RHOMBIC STAIRCASE TABLEAUX AND KOORNWINDER POLYNOMIALS

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Abstract. In this article we give a combinatorial formula for a certain class of Koornwinder polynomials, also known as Macdonald polynomials of type $\tilde{C}$. In particular, we give a combinatorial formula for the Koornwinder polynomials $K_\lambda = K_\lambda(z_1, \ldots, z_N; a, b, c, d; q, t)$, where $\lambda = (1, \ldots, 1, 0, \ldots, 0)$. We also give combinatorial formulas for all "open boundary ASEP polynomials" F_μ, where μ is a composition in $\{-1, 0, 1\}^N$; these polynomials are related to the nonsymmetric Koornwinder polynomials E_μ up to a triangular change of basis. Our formulas are in terms of *rhombic staircase tableaux*, certain tableaux that we introduced in previous work to give a formula for the stationary distribution of the two-species asymmetric simple exclusion process (ASEP) on a line with open boundaries.

Contents

1. Introduction

In recent years there has been a great deal of activity connecting the asymmetric simple exclusion process (ASEP), special functions, and combinatorics, resulting in new combinatorial formulas for various special functions. For example, Cantini, de Gier, and Wheeler [CdGW15] showed that the specialization of the Macdonald polynomial $P_\lambda(x_1 \ldots, x_N; q, t)$ at $x_1 = \cdots = x_N = 1$ and $q = 1$ is the partition function for the *multispecies ASEP on a ring*. Subsequently Martin [Mar20] gave a formula for the stationary distribution of the multispecies ASEP on a ring using some combinatorial objects he called *multiline queues*. This led to our discovery of a new formula for Macdonald polynomials and *ASEP polynomials*, which are related to the nonsymmetric Macdonald polynomials by a triangular change of basis, in terms of multiline queues [CMW22].

A few years later, Ayyer, Martin, and Mandelshtam [AMM20, AMM22] proved analogous results, connecting the *modified Macdonald polynomials*, *queue-inversion tableaux*, and the TAZRP (totally asymmetric zero range process). In particular, they gave a new formula for modified Macdonald polynomials in terms of queue-inversion tableaux.

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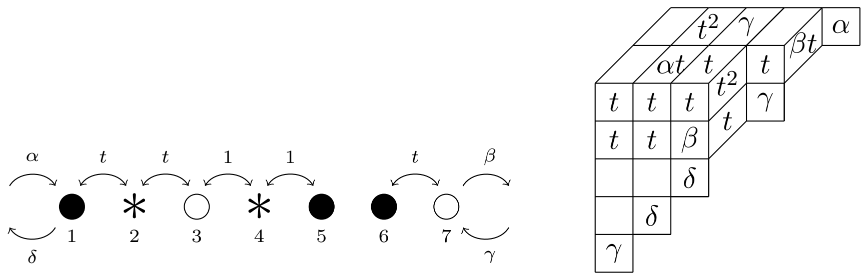


Figure 1. On the left: the two-species open ASEP state $\mu = (\bullet, *, \circ, *, \bullet, \bullet, \circ)$. On the right: a rhombic staircase tableau of type μ.

In this article we describe a parallel story, this time relating certain Koornwinder polynomials $K_\lambda = K_\lambda(\mathbf{z}; a, b, c, d; q, t)$, rhombic staircase tableaux, and the open boundary ASEP. In [CGdGW16], Cantini, Garbali, de Gier and Wheeler showed that the specialization of the Koornwinder polynomial K_λ at $z_1 = \cdots = z_N = 1$ and $q = 1$ is the partition function for the *multispecies open ASEP* (the multispecies ASEP on a line with open boundaries), generalizing a previous result of Cantini [Can17] for the two-species open ASEP. In [CMW17], we gave a formula for the stationary distribution of the two-species open ASEP in terms of some tableaux we called *rhombic staircase tableaux.* Building on the above insights, in this article we give a combinatorial formula for the Koornwinder polynomials $K_\lambda = K_\lambda(\mathbf{z}; a, b, c, d; q, t)$, where $\lambda = (1, \dots, 1, 0, \dots, 0)$, by incorporating "spectral parameters" $z_1^{\pm 1}, \dots, z_N^{\pm 1}$ into the rhombic staircase tableaux. We also give combinatorial formulas for the "open boundary ASEP polynomials" $f_\mu(\mathbf{z}; a, b, c, d; q, t)$ associated to compositions in $\{-1, 0, 1\}^N$, which are analogously related to the nonsymmetric Koornwinder polynomials $E_\mu(\mathbf{z}; a, b, c, d; q, t)$ by a triangular change of basis. While our results only treat a special case of Koornwinder polynomials, this special case is already quite nontrivial; as we explain below, Koornwinder polynomials represent a substantial generalization of Macdonald polynomials.

In the remainder of the introduction we provide some background on Koornwinder polynomials, the ASEP, and rhombic staircase tableaux, then state our main result.

1.1. **Koornwinder polynomials.** In [Mac88], Macdonald introduced a family of orthogonal symmetric polynomials in the variables $\mathbf{x} = x_1, x_2, \dots, x_N$ and parameters q, t indexed by partitions λ, associated to different Lie types. In type A, these are the classical Macdonald polynomials $P_\lambda = P_\lambda(\mathbf{x}; q, t)$, which have been studied extensively in representation theory, algebraic combinatorics, and mathematical physics. In type $\tilde{C}$, the Macdonald polynomials recover the *Koornwinder polynomials* $K_\lambda = K_\lambda(\mathbf{z}; a, b, c, d; q, t)$ (sometimes called *Macdonald-Koornwinder polynomials*), a 6-parameter family of symmetric Laurent polynomials introduced by Koornwinder in [Koo92], and further studied by van Diejen [vD95], Noumi [Nou95], Sahi [Sah99], and others. We note that the Koornwinder polynomials are a multivariate generalization of the the well-known Askey–Wilson polynomials, which in turn specialize or limit to all other families of classical hypergeometric orthogonal polynomials in one variable [AW85]. Moreover the Koornwinder polynomials give rise to the Macdonald polynomials associated to any classical root system via a limit or specialization; in particular, the usual (type A) Macdonald polynomial P_λ is the term of highest degree $|\lambda|$ in the Koornwinder polynomial K_λ [vD95].

To keep our notation compact, we will often write K_λ or $K_\lambda(\mathbf{z}; q, t)$ for $K_\lambda(\mathbf{z}; a, b, c, d; q, t)$. When the parts of λ are bounded by one, we will write $K_\lambda(\mathbf{z}; t)$, since $K_\lambda(\mathbf{z}; q, t)$ is independent of q.

There are also nonsymmetric versions of Macdonald and Koornwinder polynomials. The *nonsymmetric Koornwinder polynomials* $E_\mu = E_\mu(\mathbf{z}; a, b, c, d; q, t)$, which are indexed by compositions $\mu \in \mathbb{Z}^N$, were introduced by Sahi in [Sah99] as the joint eigenfunctions of mutually commuting difference operators constructed from generators of the affine Hecke algebra. For a partition λ, the (symmetric) Koornwinder polynomial K_λ can be obtained as a linear combination of the E_μ's, where μ ranges over all signed permutations of λ.

Although there has been a wealth of combinatorics developed to study (symmetric and nonsymmetric) Macdonald polynomials in type A, such as the tableaux formulas of Haglund, Haiman, and Loehr [HHL04], multiline queue formulas [CMW22], and vertex model formulas [BW19], so far there has been much less progress developing the combinatorics of Koornwinder polynomials. Ram and Yip gave a type-independent formula for Macdonald polynomials in terms of (quantum) alcove paths [RY11], which has been specialized to the Koornwinder case by Orr and Shimozono, see [OS18, Theorem 3.13 and Proposition 3.20]. However, these formulas are much less explicit and combinatorial than their type A counterparts.

