1 - x_{[n-1]}) \in \Delta_{r+1,n} \subset \mathbb{R}^n$. We would like to find a bicolored subdivision $\sigma \in \mathcal{K}$ of area r such that $y \in \Gamma_\sigma$. To do so, we will color the grey polygons of τ one by one, in the same order as we chose the distinguished vertices.

We begin with grey polygon p, with distinguished vertex $v_p = i$ and boundary arc $i \to j$ with no other grey polygons to its left. Triangulate p with edges with one endpoint v_p. Once we color each triangle black or white, we will obtain a kermit subdivision of p based at v_p.

Suppose that going clockwise around p, we see vertex $v_p = i$, then a then b, so $i \to b$ is an arc in the triangulation. Note that $i \to a$ and $a \to b$ are compatible with τ and

$$\text{area}_\tau(i \to b) = \text{area}_\tau(i \to a) + \text{area}_\tau(a \to b).$$

Meanwhile the inequalities of $\tilde{\Gamma}_\tau$ for arcs $i \to a$ and $a \to b$ say that

$$\text{area}_\tau(i \to a) \leq y_{[i,a-1]} \leq \text{area}_\tau(i \to a) + 1, \text{ and}$$
$$\text{area}_\tau(a \to b) \leq y_{[a,b-1]} \leq \text{area}_\tau(a \to b) + 1,$$

and hence y satisfies

$$\text{area}_\tau(i \to b) \leq y_{[i,b-1]} \leq \text{area}_\tau(i \to b) + 2.$$

By assumption y has non-integral consecutive coordinate sums, so we have either

$$\text{area}_\tau(i \to b) < y_{[i,b-1]} < \text{area}_\tau(i \to b) + 1 \quad \text{or} \quad \text{area}_\tau(i \to b) + 1 < y_{[i,b-1]} < \text{area}_\tau(i \to b) + 2.$$

In the former case, we color the triangle on i, a, b white in σ and in the latter case we color the triangle black. In either case, y satisfies

$$\text{area}_\sigma(i \to b) < y_{[i,b-1]} < \text{area}_\sigma(i \to b) + 1.$$

Repeat this for all triangles of the chosen triangulation for p, starting with the triangle adjacent to the one we just colored. This gives a new tricolored subdivision τ' with one fewer grey polygon than τ. Repeating the above procedure on τ' produces the desired bicolored subdivision σ.

This shows that a dense subset of points in $\tilde{\Gamma}_\tau$ is contained in $\bigcup_{\sigma \in \mathcal{K}} \tilde{\Gamma}_\sigma$. Since this union is closed, the desired containment follows.

Now, we have

$$\bigcup_{\sigma \in \mathcal{K}} \tilde{\Gamma}_\sigma \subset \bigcup_{\sigma \in \mathcal{S}} \tilde{\Gamma}_\sigma \subset \tilde{\Gamma}_\tau \subset \bigcup_{\sigma \in \mathcal{K}} \tilde{\Gamma}_\sigma$$

where the leftmost containment is because $\mathcal{K} \subset \mathcal{S}$ and the other containments have been argued above. This shows both (11) and (12).

Finally, the polytopes $\{\tilde{\Gamma}_\sigma\}_{\sigma \in \mathcal{K}}$ have disjoint interiors. Indeed, it follows from Corollary 4.11 that a projected w-simplex $\tilde{\Delta}_w$ lies in at most one $\tilde{\Gamma}_\sigma$ for $\sigma \in \mathcal{K}$. The polytopes $\tilde{\Gamma}_\sigma$ are triangulated by projected w-simplices, so this implies no two polytopes in $\{\tilde{\Gamma}_\sigma\}_{\sigma \in \mathcal{K}}$ have full-dimensional intersection and thus they have disjoint interiors. □

We also have a triangulation of $\tilde{\Gamma}_\tau$ into projected w-simplices; this is a generalization of Corollary 4.8.