In this paper we aim to leverage the connection between Koornwinder polynomials and the open boundary ASEP [Can17, CW18, CGdGW16, FV17] together with the tableaux formula for the two-species open boundary ASEP [CMW17] to give tableaux formulas for certain Koornwinder polynomials.

1.2. The two-species open boundary ASEP.

The *asymmetric simple exclusion process (ASEP)* is a canonical example of an out-of-equilibrium system of interacting particles on a one-dimensional lattice, originally introduced by Spitzer [Spi70]. In the ASEP with open boundaries, the particles hop left and right on a one-dimensional finite lattice, subject to the constraint that there is at most one particle at each site; particles can also enter and exit the lattice at the left and right endpoints of the lattice, at rates α, β, γ and δ.

There is a multispecies version of the open boundary ASEP, in which particles have *weights* $\{0, \pm 1, \dots, \pm m\}$. For the purpose of this article, it is enough to define the two-species open boundary ASEP, shown at the left of Figure 1.

Definition 1.1. The *two-species asymmetric simple exclusion process (ASEP) with open boundaries* (or *two-species open ASEP*) is a model of interacting particles on a one-dimensional lattice of N sites, such that each site is either vacant (represented by $\circ$ or "-1") or occupied by a first class particle (represented by $\bullet$ or "1") or a second class particle (represented by $*$ or "0"). States of the ASEP are given by words $\mu = (\mu_1, \dots, \mu_N) \in \{\bullet, \circ, *\}^N = \{-1, 0, 1\}^N$. We say that such a word μ has *length* $|\mu| = N$. First class particles may enter and exit at the left and right boundaries, and both types of particles can hop left or right to adjacent (vacant) sites. The transitions in this Markov chain are as follows:

(1) $X{\bullet}{\circ}Y \to X{\circ}{\bullet}Y$, $X{\bullet}{*}Y \to X{*}{\bullet}Y$, and $X{*}{\circ}Y \to X{\circ}{*}Y$ with probability $\frac{t}{N+1}$,
(2) $X{\circ}{\bullet}Y \to X{\bullet}{\circ}Y$, $X{*}{\bullet}Y \to X{\bullet}{*}Y$, and $X{\circ}{*}Y \to X{*}{\circ}Y$ with probability $\frac{1}{N+1}$,
(3) ${\circ}Z \to {\bullet}Z$ with probability $\frac{\alpha}{N+1}$,

(4) $\bullet Z \to \bullet Z$ with probability $\frac{\delta}{N+1}$,
(5) $Z\bullet \to Z\circ$ with probability $\frac{\beta}{N+1}$,
(6) $Z\circ \to Z\bullet$ with probability $\frac{\gamma}{N+1}$,

where $X, Y, Z \in \{\circ, *, \bullet\}^*$ such that $|X| + |Y| = N - 2$ and $|Z| = N - 1$.

Note that when there is a $*$ at a boundary, no particles may enter or exit at that boundary. In particular, the number of second class particles is conserved.

Building upon earlier work of [Uch08], the first and third author discovered that the partition function $Z_{N,r}$ of the two-species open ASEP on a lattice of N sites with r second class particles can be described as a *Koornwinder moment* of type $\lambda = ((N-r), 0^r)$ [CW18]. Concurrently, Cantini [Can17] found that the partition function $Z_{N,r}$ coincides with the specialization of a Koornwinder polynomial $K_\lambda(\mathbf{z}; a, b, c, d; q, t)$ of type $\lambda = (1^{N-r}, 0^r)$ when $z_1 = \cdots = z_N = 1$ after a change of variables according to [Can17, (9)]. [1] In subsequent work, we introduced rhombic staircase tableaux in order to describe the stationary distribution (and in particular the partition function) of the two-species open ASEP [CMW17].

1.3. **Rhombic staircase tableaux.** In [CW11], Corteel and Williams introduced *staircase tableaux* and used them to give a combinatorial formula for the stationary distribution of the open boundary ASEP. Subsequently, Mandelshtam and Viennot [MV18] introduced *rhombic alternative tableaux*, and used them to give a combinatorial formula for the stationary distribution of the two-species open ASEP, when $\gamma = \delta = 0$. Finally in [CMW17], we introduced *rhombic staircase tableaux*, and used them to give a combinatorial formula for the stationary distribution of the two-species open ASEP (with all parameters general). We now define these tableaux.

Definition 1.2. Let $\mu = (\mu_1, \ldots, \mu_N) \in \{\circ, *, \bullet\}^N$. We define the *rhombic diagram* $\Gamma(\mu)$ of shape μ to be the piecewise linear curve whose southeast border is obtained by reading μ from left to right and adding a south step followed by a west step for each $\circ$ or $\bullet$, and a southwest step for each $*$. We label the squares and rhombi adjacent to the southeast border from top to bottom with $\{1, \ldots, N\}$. Then a square labelled j corresponds to $\mu_j \in \{\circ, \bullet\}$, and a rhombus labeled j corresponds to $\mu_j = *$. The northwest border is obtained by adding $N - r$ west steps followed by r southwest steps followed by $N - r$ south steps to the northeast end of the southeast border, where r are the total numbers of $*$ particles in μ. We say such $\Gamma(\mu)$ has *size* (N, r).

See Figure 2 for an example.

Definition 1.3. We can tile $\Gamma(\mu)$ using three types of tiles:

- a *square* tile consisting of horizontal and vertical edges,
- a *horizontal rhombic* tile (or "horizontal rhombus" or "short rhombus") consisting of horizontal and diagonal edges,
- and a *vertical rhombic* tile (or "vertical rhombus" or "tall rhombus") consisting of vertical and diagonal edges.

We will always choose a distinguished tiling of $\Gamma(\mu)$, shown in Figure 2.

[1] We could also add here the condition $q = 1$, but Koornwinder polynomials $K_{(1,\ldots,1,0,\ldots,0)}$ in fact have no dependence on q.

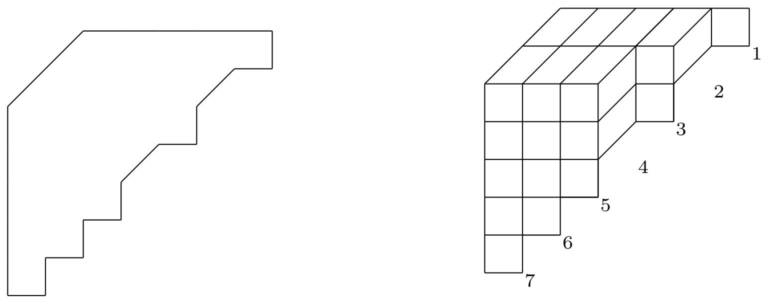


FIGURE 2. $\Gamma(\mu)$ and its distinguished tiling when $\mu = (\bullet, *, \circ, *, \bullet, \bullet, \circ) \in \{\circ, *, \bullet\}^7$, with labels assigned to the tiles on the southeast border.

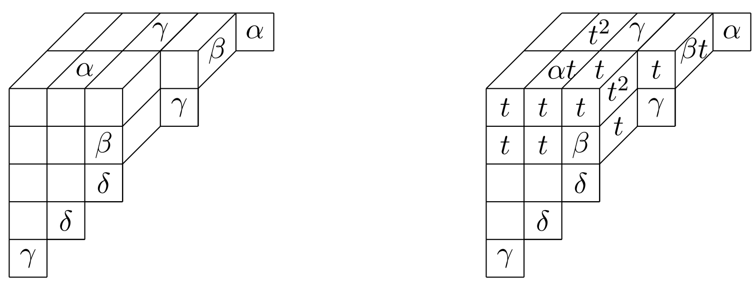


FIGURE 3. On the left, we show a tableau of type $\mu = (\bullet, *, \circ, *, \bullet, \bullet, \circ)$, and on the right we show the weights in t associated to each tile. The total weight of the tableau is $\alpha^2\beta^2\delta^2\gamma^3t^{14}$.