Proposition 7.8. *Let τ be a tricolored subdivision of type (k, ℓ, n). Then*

$$\tilde{\Gamma}_\tau = \bigcup_{(w) \in \text{Ext}(C_\tau)} \tilde{\Delta}_w.$$

It follows that the volume of $\tilde{\Gamma}_\tau$ is the number of circular extensions of the cyclic order C_τ. That is, $\text{Vol}(\Gamma_\tau) = |\,\text{Ext}(C_\tau)|$.

Proof. By Proposition 7.6,

$$\tilde{\Gamma}_\tau = \bigcup_\sigma \tilde{\Gamma}_\sigma$$

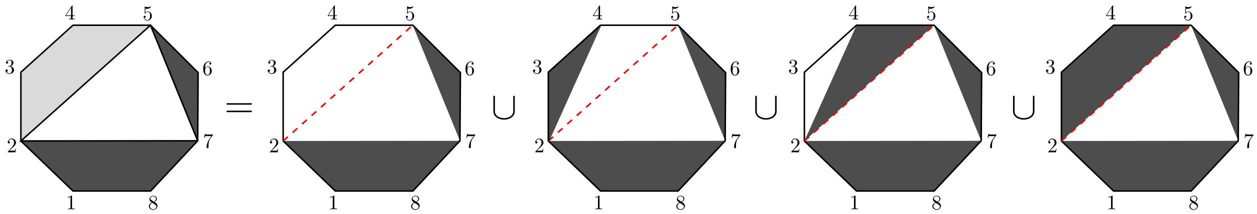

FIGURE 12. An illustration of the second part of Proposition 7.6. The polytope $\tilde{\Gamma}_\sigma$ whose σ is on the left hand side and in Figure 6 is the union of the polytopes $\tilde{\Gamma}_\tau$ for τ on the right hand side. The distinguished vertex of the grey polygon is 2 in this case.

where the union is over the collection $\mathcal{K}$ of bicolored subdivisions obtained from τ by choosing a distinguished vertex v_p of each grey polygon p of τ and replacing p with all possible kermit subdivisions of p (based at v_p). Therefore if $\tilde{\Delta}_w$ lies in $\tilde{\Gamma}_\tau$, then $\tilde{\Delta}_w$ lies in some $\tilde{\Gamma}_\sigma$ for some $\sigma \in \mathcal{K}$, which implies that C_w is a circular extension of C_σ. But $C_\tau \subset C_\sigma$, and hence C_w is a circular extension of C_τ as well.

Given a circular extension C_w of C_τ, let $C_w|_p$ be the restriction of C_w to the vertices of a grey polygon p of τ. By Corollary 4.11, there is exactly one kermit subdivision $\sigma(p)$ of p (using the distinguished vertex v_p) for which $C_w|_p$ is a circular extension of $C_{\sigma(p)}$. This means that C_w is a circular extension of C_σ, where σ is obtained by subdividing each grey polygon p with $\sigma(p)$. Such a subdivision σ is one of the subdivisions described in the previous paragraph. □

Example 7.9. For the tricolored subdivision τ in Figure 6, we have

$$\mathrm{Vol}(\tilde{\Gamma}_\tau) = 55 + 62 + 127 + 127 = 371$$

which is the number of circular extensions of the cyclic order $C_\tau = C_{(2,5,7)} \cup C_{(5,7,6)} \cup C_{(1,8,7,2)}$. Each term in the sum above is the volume of $\tilde{\Gamma}_\sigma$ for σ on the right hand side of Figure 12.

Remark 7.10. Proposition 7.8 is quite analogous to a result from [AJVR20], which shows that the (normalized) volume of a certain polytope $B_{I,n}$ equals the number of circular extensions of a partial cyclic order $A_{I,n}$. Here I is a subset of $\binom{[n]}{2}$, the polytope $B_{I,n}$ is defined as

$$B_{I,n} = \{(x_1, \dots, x_n) \in [0,1]^n \mid x_{[i+1,j]} \leq 1 \text{ for each } \{i < j\} \in I\},$$

and the cyclic order $A_{I,n}$ is defined as the union of the chains $C_{(i,i+1,\dots,j)}$ for each $\{i < j\} \in I$. In the special case when the intervals $[i,j]$ (for $\{i < j\} \in I$) are noncrossing (i.e. we do not have $i < i' < j < j'$), $A_{I,n}$ is equal to C_τ, where τ is a tricolored subdivision with no black polygons such that the vertices of every white polygon form an interval $[i,j]$; and $B_{I,n} \subset \mathbb{R}^n$ is, up to renaming variables, equal to $\tilde{\Gamma}_\tau$ times the interval $[0,1]$. For other $I \subset \binom{[n]}{2}$ and for other subdivisions τ, the cyclic orders $A_{I,n}$ and C_τ, and the polytopes $B_{I,n}$ and $\tilde{\Gamma}_\tau$, are not obviously related.

7.2. **Parke-Taylor identities.** In this section we derive identities for Parke–Taylor functions, one for each tricolored subdivision. Parke-Taylor identities have previously been studied and connected to: non-planar generalizations of *plabic graphs* [AHBC+15, CEGM19]; the *momentum amplituhedron* [DFLM21]; *Lie polynomials* [FM21]; and *CGEM amplitudes* in relation to the configuration of n points in $\mathbb{CP}^{k-1}$ [CEZ23].

Theorem 7.11. *Let τ be a tricolored subdivision of $\mathbf{P}_n$ which contains at least one grey polygon, and let C_τ be the corresponding cyclic partial order. Recall that $\mathrm{Ext}(C_\tau)$ is the set of cyclic extensions of C_τ. Then we have that*

$$\sum_{(w)\in \mathrm{Ext}(C_\tau)} \mathrm{PT}(w) = 0.$$

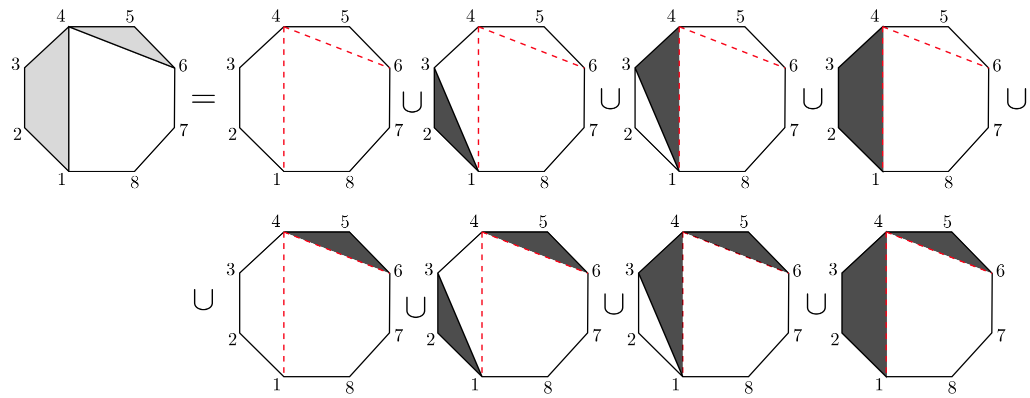

FIGURE 13. Example for both Theorem 7.11 and Corollary 7.14 (with $S = \{1,4,6,7,8\}$). The dashed lines are to help the reader and are not part of the tricolored subdivision.

Proof. Let us label the grey polygons of τ by $P_1,\dots,P_\ell$, and for each one, choose a distinguished vertex $v_1,\dots,v_\ell$. We will apply Corollary 4.11 and Remark 4.12 (see also Figure 7) to each grey polygon P_i. In particular, each total cyclic order on the vertices of P_i must be a cyclic extension of one of the partial cyclic orders associated to a "kermit" bicolored subdivision (based at v_i) of P_i.