Definition 1.4. A *west-strip* is the maximal contiguous strip consisting of squares and vertical rhombic tiles, where all tiles are adjacent along a vertical edge. Likewise, a *north-strip* is the maximal contiguous strip consisting of squares and horizontal rhombi, where all tiles are adjacent along a horizontal edge. The total number of west- and north-strips in a rhombic diagram of size (N, r) is equal to $N - r$. The *distinguished tiling* of a rhombic diagram is the tiling in which all north strips consist of squares then horizontal rhombi from bottom to top.

See Figure 2 for an example.

Definition 1.5. For $\mu \in \{\circ, *, \bullet\}^N$, a *rhombic staircase tableau (RST)* of *type* μ is a filling of the tiles of $\Gamma(\mu)$ with the letters $\alpha, \beta, \gamma, \delta$ which satisfies the following conditions:

- A square on the southeast border corresponding to $\mu_j = \circ$ must contain a β or a γ.
- A square on the southeast border corresponding to $\mu_j = \bullet$ must contain an α or a δ.
- Each tile is either empty or contains one letter.
- A horizontal rhombus may contain the letters α or γ.
- A vertical rhombus may contain the letters β or δ.
- Every tile in the same north-strip and above α or γ must be empty.
- Every tile in the same west-strip and to the left of β or δ must be empty.

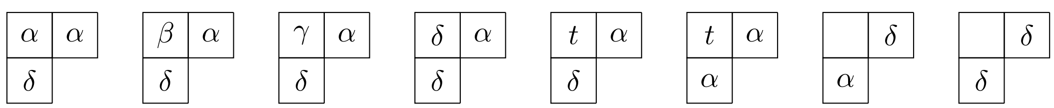
Definition 1.6. The *weight* $wt(T)$ of a rhombic staircase tableau T is a monomial in $\alpha, \beta, \gamma, \delta, t$, obtained by scanning each tile and giving it a weight based on the nearest nonempty tile to its right (*resp.* below) in its west-strip (*resp.* north-strip), if those exist:

- Each horizontal rhombus containing α gets the weight t,
- Each vertical rhombus containing β gets the weight t,
- Each empty square that sees α or γ to its right and α or δ below gets the weight t
- Each empty square that sees β to its right gets the weight t
- Each empty vertical rhombus that sees β to its right gets the weight t^2
- Each empty vertical rhombus that sees α or γ to its right gets the weight t
- Each empty horizontal rhombus that sees α below gets the weight t^2
- Each empty horizontal rhombus that sees β or δ below gets the weight t

Finally, $wt(T)$ is the product of all Greek letters in the filling of T times the product of all weights in t assigned to the tiles according to the rules above. See Figure 3 for an example of a tableau of weight $\alpha^2\beta^2\delta^2\gamma^3t^{14}$.

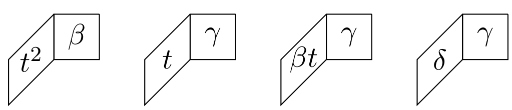
Definition 1.7. Given a word $\mu = (\mu_1, \dots, \mu_N) \in \{-1, 0, 1\}^N = \{\circ, *, \bullet\}^N$, let $R(\mu) = \sum_T wt(T)$, where the sum is over all rhombic staircase tableaux of type μ. In particular, $R(\mu)$ is a polynomial in $\alpha, \beta, \gamma, \delta$ and t.

Example 1.8. The following figure shows all rhombic staircase tableaux of type $\bullet\bullet$:



Thus $R(\bullet\bullet) = \alpha\delta(1 + t + \alpha + \beta + \gamma + \delta) + \alpha^2 t + \delta^2$.

To give a second example, the following figure shows all rhombic staircase tableaux of type $\circ*$:



Thus $R(\circ*) = \beta t^2 + \gamma t + \beta\gamma t + \gamma\delta$.

The main result of [CMW17] was the following.

Theorem 1.9. [CMW17] *Consider the two-species open boundary ASEP on a lattice of N sites, with r second class particles. Let $Z_{N,r} = \sum_\sigma R(\sigma)$, where the sum is over all words $\sigma \in \{-1, 0, 1\}^N$ containing exactly r 0's. Then $Z_{N,r}$ is the partition function for the two-species open boundary ASEP, and for each $\mu \in \{-1, 0, 1\}^N = \{\circ, *, \bullet\}^N$ containing exactly r 0's (or $*$'s), the steady state probability of being in state μ equals*

$$\frac{R(\mu)}{Z_{N,r}}.$$

Using Theorem 1.9, it now follows from the result of Cantini [Can17] that $Z_{N,r}$ is the specialization of a Koornwinder polynomial at $z_1 = \cdots = z_N$ and $q = 1$. This gives rise to the natural question: how can we incorporate the "spectral parameters" $z_1, \dots, z_N$ into the rhombic staircase tableaux, so as to give a formula for the Koornwinder polynomials themselves?

1.4. **Main result.** The above question leads us to our main result, described in Theorem 1.10, which is a formula for each "open boundary ASEP polynomial" $F_\mu(\mathbf{z};t)$, where μ is a composition in $\{-1,0,1\}^N$. We note that when $\mu = ((-1)^{N-r}, 0^r)$, $F_\mu(\mathbf{z};t)$ coincides with the nonsymmetric Koornwinder polynomial $E_\mu(\mathbf{z};t)$. Additionally the sum over all $\mu \in \{-1,0,1\}^N$ with r 0's is the symmetric Koornwinder polynomial $K_{1^{N-r},0^r}(\mathbf{z};t)$, so we get combinatorial formulas for these polynomials as well.

Recall from Definition 1.7 that for $\mu = (\mu_1,\dots,\mu_N) \in \{-1,0,1\}^N = \{\circ, *, \bullet\}^N$, $R(\mu)$ denotes the generating polynomial in the variables $\alpha,\beta,\gamma,\delta$ and t for the rhombic staircase tableaux of type μ. We also let

$$\tilde{R}(\mu) = \frac{(t-1)^{N-r}}{\prod_{i=2r}^{N+r-1}(\alpha\beta t^i - \gamma\delta)} R(\mu), \tag{1.1}$$

where r is the number of zeroes (or $*$) in μ.

Let μ be a word of length $N := |\mu|$, and let $[N] = \{1,\dots,N\}$. For subsets of indices $I, T \subseteq [N]$ and $u = \mu|_I$ a subword of μ restricted to the indices I, define $u|_T := \mu|_{T\cap I}$ to be the maximal subword of u supported on T. For example, if $\mu = \mu_1\mu_2\mu_3\mu_4\mu_5$, $I = \{2,4,5\}$, $u = \mu_2\mu_4\mu_5$, and $T = \{1,2,4\}$, then $u|_T = \mu_2\mu_4$.

We introduce the following change of variables using the parameters a,b,c,d[2]

$$\alpha = \frac{-ac(1-t)}{(a-1)(c-1)}, \qquad \gamma = \frac{1-t}{(a-1)(c-1)}$$
$$\beta = \frac{-bd(1-t)}{(b-1)(d-1)}, \qquad \delta = \frac{1-t}{(b-1)(d-1)} \tag{1.2}$$

Let $W_0 = \langle s_1,\dots,s_N\rangle$ be the finite Weyl group of type C_N, as in Definition 2.1. For $S \subseteq [N]$, we let $\overline{S} = [N] \setminus S$.

Theorem 1.10. *Let $\lambda \in \{1,0\}^N$ be a partition, and let $\boldsymbol{\delta}$ be the signed permutation of λ such that $\boldsymbol{\delta}_1 \le \boldsymbol{\delta}_2 \le \cdots \le \boldsymbol{\delta}_N \le 0$. Choose $\mu = (\mu_1,\dots,\mu_N) \in W_0\cdot\lambda \subset \{-1,0,1\}^N = \{\circ,*,\bullet\}^N$ and let $V = \{i \mid \mu_i \in \{\pm 1\}\}$.*

We define

$$F_\mu(\mathbf{z};t) := \sum_{S\subseteq V} \tilde{R}(\mu|_{\overline{S}}) \cdot \prod_{i\in S}(z_i^{\mu_i} - 1) = \sum_{S\subseteq [N]} \tilde{R}(\mu|_{\overline{S}}) \cdot \prod_{i\in S}(z_i^{\mu_i} - 1). \tag{1.3}$$

Then as μ ranges over $W_0\cdot\lambda$, the Laurent polynomials

$$\{F_\mu(\mathbf{z};t) \mid \mu \in W_0\cdot\lambda\}$$

form a qKZ family *(in the sense of Definition 2.4).*

Moreover, if we use the change of variables from (1.2),

- $F_{\boldsymbol{\delta}}(\mathbf{z};t)$
  equals the nonsymmetric Koornwinder polynomial $E_{\boldsymbol{\delta}}(\mathbf{z};a,b,c,d;t)$, and

[2]This change of variables is nearly the same as [Can17, (9)] but is different from [CGdGW16, (24), (25)]).

- *the symmetric Koornwinder polynomial* K_λ *is equal to*
$$K_\lambda(\mathbf{z}; a, b, c, d; q, t) = \sum_{\mu \in W_0 \cdot \lambda} F_\mu(\mathbf{z}; t),$$
where the sum runs over all distinct signed permutations of λ.

Remark 1.11. We refer to the Laurent polynomials $\{F_\mu(\mathbf{z}; t)\}$ as *open boundary ASEP polynomials*; each F_μ specializes at $z_1 = \cdots = z_N = 1$ to the steady state probability that the two-species ASEP is in state μ.

Remark 1.12. It follows from the definition of F_μ that the coefficient of z^μ in F_μ is 1: we get this term when $S = V$ because $\widetilde{R}(\mu) = 1$ when $\mu = *^r$.

Remark 1.13. We refer to the Laurent polynomials $F_\mu(z_1, \ldots, z_N; t)$ as *open boundary ASEP polynomials* because they are the open boundary analogue of the *ASEP polynomials* (on a ring) which were first studied in [CdGW15, CMW22] (and which were subsequently dubbed "ASEP polynomials" in [CdGW18]). We note that the Laurent polynomials F_μ are related to the nonsymmetric Koornwinder polynomials E_μ via a triangular change of basis, see Proposition 2.9.

Example 1.14. For $N = 2$, the ASEP polynomials are:

- $F_{(1,1)} = \widetilde{R}(\bullet\bullet) + \widetilde{R}(\bullet)(z_1 + z_2 - 2) + (z_1 - 1)(z_2 - 1)$.
- $F_{(1,-1)} = \widetilde{R}(\bullet\circ) + \widetilde{R}(\bullet)(z_2^{-1} - 1) + \widetilde{R}(\circ)(z_1 - 1) + (z_1 - 1)(z_2^{-1} - 1)$.
- $F_{(-1,1)} = \widetilde{R}(\circ\bullet) + \widetilde{R}(\circ)(z_2 - 1) + \widetilde{R}(\bullet)(z_1^{-1} - 1) + (z_1^{-1} - 1)(z_2 - 1)$.
- $F_{(-1,-1)} = \widetilde{R}(\circ\circ) + \widetilde{R}(\circ)(z_1^{-1} + z_2^{-1} - 2) + (z_1^{-1} - 1)(z_2^{-1} - 1)$.
- $F_{(1,0)} = \widetilde{R}(\bullet *) + \widetilde{R}(*)(z_1 - 1)$.
- $F_{(0,1)} = \widetilde{R}(* \bullet) + \widetilde{R}(*)(z_2 - 1)$.
- $F_{(-1,0)} = \widetilde{R}(\circ *) + \widetilde{R}(*)(z_1^{-1} - 1)$.
- $F_{(0,-1)} = \widetilde{R}(* \circ) + \widetilde{R}(*)(z_2^{-1} - 1)$.
- $F_{(0,0)} = \widetilde{R}(**)$.

For example, since $\widetilde{R}(\bullet\bullet) = \frac{(t-1)^2}{(\alpha\beta t - \gamma\delta)(\alpha\beta - \gamma\delta)}\left(\alpha\delta(1 + t + \alpha + \beta + \gamma + \delta) + \alpha^2 t + \delta^2\right)$ and $\widetilde{R}(\bullet) = \frac{t-1}{\alpha\beta - \gamma\delta}(\alpha + \delta)$, we have that

$$F_{(1,1)} = \frac{(t-1)^2}{(\alpha\beta t - \gamma\delta)(\alpha\beta - \gamma\delta)}\left(\alpha\delta(1 + t + \alpha + \beta + \gamma + \delta) + \alpha^2 t + \delta^2\right) + \frac{t-1}{\alpha\beta - \gamma\delta}(\alpha + \delta)(z_1 + z_2 - 2) + (z_1 - 1)(z_2 - 1).$$

We then have that $K_{(1,1)} = F_{(1,1)} + F_{(1,-1)} + F_{(-1,1)} + F_{(-1,-1)}$, and $K_{(1,0)} = F_{(1,0)} + F_{(-1,0)} + F_{(0,1)} + F_{(0,-1)}$.

Using Theorem 1.10, we can also give a simpler combinatorial formula for $K_\lambda(\mathbf{z}; q, t)$, where $\lambda = (1^{N-r}, 0^r)$.

Theorem 1.15. *Let* $\lambda = (1^{N-r}, 0^r)$ *and let*

$$\tilde{Z}_{N,r} = \frac{(t-1)^{N-r}}{\prod_{i=2r}^{N+r-1}(\alpha\beta t^i - \gamma\delta)} Z_{N,r}(\alpha,\beta,\gamma,\delta;t),$$

where $Z_{N,r} = Z_{N,r}(\alpha,\beta,\gamma,\delta;t)$ *is defined in Theorem 1.9 as the generating polynomial for staircase tableaux of size* N *with* r *diagonal steps. Then*

$$K_\lambda(\mathbf{z};q,t) = \sum_{k=0}^{N-r} \tilde{Z}_{N-k,r} \cdot e_k(y_1,\ldots,y_N), \tag{1.4}$$

where $y_i = z_i + 1/z_i - 2$ *for* $1 \le i \le N$ *and* $e_k(y_1,\ldots,y_N)$ *is the elementary symmetric polynomial.*

We now recall that Koornwinder polynomials $K_\lambda(\mathbf{z};q,t)$ factor when $q=1$. This is a Koornwinder analogue of the corresponding factorization for Macdonald polynomials when $q=1$, see (4.1). (In Section 2.4 we will sketch a proof of Proposition 1.16 which we learned from Eric Rains.)

Proposition 1.16 ([Rai24])**.** *Let* $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_N)$ *be any partition. Then at* $q=1$*, we have the following formula for the Koornwinder polynomial* K_λ*:*

$$K_\lambda(\mathbf{z};1,t) = \prod_{i=1}^{\lambda_1} K_{(1^{\lambda'_i},0^{N-\lambda'_i})}(\mathbf{z};t) = \prod_{i=1}^{\lambda_1} K_{1^{\lambda'_i}}(\mathbf{z};t),$$

where λ' *is the partition conjugate to* λ*.*

Proposition 1.16 allows us to give a combinatorial formula for $K_\lambda(\mathbf{z};1,t)$ for any partition λ.

Corollary 1.17. *The Koornwinder polynomials* $K_\lambda(\mathbf{z};q,t)$ *at* $q=1$ *can be written as*

$$K_\lambda(\mathbf{z};1,t) = \sum_\mu e_\mu(y_1,\ldots,y_N) \prod_{i=1}^{\lambda_1} \tilde{Z}_{N-\mu_i,N-\lambda'_i},$$

where $y_i = z_i + 1/z_i - 2$*, and the sum is over compositions* $\mu = (\mu_1,\ldots,\mu_\ell)$ *such that* $0 \le \mu_i \le \lambda'_i$ *for* $1 \le i \le \lambda_1$*.*

We can consider Corollary 1.17 as giving a combinatorial formula for Koornwinder polynomials in terms of sequences of ℓ rhombic staircase tableaux.