Let $\sigma_1,\dots,\sigma_r$ be the set of all bicolored subdivisions of $\mathbf{P}_n$ obtained from τ by replacing each grey polygon P_i (for $1 \le i \le \ell$) by a bicolored kermit subdivision of that polygon based at v_i, see Figure 13. By Corollary 4.11, each circular extension C_w of C_τ must be a circular extension of exactly one of the partial cyclic orders C_{σ_i} for $1 \le i \le r$, which by Corollary 4.8 corresponds to a w-simplex from the corresponding tile Γ_{σ_i}. Thus we have

$$\sum_{(w)\in \mathrm{Ext}(C_\tau)} \mathrm{PT}(w) = \sum_{i=1}^{r} \Omega(\Gamma_{\sigma_i}).$$

But by Proposition 5.4, the Parke-Taylor function of a tile Γ_{σ_i} is $(-1)^k\,\mathrm{PT}(\mathbf{I}_n)$, where k is the number of black triangles in any triangulation of the black polygons of σ_i. Therefore, if d_i is the number of vertices of P_i and k' is the area of τ, we get

$$\sum_{(w)\in \mathrm{Ext}(C_\tau)} \mathrm{PT}(w) = \sum_{i=1}^{r} \Omega(\Gamma_{\sigma_i}) = (-1)^{k'}\,\mathrm{PT}(\mathbf{I}_n)\cdot \prod_{i=1}^{\ell}\left(\sum_{k_i=0}^{d_i-2}\binom{d_i-2}{k_i}(-1)^{k_i}\right) = 0,$$

where the last equality holds because each term in the product is 0. $\square$

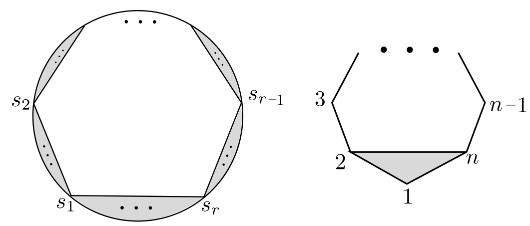

FIGURE 14. Tricolored subdivisions generating the relation in Corollary 7.14 (left) and $U(1)$ decoupling identities as a special case (right).

Remark 7.12. We can extend Definition 5.3 and define the weight function $\Omega(\tilde{\Gamma}_\tau)$ of a Parke-Taylor polytope $\tilde{\Gamma}_\tau$ as

$$\Omega(\tilde{\Gamma}_\tau) := \sum_{\tilde{\Delta}_{(w)} \subset \tilde{\Gamma}_\tau} \Omega(\Delta_{(w)}) = \sum_{(w) \in \mathrm{Ext}(C_\tau)} \mathrm{PT}(w),$$

where the second equality follows from Proposition 7.8. Theorem 7.11 is then equivalent to say that the weight function of a Parke-Taylor polytope $\tilde{\Gamma}_\tau$ vanishes, if τ has at least a grey polygon.

We highlight a few special cases of Theorem 7.11.

If we consider a tricolored subdivision with $\mathbf{P}_n$ all colored grey, then by Example 7.4, Theorem 7.11 and Remark 7.12, we have the following.

Corollary 7.13. *The weight function of the unit hypercube* $⧈_{n-1}$ *is*

$$\Omega(⧈_{n-1}) = \sum_{k=0}^{n-2} \Omega(\Delta_{k+1,n}) = \sum_{w \in D_n} \Omega(\Delta_{(w)}) = 0.$$

Equivalently, the sum over $w \in D_n$ of the weight functions of w-simplices is zero.

If we consider a tricolored subdivision τ with one white polygon and all others grey (see Figure 14), we obtain the following.

Corollary 7.14. *Let $S = \{s_1 < \cdots < s_r\} \subseteq [n]$, and let $\mathcal{D}_S$ be the set of permutations in D_n in which $s_1, \ldots, s_r$ appear in order. Then we have that*

$$\sum_{w \in \mathcal{D}_S} \mathrm{PT}(w) = 0.$$

Proof. Consider the n-gon $\mathbf{P}_n$ with vertices $1, 2, \ldots, n$ and let R be the subpolygon with vertices $s_1, \ldots, s_r$. Let τ be the tricolored subdivision of $\mathbf{P}_n$ in which the subpolygon R is white and the rest of $\mathbf{P}_n$ is grey, see Figure 13. By Definition 4.6, $\mathcal{D}_S$ exactly corresponds to the total cyclic orders which are circular extensions of the circular order C_τ. Now the result follows from Theorem 7.11. □

In the specific case of Corollary 7.14 when $r = n - 1$ and $S = [n] \setminus \{1\}$, we obtain the known $U(1)$ *decoupling identities*[9] for Parke-Taylor functions.