The structure of this paper is as follows. In Section 2 we define the affine Hecke algebra of type C, Koornwinder polynomials, and the notion of a qKZ family. In Section 3 we prove our main result by showing that our polynomials $\{F_\mu\}$ form a qKZ family. And in Section 4 we compare this story to its counterpart in type A, and discuss further directions and generalizations.

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2. The Hecke algebra and Koornwinder polynomials

In this section we review the affine Hecke algebra of type C, nonsymmetric and symmetric Koornwinder polynomials, the notion of qKZ family, and we explain the relationship between Koornwinder polynomials and qKZ family. For more background on these topics, see [Lus89], [Koo92], [Sah99], and [NNSY01].

2.1. The affine Hecke algebra of type C.

Definition 2.1. The *affine Weyl group* $\mathcal{W}_N$ of type $\tilde{C}_N$ is the group generated by the elements $s_0, s_1, \ldots, s_N$ subject to the relations

$$\begin{aligned} s_i^2 &= 1 \text{ for all } 0 \le i \le N \\ s_i s_j &= s_j s_i \text{ if } |i-j| > 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ for } 1 \le i \le N-2 \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0 \\ s_N s_{N-1} s_N s_{N-1} &= s_{N-1} s_N s_{N-1} s_N. \end{aligned}$$

The *finite Weyl group of type* C_N is $W_0 = \langle s_1, \ldots, s_N \rangle$.

Given a parameter q, the affine Weyl group $\mathcal{W}_N$ acts on the space $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ of Laurent polynomials in N variables as follows:

$$\begin{aligned} s_i f(z_1, \ldots, z_N) &= f(z_1, \ldots, z_{i+1}, z_i, \ldots, z_N), \qquad 1 \le i \le N-1 \\ s_0 f(z_1, \ldots, z_N) &= f(q z_1^{-1}, z_2, \ldots, z_N) \\ s_N f(z_1, \ldots, z_N) &= f(z_1, \ldots, z_{N-1}, z_N^{-1}) \end{aligned}$$

The affine Hecke algebra $\mathcal{H}_N$ of type $\tilde{C}_N$ is a deformation of the group algebra of $\mathcal{W}_N$, which depends on three parameters t_0, t_N and t. It is generated by elements $T_0, T_1, \ldots, T_N$ subject to the relations

$$T_i - T_i^{-1} = t_i^{1/2} - t_i^{-1/2} \text{ for } 0 \le i \le N \tag{2.1}$$

$$T_i T_j = T_j T_i \text{ if } |i-j| > 1 \tag{2.2}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } 1 \le i \le N-2 \tag{2.3}$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0 \tag{2.4}$$

$$T_N T_{N-1} T_N T_{N-1} = T_{N-1} T_N T_{N-1} T_N \tag{2.5}$$

where $t_1 = t_2 = \cdots = t_{N-1} = t$.

For $1 \le i \le N$, define the operators

$$Y_i = (T_i \ldots T_{N-1})(T_N \ldots T_0)(T_1^{-1} \ldots T_{i-1}^{-1}), \tag{2.6}$$

which form an abelian subalgebra, and hence share a common set of eigenfunctions, which are precisely the nonsymmetric Koornwinder polynomials in the polynomial representation of the Hecke algebra $\mathcal{H}_N$.

The following operators, introduced by Noumi [Nou95], give a polynomial representation of $\mathcal{H}_N$. These operators also appear as the T_i's in [CGdGW16, (73)] and are closely related to the $\hat{T}_i$'s in [Can17] (up to swapping b and c).

Definition 2.2. We fix parameters a, b, c, d, t, q, as well as positive integer N. The following operators act on the space $\mathbb{C}[z_1^{\pm 1}, \dots, z_N^{\pm 1}]$ of Laurent polynomials in N variables.

$$\widetilde{T}_0 = -\frac{ac}{q} - \frac{(z_1 - a)(z_1 - c)}{z_1} \cdot \frac{1 - s_0}{z_1 - qz_1^{-1}} \tag{2.7}$$

$$\widetilde{T}_i = t - (tz_i - z_{i+1}) \cdot \frac{1 - s_i}{z_i - z_{i+1}} \tag{2.8}$$

$$= s_i + \frac{(t-1)(z_i s_i - z_{i+1})}{z_i - z_{i+1}} \qquad \text{for } 1 \leq i < N \tag{2.9}$$

$$\widetilde{T}_N = -bd + \frac{(bz_N - 1)(dz_N - 1)}{z_N} \cdot \frac{1 - s_N}{z_N - z_N^{-1}}. \tag{2.10}$$

It is straightforward to check that these operators satisfy the relations (2.1) to (2.5).

2.2. **Koornwinder polynomials.**

We follow the exposition of [Kas11, Definition 4.1] and [CGdGW16, Definition 1] for the following characterization of nonsymmetric Koornwinder polynomials.

Definition 2.3. [CGdGW16, Definition 1] Let $\lambda \in \mathbb{Z}^N$ be a composition, and let λ^+ denote the unique dominant element in $W_0 \cdot \lambda$, that is, $\lambda_1^+ \geq \lambda_2^+ \geq \dots \geq \lambda_N^+ \geq 0$. Take the shortest element $w \in W_0$ such that $w \cdot \lambda^+ = \lambda$, and denote it by w_λ^+. Let $\rho = (N-1, N-2, \dots, 1, 0)$, and $\rho(\lambda) = w_\lambda^+ \cdot \rho$.

Then the nonsymmetric Koornwinder polynomial E_λ is the unique polynomial which solves the eigenvalue equations

$$Y_i E_\lambda = y_i(\lambda) E_\lambda \text{ for } i = 1 \dots N, \text{ where} \tag{2.11}$$

$$t_0 = -acq^{-1}, \qquad t_N = -bd, \text{ and} \tag{2.12}$$

$$y_i(\lambda) = q^{\lambda_i} t^{N-i+\rho(\lambda)_i} (t_0 t_N)^{\epsilon_i(\lambda)} \text{ and } \epsilon_i(\lambda) = \begin{cases} 1 \text{ if } \lambda_i \geq 0 \\ 0 \text{ if } \lambda_i < 0 \end{cases} \tag{2.13}$$

and whose coefficient of the term $z^\lambda = z_1^{\lambda_1} \dots z_N^{\lambda_N}$ is equal to 1.

Definition 2.4. [Kas11, Definition 3.1] Let $\lambda \in \mathbb{Z}^N$ be a composition. Suppose that for each signed permutation μ of λ, we have a Laurent polynomial $f_\mu \in \mathbb{C}[z_1^{\pm 1}, \dots, z_N^{\pm 1}]$ such that the following relations hold:

$$\widetilde{T}_0 f_{\mu_1,\dots} = q^{\mu_1} f_{-\mu_1,\dots} \text{ if } \mu_1 < 0 \tag{2.14}$$
$$\widetilde{T}_0 f_{\mu_1,\dots} = t_0 f_{\mu_1,\dots} \text{ if } \mu_1 = 0 \tag{2.15}$$
$$\widetilde{T}_i f_{\dots,\mu_i,\mu_{i+1},\dots} = t f_{\dots,\mu_i,\mu_{i+1},\dots} \text{ if } \mu_i = \mu_{i+1} \tag{2.16}$$
$$\widetilde{T}_i f_{\dots,\mu_i,\mu_{i+1},\dots} = f_{\dots,\mu_{i+1},\mu_i,\dots} \text{ if } \mu_i > \mu_{i+1} \tag{2.17}$$
$$\widetilde{T}_N f_{\dots,\mu_N} = t_N f_{\dots,\mu_N} \text{ if } \mu_N = 0 \tag{2.18}$$
$$\widetilde{T}_N f_{\dots,\mu_N} = f_{\dots,-\mu_N} \text{ if } \mu_N > 0 \tag{2.19}$$

Then we call the polynomials $\{f_\mu\}$ a *qKZ family.*

The following result appears in [CGdGW16], see [CGdGW16, Lemma 3] and the discussion that follows. We take Proposition 2.5 as the definition of the symmetric Koornwinder polynomial.

Proposition 2.5. *Let $\lambda = (\lambda_1, \dots, \lambda_N)$ be a partition. Suppose that we have a qKZ family $\{f_\mu\}$ as in Definition 2.4, and suppose further that for $\delta := -\lambda$, we have that f_δ equals the nonsymmetric Koornwinder polynomial E_δ. Then the symmetric Koornwinder polynomial K_λ is equal to*

$$K_\lambda(z_1, \dots, z_N; q, t) = \sum_{\mu \in W_0 \cdot \lambda} f_\mu(z_1, \dots, z_N; q, t),$$

where the sum runs over all distinct signed permutations of λ.

The following result will be a key ingredient in the proof of our main result.

Proposition 2.6. *Let $\lambda \in \mathbb{Z}^N$ be a composition, and let $\boldsymbol{\delta}$ be the signed permutation of λ such that $\boldsymbol{\delta}_1 \leq \boldsymbol{\delta}_2 \leq \dots \leq \boldsymbol{\delta}_N \leq 0$. Suppose that we have a qKZ family $\{f_\mu\}$ as in Definition 2.4. Then*

$$Y_i f_{\boldsymbol{\delta}} = y_i(\boldsymbol{\delta}) f_{\boldsymbol{\delta}}$$

for $i = 1, \dots, N$, i.e. (2.11) holds with $f_{\boldsymbol{\delta}}$ in place of E_λ. Therefore if the coefficient of the term $z^{\boldsymbol{\delta}}$ in $f_{\boldsymbol{\delta}}$ is equal to 1, then $f_{\boldsymbol{\delta}}$ equals the nonsymmetric Koornwinder polynomial $E_{\boldsymbol{\delta}}$.

Example 2.7. As an example, we compute $\tilde{R}(\circ *)$. From (1.3), we have that $F_{\circ *} = (z_1^{-1} - 1)\tilde{R}(*) + \tilde{R}(\circ *) = (z_1^{-1} - 1) + \tilde{R}(\circ *)$. From Proposition 2.6, $Y_1(F_{\circ *}) = y_i F_{\circ *} = q^{-1} F_{\circ *}$, where $Y_1 = T_1 T_2 T_1 T_0$ for $N = 2$. We compute

$$T_1 T_2 T_1 T_0 (z_1^{-1} - 1) = q^{-1} \left((z_1^{-1} - 1) - \frac{(t-1)(\beta t^2 + \gamma\delta + \beta\gamma t + \gamma t)}{\gamma\delta} \right)$$
$$T_1 T_2 T_1 T_0 (\tilde{R}(\circ *)) = \frac{\alpha\beta t^2}{\gamma\delta q} \tilde{R}(\circ *)$$

and thus

$$\begin{aligned} T_1 T_2 T_1 T_0 (F_{\circ *}) &= q^{-1} \left((z_1^{-1} - 1) - \frac{(t-1)(\beta t^2 + \gamma\delta + \beta\gamma t + \gamma t)}{\gamma\delta} + \frac{\alpha\beta t^2}{\gamma\delta} \tilde{R}(\circ *) \right) \\ &= q^{-1} \left((z_1^{-1} - 1) + \tilde{R}(\circ *) \right) \end{aligned}$$

Solving for $\tilde{R}(\circ *)$, we get

$$\tilde{R}(\circ *) = \frac{(t-1)(\beta t^2 + \gamma t + \gamma\beta t + \delta\gamma)}{\alpha\beta t^2 - \gamma\delta},$$

which matches the combinatorial computation for $R(\circ *)$ from Example 1.8.

Proof. We start by writing $\boldsymbol{\delta} = (\lambda_1^{k_1}, \dots, \lambda_r^{k_r}, 0^{N-k})$, where $\lambda_1 < \cdots < \lambda_r < 0$ and $(k_1, \dots, k_r)$ is a (strong) composition of k, i.e. $\sum k_i = k$, and $k_i > 0$ for all i. So $\boldsymbol{\delta}^+ = (-\lambda_1^{k_1}, \dots, -\lambda_r^{k_r}, 0^{N-k})$. The permutation $w_{\boldsymbol{\delta}}^+$ is the signed permutation which divides $\{1, 2, \dots, N\}$ into consecutive blocks of sizes $k_1, k_2, \dots, k_r, N-k$, and on the first r blocks, it reverses then negates the elements, and on the last block, it acts as the identify. (E.g. if $\boldsymbol{\delta} = (-3, -3, -1, -1, -1, 0, 0)$, then $w_{\boldsymbol{\delta}}^+$ is the signed permutation mapping $(1, 2, 3, 4, 5, 6, 7)$ to $(-2, -1, -5, -4, -3, 6, 7)$.) So $\rho(\boldsymbol{\delta})$ is the signed permutation obtained from $\rho = (N-1, N-2, \dots, 1, 0)$ by dividing up the domain $(1, 2, \dots, N)$ into consecutive blocks of sizes $k_1, k_2, \dots, k_r, N-k$, and on the first r blocks, it reverses then negates the values of ρ. In our example, $\rho = (6, 5, 4, 3, 2, 1, 0)$ and $\rho(\boldsymbol{\delta}) = (-5, -6, -2, -3, -4, 1, 0)$.

Thus we have

$$\rho(\boldsymbol{\delta})_i = -(N - 1 - k_1 - \cdots - k_j + (i - k_1 - \cdots - k_{j-1})) = -N - i + 1 + 2(k_1 + \cdots + k_{j-1}) + k_j$$

for $k_1 + \cdots + k_{j-1} < i \leq k_1 + \cdots + k_j$, and for $k < i \leq N$,

$$\rho(\boldsymbol{\delta})_i = N - i.$$

For $1 \leq i \leq k$, let j be such that $\lambda_j = \boldsymbol{\delta}_i$.

$$\begin{aligned}
Y_i f_{(\lambda_1^{k_1}, \dots, \lambda_r^{k_r}, 0^{N-k})} &= (T_i \cdots T_{N-1})(T_N \cdots T_0)(T_1^{-1} \cdots T_{i-1}^{-1}) f_{(\lambda_1^{k_1}, \dots, \lambda_j^{k_j}, \dots, \lambda_r^{k_r}, 0^{N-k})} \\
&= (T_i \cdots T_{N-1})(T_N \cdots T_0) t^{-(i-k_1-\cdots-k_{j-1}-1)} f_{(\lambda_j \lambda_1^{k_1}, \dots, \lambda_j^{k_j-1}, \dots, \lambda_r^{k_r}, 0^{N-k})} && (2.20) \\
&= t^{-(i-k_1-\cdots-k_{j-1}-1)} (T_i \cdots T_{N-1})(T_N \cdots T_1) q^{\boldsymbol{\delta}_i} f_{((-\lambda_j) \lambda_1^{k_1}, \dots, \lambda_j^{k_j-1}, \dots, \lambda_r^{k_r}, 0^{N-k})} && (2.21) \\
&= q^{\boldsymbol{\delta}_i} t^{-(i-k_1-\cdots-k_{j-1}-1)} (T_i \cdots T_{N-1}) T_N f_{(\lambda_1^{k_1}, \dots, \lambda_j^{k_j-1}, \dots, \lambda_r^{k_r}, 0^{N-k}, (-\lambda_j))} && (2.22) \\
&= q^{\boldsymbol{\delta}_i} t^{-(i-k_1-\cdots-k_{j-1}-1)} (T_i \cdots T_{N-1}) f_{(\lambda_1^{k_1}, \dots, \lambda_j^{k_j-1}, \dots, \lambda_r^{k_r}, 0^{N-k}, \lambda_j)} && (2.23) \\
&= q^{\boldsymbol{\delta}_i} t^{-(i-k_1-\cdots-k_{j-1}-1)} t^{k_1+\cdots+k_j-i} f_{(\lambda_1^{k_1}, \dots, \lambda_j^{k_j}, \dots, \lambda_r^{k_r}, 0^{N-k})} && (2.24) \\
&= q^{\boldsymbol{\delta}_i} t^{2(k_1+\cdots+k_{j-1})+k_j-2i+1} f_{\boldsymbol{\delta}},
\end{aligned}$$

where (2.20), (2.22), and (2.24) are due to (2.16), (2.17), and the fact that $-\boldsymbol{\delta}_i > 0 \geq \boldsymbol{\delta}_\ell$ for all ℓ, (2.21) is due to (2.14), and (2.23) is due to (2.19).