Corollary 7.15. *We have*

$$\mathrm{PT}(1, 2, 3, \ldots, n) + \mathrm{PT}(2, 1, 3, \ldots, n) + \mathrm{PT}(2, 3, 1, \ldots, n) + \ldots + \mathrm{PT}(2, 3, \ldots, 1, n) = 0.$$

[9]This relation has the interpretation of a scattering amplitude of $n-1$ $SU(N)$ 'gluons' and a $U(1)$ 'photon' that vanishes because the latter is decoupled from (i.e. does not interact with) the former.

Appendix A. The G-amplituhedron and Parke-Taylor functions

In this section, we first give the definition of w-chambers in $\mathcal{A}_{n,k,2}(Z)$. Then we describe the interpretation of some of our results in the *G-amplituhedron* $\mathcal{G}_{n,k,2}$, which is a "Z-independent" version of $\mathcal{A}_{n,k,2}(Z)$ introduced in [LPW23]. In this context, the Parke-Taylor functions $\mathrm{PT}(w)$ have a geometric interpretation as *canonical functions* of certain *positive geometries*.

Recall the definition of twistor coordinates $\langle\langle Yab\rangle\rangle$ from Definition 2.15.

Definition A.1 ([LPW23, Definition 10.7]). Fix $Z \in \mathrm{Mat}_{n,k+2}^{>0}$. Let $w \in D_{k+1,n}$, $I_a = \mathrm{cDes}_{\mathrm{L}}(w^{(a)})$, and define

$$(\Delta_{(w)}^{Z})^{\circ} := \left\{Y \in \mathrm{Gr}_{k,k+2}\colon\ \mathrm{sgn}\langle\langle Yab\rangle\rangle = (-1)^{|I_a \cap [a,b-1]|-1} \text{ for all } a<b\right\}$$

The closure $\Delta_{(w)}^{Z} := \overline{(\Delta_{(w)}^{Z})^{\circ}}$ in $\mathrm{Gr}_{k,k+2}$ is a *w-chamber* of $\mathcal{A}_{n,k,2}(Z)$.

Depending on the choice of Z, $\Delta_{(w)}^{Z}$ may be empty [PSBW23, Section 11.3], though there always exists a Z such that $\Delta_{(w)}^{Z}$ is nonempty [PSBW23, Theorem 11.5]. To avoid this irregularity, one may pass to the *G-amplituhedron* and define w-chambers there.

Given $v \in \mathbb{R}^n$, let $\mathrm{var}(v)$ be the number of times the entries of v changes sign when we read the entries from left to right and ignore any zeros. For example, if $v := (4,-1,0,-2)$ then $\mathrm{var}(v)=1$.

Definition A.2 ([LPW23, Definition 11.9]). Fix $k<n$ and let

$$\mathcal{G}_{n,k,2}^{\circ} := \{z \in \mathrm{Gr}_{2,n}\ :\ P_{i,i+1}(z) > 0 \text{ for } 1 \le i \le n-1, (-1)^k P_{1n}(z) > 0, \text{ and } \mathrm{var}((P_{12}(z), P_{13}(z), \ldots P_{1n}(z)) = k\}.$$

The closure $\mathcal{G}_{n,k,2} := \overline{\mathcal{G}_{n,k,2}^{\circ}}$ in $\mathrm{Gr}_{2,n}$ is the *G-amplituhedron*. The *total G-amplituhedron* is the union

$$\mathcal{G}_n := \bigcup_{k=0}^{n-2} \mathcal{G}_{n,k,2} \subset \mathrm{Gr}_{2,n}\,.$$

One should think of $\mathcal{G}_n$ as an analogue of the hypercube: just as the hypercube is the union of all (projected) hypersimplices, the total G-amplituhedron is the union of all G-amplituhedra.