Since $\rho(\boldsymbol{\delta})_i = -N + 1 - i + 2(k_1 + \cdots + k_{j-1}) + k_j$, the exponent of t is equal to $N - i + \rho(\boldsymbol{\delta})_i$, so $Y_i f_{\boldsymbol{\delta}} = y_i(\boldsymbol{\delta}) f_{\boldsymbol{\delta}}$ for $1 \leq i \leq k$.

For $k+1 \leq i \leq N$, we have

$$Y_i f_{(\lambda_1^{k_1},\dots,\lambda_r^{k_r},0^{N-k})} = (T_i \cdots T_{N-1})(T_N \cdots T_0) t^{-(i-1-k)} f_{(0,\lambda_1^{k_1},\dots,\lambda_r^{k_r},0^{N-k-1})} \tag{2.25}$$

$$= t^{k+1-i}(T_i \cdots T_{N-1})(T_N \cdots T_1) t_0 f_{(0,\lambda_1^{k_1},\dots,\lambda_r^{k_r},0^{N-k-1})} \tag{2.26}$$

$$= t_0 t^{k+1-i}(T_i \cdots T_{N-1}) T_N t^{N-k-1} f_{(\lambda_1^{k_1},\dots,\lambda_r^{k_r},0^{N-k})} \tag{2.27}$$

$$= t_0 t^{N-i}(T_i \cdots T_{N-1}) t_N f_{(\lambda_1^{k_1},\dots,\lambda_r^{k_r},0^{N-k})} \tag{2.28}$$

$$= t_0 t_N t^{N-i} t^{N-i} f_{\boldsymbol{\delta}} \tag{2.29}$$

(2.30)

where (2.25), (2.26), and (2.29) are due to (2.16) and (2.17), (2.26) is due to (2.15), and (2.28) is due to (2.18).

For $k+1 \leq i \leq N$, we have $y_i(\boldsymbol{\delta}) = t^{N-i+\rho(\boldsymbol{\delta})_i}(t_0 t_N) = t^{2N-2i} t_0 t_N$.

□

2.3. **The monomials in the qKZ family.** We define two partial orders on $\mathbb{Z}^N$.

Definition 2.8. The *dominance order* on compositions in $\mathbb{Z}^N$ is defined as follows: $\mu \geq \nu$ if for $1 \leq j \leq N$ we have $\sum_{i=1}^{j}(\mu_i - \nu_i) \geq 0$. We define a second order $\preceq$ on compositions as follows. Let μ^+ be the unique element in $W_0 \cdot \mu$ such that μ^+ is a partition, i.e. such that $\mu_1 \geq \mu_2 \geq \cdots \geq 0$. Then we say that $\mu \preceq \nu$ if $\mu^+ < \nu^+$, or if $\mu^+ = \nu^+$ and $\mu \leq \nu$.

For example, if $\mu = (-2, 0)$, then the compositions $\nu \in \mathbb{Z}^2$ such that $\nu \preceq \mu$ are:

$$(-2,0),(1,1),(1,-1),(-1,1),(-1,-1),(1,0),(-1,0),(0,1),(0,-1),(0,0).$$

It is well-known [Sah99] (see also [Can17]) that the non-symmetric Koornwinder polynomials $E_\mu(\mathbf{z})$ have the form

$$E_\mu(\mathbf{z}) = \mathbf{z}^\mu + \sum_{\nu \prec \mu} c_\nu \mathbf{z}^\nu.$$

Moreover, it follows from the definitions that the nonsymmetric Koornwinder polynomials E_μ and any qKZ family f_μ are related via a triangular change of basis.

Proposition 2.9. [CGdGW16, Proposition 1] *Let $\{f_\mu\}$ be a qKZ family. Then the nonsymmetric Koornwinder polynomials E_μ and the qKZ family f_μ are related via an invertible triangular change of basis:*

$$E_\mu = \sum_{\nu \preceq \mu} c_{\mu\nu}(q,t) f_\nu, \text{ and } f_\mu = \sum_{\nu \preceq \mu} d_{\mu\nu}(q,t) E_\nu$$

for suitable rational coefficients $c_{\mu\nu}(q,t)$ and $d_{\mu\nu}(q,t)$.

It follows that the Laurent monomials appearing in the support of each f_μ can be characterized as follows.

Corollary 2.10. *Let $\{f_\mu\}$ be a qKZ family. Then $f_\mu(\mathbf{z}; q, t)$ has the form*

$$f_\mu(\mathbf{z}; q, t) = \sum_{\nu \preceq \mu} e_{\mu\nu}(q,t) \mathbf{z}^\nu.$$

2.4. **Factorization of Koornwinder polynomials at $q = 1$.** In this section we sketch the proof of the factorization of Koornwinder polynomials stated in Proposition 1.16. We thank Eric Rains for explaining this argument to us.

Define $\mathbb{T}^M := \{(z_1, \dots, z_M) \in \mathbb{C}^M : |z_1| = \dots = |z_M| = 1\}$ to be the M-dimensional complex torus, denote $\mathbf{z} = z_1, \dots, z_M$, and $dT(\mathbf{z}) := \frac{1}{2^M M!(2\pi i)^M} \frac{dz_1}{z_1} \cdots \frac{dz_M}{z_M}$. The Koornwinder polynomial $K_\lambda(\mathbf{z}; q, b, c, d; q, t)$ is characterized as the unique symmetric Laurent polynomial with leading monomial z^λ that satisfies the orthogonality condition

$$\int_{\mathbb{T}^M} K_\lambda(\mathbf{z}; a, b, c, d; q, t) K_\mu(\mathbf{z}; a, b, c, d; q, t) \Delta^M(\mathbf{z}; a, b, c, d; q, t) dT(\mathbf{z}) = 0$$

for $\mu \neq \lambda$, where $\Delta^M(\mathbf{z}; a, b, c, d; q, t) = \Delta^M(z_1, \dots, z_M; a, b, c, d; q, t)$ is the Koornwinder density.

At $t = 1$, the Koornwinder orthogonality density has the form $\prod_i \Delta(z_i; a, b, c, d; q)$, where Δ is the Askey-Wilson density. This leads to the following.

Claim 2.11. *Let $\lambda = (\lambda_1, \dots, \lambda_N)$. The following quantities are proportional:*

$$K_\lambda(z_1, \dots, z_N; a, b, c, d; q, t = 1) \propto \sum_{\pi \in S_N} \prod_{1 \leq i \leq N} p_{\lambda_i}(z_{\pi(i)}; a, b, c, d; q),$$

where $p_k(z; a, b, c, d; q)$ is the Askey-Wilson polynomial.

The constant of proportionality is the size of the stabilizer of the composition λ. The following theorem comes from [Mim01], see also [Rai05, Theorem 5.18]

Theorem 2.12. *The Koornwinder polynomials satisfy a Cauchy identity*

$$\sum_{\mu \subseteq M^N} (-1)^{NM - |\mu|} K_\mu(x_1, \dots, x_N; a, b, c, d; q, t) K_{N^M - \mu'}(y_1, \dots, y_M; a, b, c, d; t, q)$$
$$= \prod_{1 \leq i \leq N} \prod_{1 \leq j \leq M} (x_i + \frac{1}{x_i} - y_j - \frac{1}{y_j}).$$

(Note that q and t are swapped in the second Koornwinder polynomial above.)