Motivated by the decomposition of $\mathcal{A}_{n,k,2}(Z)$ into w-chambers, we analogously define w-chambers for $\mathcal{G}_{n,k,2}$ which are (closures of) certain uniform oriented matroid strata in $\mathrm{Gr}_{2,n}$.

Definition A.3 ([LPW23, Definition 11.13]). Let $w \in D_{k+1,n}$, $I_a = \mathrm{cDes}_{\mathrm{L}}(w^{(a)})$, and define

$$(\Delta_{(w)}^{\mathcal{G}})^{\circ} := \left\{z \in \mathrm{Gr}_{2,n}\colon\ \mathrm{sgn}\, P_{ab}(z) = (-1)^{|I_a \cap [a,b-1]|-1} \text{ for all } a<b\right\}$$

The closure $\Delta_{(w)}^{\mathcal{G}} := \overline{(\Delta_{(w)}^{\mathcal{G}})^{\circ}}$ in $\mathrm{Gr}_{2,n}$ is a *w-chamber* of the G-amplituhedron $\mathcal{G}_{n,k,2}$.

Remark A.4. In analogy with the amplituhedron $\mathcal{A}_{n,k,2}(Z)$, the G-amplituhedron $\mathcal{G}_{n,k,2}$ is the union of the w-chambers $\Delta_{(w)}^{\mathcal{G}}$, with w ranging over $D_{k+1,n}$ [PSBW23, Theorem 11.21]. The amplituhedron $\mathcal{A}_{n,k,2}(Z)$ can be seen as a $2k$-dimensional linear slice of $\mathcal{G}_{n,k,2}$ and the w-chamber $\Delta_{(w)}^{Z}$ in $\mathcal{A}_{n,k,2}(Z)$ as a $2k$-dimensional linear slice of the w-chamber $\Delta_{(w)}^{\mathcal{G}}$ (see [PSBW23, Proposition 11.11, Remark 11.17]).

Proposition A.5 ([PSBW23, Proposition 11.15]). *Let $w \in D_{k+1,n}$. Then $(\Delta_{(w)}^{\mathcal{G}})^{\circ}$ is nonempty and is contractible.*

From the proof of [PSBW23, Proposition 11.15], we can find points in $(\Delta^{\mathcal{G}}_{(w)})^\circ$ explicitly. Consider n vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^2$ so that the matrix

$$\begin{pmatrix} v_1 & v_{w^{(1)}_1} & v_{w^{(1)}_2} & \cdots & v_{w^{(1)}_{n-1}} \end{pmatrix} \in \mathrm{Mat}^{>0}_{2,n}, \tag{13}$$

equivalently, the vectors $v_1, v_{w^{(1)}_1}, v_{w^{(1)}_2}, \ldots, v_{w^{(1)}_{n-1}}$ drawn in a plane are ordered counterclockwise. (Recall that $w^{(1)}$ denotes the rotation of w ending in 1, and hence $w^{(1)}_n = 1$.) Now, set $z_1 := v_1$ and $z_b := (-1)^{|I_1 \cap [1,b-1]|-1} v_b$, for $b > 2$. Then

$$z = \begin{pmatrix} z_1 & z_2 & z_3 & \ldots & z_n \end{pmatrix} \tag{14}$$

represents a point in $(\Delta^{\mathcal{G}}_{(w)})^\circ$. Moreover, all points in $(\Delta^{\mathcal{G}}_{(w)})^\circ$ arise this way.

Example A.6. Let $w = 2564137 \in D_{4,7}$, so $w^{(1)} = 3725641$. We have $I_1 = \{1, 2, 4, 6\}$ and

$$(v_1, v_{w^{(1)}_1}, v_{w^{(1)}_2}, v_{w^{(1)}_3}, v_{w^{(1)}_4}, v_{w^{(1)}_5}, v_{w^{(1)}_6}) = (v_1, v_3, v_7, v_2, v_5, v_6, v_4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

We then get

$$z = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 4 & -2 & -7 & 5 & 6 & -3 \end{pmatrix} \in (\Delta^{\mathcal{G}}_{(w)})^\circ.$$

Remark A.7. The positive Grassmannian $\mathrm{Gr}^{\geq 0}_{2,n}$ is a *positive geometry* in the sense of [AHBL17, Lam22]. Furthermore, the Parke-Taylor function $\mathrm{PT}(\mathbf{I}_n)$ is the *canonical function* of $\mathrm{Gr}^{\geq 0}_{2,n}$; that is multiplying $\mathrm{PT}(\mathbf{I}_n)$ by the standard top form of $\mathrm{Gr}_{2,n}$ (cf. [AHBL17, Appendix C.2]) gives the *canonical form* of $\mathrm{Gr}^{\geq 0}_{2,n}$. The poles $P_{i,i+1} = 0$ of $\mathrm{PT}(\mathbf{I}_n)$ correspond to the facets of $\mathrm{Gr}^{\geq 0}_{2,n}$.

More generally, $\mathrm{Gr}^{\geq 0}_{k,n}$ is a positive geometry [AHBL17, Lam22] and the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is conjectured to be a positive geometry. This was recently proved for $k = 2, m = 2$ [RST24].

Remark A.8. Each w-chamber $\Delta^{\mathcal{G}}_{(w)}$ of the G-amplituhedron is isomorphic to $\mathrm{Gr}^{\geq 0}_{2,n}$ by a map induced by permuting and rescaling columns [PSBW23, Proposition 11.15]. Hence $\Delta^{\mathcal{G}}_{(w)}$ is also a positive geometry and the weight function $\Omega(\Delta_{(w)}) = \mathrm{PT}(w)$ of the w-simplex $\Delta_{(w)}$ is the canonical function of $\Delta^{\mathcal{G}}_{(w)}$. Indeed, the poles $P_{w_i, w_i+1} = 0$ of $\mathrm{PT}(w)$ correspond to the facets of $\Delta^{\mathcal{G}}_{(w)}$. From the definition of z_a and (13), the facets of $\Delta^{\mathcal{G}}_{(w)}$ are in the locus where $v_{w^{(1)}_i}, v_{w^{(1)}_{i+1}}$ are parallel and are therefore cut out by the equation $P_{w_i, w_i+1} = 0$ in the Plückers of the matrix z of (14).

Given a (k, n)-bicolored subdivision σ, one can define *tiles* in the G-amplituhedron by considering the union of the w-chambers $\Delta^{\mathcal{G}}_{(w)}$ over $(w) \in \mathrm{Ext}(C_\sigma)$. One can use these tiles to tile the G-amplituhedron $\mathcal{G}_{n,k,2}$ and such tilings of $\mathcal{G}_{n,k,2}$ are in bijection with tilings of $\Delta_{k+1,n}$ (and with all-Z tilings of $\mathcal{A}_{n,k,2}(Z)$).

The total G-amplituhedron $\mathcal{G}_n$ is given by the union of the w-chambers $\Delta^{Z}_{(w)}$ over $w \in D_n$ [PSBW23, Theorem 11.21]. Then one can consider tricolored subdivision τ and define the region in $\mathcal{G}_n$ which is the union of the w-chambers $\Delta^{Z}_{(w)}$ with $w \in \mathrm{Ext}(C_\tau)$. This is the analogue of the Parke-Taylor polytope $\tilde{\Gamma}_\tau$. As the weight function $\Omega(\Delta_{(w)})$ of w-simplices gives the canonical function of w-chambers $\Delta^{\mathcal{G}}_{w}$, it would be interesting to interpret the weight function of Parke-Taylor polytopes $\Omega(\tilde{\Gamma}_\tau)$ as canonical functions of some positive geometries inside the (total) G-amplituhedron.

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