If we fix a partition $\lambda \subseteq M^N$, with conjugate partition denoted λ', then multiply both sides of Theorem 2.12 by $K_{N^M - \lambda'}(y_1, \dots, y_M; a, b, c, d; t, q)$, and integrate over $\mathbb{T}^M$ against the Koornwinder density $\Delta^{(M)}(y_1, \dots, y_M; a, b, c, d; t, q)$, we obtain Corollary 2.13. (For ease of reading, in what follows, we omit the bounds of integration and write $\int f(\mathbf{z})$ to mean $\int_{\mathbb{T}_n} f(\mathbf{z}) dT(\mathbf{z})$.)

Corollary 2.13. *$K_\lambda(x_1, \dots, x_N; a, b, c, d; q, t)$ is proportional to*

$$\int \prod_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M}} (x_i + \frac{1}{x_i} - z_j - \frac{1}{z_j}) K_{N^M - \lambda'}(z_1, \dots, z_M; a, b, c, d; t, q) \Delta^{(M)}(\mathbf{z}; a, b, c, d; t, q)$$

Taking $q \to 1$ in Corollary 2.13, using Claim 2.11 to expand the Koornwinder polynomial in the integrand, and factoring Δ^M, we obtain that $K_\lambda(x_1, \dots, x_N; a, b, c, d; 1, t)$ is proportional to

$$\int \prod_{\substack{1\le i\le N\\1\le j\le M}} (x_i+1/x_i-z_j-1/z_j) \sum_{\pi\in S_M} \prod_{1\le k\le M} p_{(N^M-\lambda')_k}(z_{\pi(k)};a,b,c,d;t) \prod_{1\le k\le M} \Delta(z_k;a,b,c,d;t).$$

Now observing that all $M!$ terms in the sum over $\pi \in S_M$ give the same integral tells us that (assuming $\lambda_1 \le M$)

$$\begin{aligned} &K_\lambda(x_1,\dots,x_N;a,b,c,d;1,t) \propto \\ &\quad \int \prod_{\substack{1\le i\le N\\1\le j\le M}} (x_i+1/x_i-z_j-1/z_j) \left(\prod_{1\le k\le M} p_{(N^M-\lambda')_k}(z_k;a,b,c,d;t)\Delta(z_k;a,b,c,d;t)\right) \\ &\quad = \prod_{1\le j\le M} \int \prod_{1\le i\le N} (x_i+1/x_i-z-1/z) p_{N-\lambda'_{M+1-j}}(z;a,b,c,d;t)\Delta(z;a,b,c,d;t), \end{aligned}$$

where the last equality comes from the Fubini theorem.

The requirement that $\lambda_1 \le M$ can be eliminated by rewriting the univariate integral as a coefficient in an expansion in Askey-Wilson polynomials, and noting that the coefficient of $p_N(z;a,b,c,d;t)$ in $\prod_{1\le j\le N}(z+\frac{1}{z}-x_j-\frac{1}{x_j})$ is 1.

Let $[p_i]F(z)$ denote the coefficient of p_i in the polynomial $F(z)$. We can thus rewrite this as an infinite product:

$$K_\lambda(x_1,\dots,x_N;a,b,c,d;1,t) \propto \prod_{1\le i} [p_{N-\lambda'_i}(z;a,b,c,d;t)] \prod_{1\le j\le N} (z+\frac{1}{z}-x_j-\frac{1}{x_j})$$

In particular, we have

$$K_{1^\ell}(x_1,\dots,x_N;a,b,c,d;1,t) \propto [p_{N-\ell}(z;a,b,c,d;t)] \prod_{1\le j\le N} (z+\frac{1}{z}-x_j-\frac{1}{x_j}).$$

Thus one can write

$$K_\lambda(x_1,\dots,x_N;a,b,c,d;1,t) \propto \prod_{1\le i} K_{1^{\lambda'_i}}(x_1,\dots,x_N;a,b,c,d;1,t),$$

where the constant can be seen to be 1 by comparing the leading terms.

3. The proof of Theorem 1.10

The main ingredient in our proof of Theorem 1.10 is the following theorem, which says that the Laurent polynomials $\{F_\mu(\mathbf{z};t)\}$ are a qKZ family. In what follows, if μ is a word in $\{1,-1,0\}^N$ then we define $||\mu|| = N + r$, where r is the number of 0's in μ.

Theorem 3.1. *Let $\lambda \in \{1,-1,0\}^N$ be a composition. Then if we let μ range over all signed permutations of λ, the collection $\{F_\mu(\mathbf{z};t)\}_\mu \subset \mathbb{C}[z_1^{\pm 1},\dots,z_N^{\pm 1}]$ of Laurent polynomials is a qKZ family, in the sense of Definition 2.4.*

We will prove Theorem 3.1 over the course of this section. Recall that we identify $\bullet = 1$, $\circ = -1$, and $* = 0$. We start by recalling some "Matrix Ansatz"-style relations among the generating polynomials $R(\mu)$ that previously appeared in [CMW17]. It will turn out that the Matrix Ansatz relations are closely related to the action of the Hecke operators.

Theorem 3.2 ([CMW17, Theorem 5.4])**.** *For any words x and y in the letters $\{\circ, *, \bullet\} = \{-1, 0, 1\}$,*

$$tR(x\bullet\circ y) = R(x\circ\bullet y) + \lambda_{||x||+||y||+2}(R(x\bullet y) + R(x\circ y)) \tag{3.1}$$

$$tR(x*\circ y) = R(x\circ * y) + \lambda_{||x||+||y||+3}R(x*y) \tag{3.2}$$

$$tR(x\bullet * y) = R(x*\bullet y) + \lambda_{||x||+||y||+3}R(x*y) \tag{3.3}$$

$$\beta R(x\bullet) = \delta R(x\circ) + \lambda_{||x||+1}R(x) \tag{3.4}$$

$$\alpha R(\circ x) = \gamma R(\bullet x) + \lambda_{||x||+1}R(x) \tag{3.5}$$

where $\lambda_N = (\alpha\beta t^{N-1} - \gamma\delta)$.

We can rewrite Theorem 3.2 in terms of the $\tilde{R}(\mu)$, obtaining the following statements.

Corollary 3.3. *For any words x and y in the letters $\{\circ, *, \bullet\} = \{-1, 0, 1\}$,*

$$t\tilde{R}(x\bullet\circ y) = \tilde{R}(x\circ\bullet y) + (t-1)(\tilde{R}(x\bullet y) + \tilde{R}(x\circ y)) \tag{3.6}$$

$$t\tilde{R}(x*\circ y) = \tilde{R}(x\circ * y) + (t-1)\tilde{R}(x*y) \tag{3.7}$$

$$t\tilde{R}(x\bullet * y) = \tilde{R}(x*\bullet y) + (t-1)\tilde{R}(x*y) \tag{3.8}$$

$$\alpha\tilde{R}(\circ x) = \gamma\tilde{R}(\bullet x) + (t-1)\tilde{R}(x). \tag{3.9}$$

$$\beta\tilde{R}(x\bullet) = \delta\tilde{R}(x\circ) + (t-1)\tilde{R}(x) \tag{3.10}$$

The following lemmas, which are straightforward to prove, will be helpful for working with the operators.

Lemma 3.4. *Let $1 \leq i \leq N-1$, and suppose that G is a polynomial that is independent of z_i and z_{i+1}. Then we have the following.*

$$\widetilde{T}_i\left((z_i - 1)\left(\frac{1}{z_{i+1}} - 1\right)G\right) = (z_{i+1} - 1)\left(\frac{1}{z_i} - 1\right)G. \tag{3.11}$$

$$\widetilde{T}_i\left((z_i - 1)G\right) = \left((z_{i+1} - 1) - (t-1)\right)G. \tag{3.12}$$

$$\widetilde{T}_i\left(\left(\frac{1}{z_{i+1}} - 1\right)G\right) = \left(\left(\frac{1}{z_i} - 1\right) - (t-1)\right)G. \tag{3.13}$$

$$\widetilde{T}_i(G) = tG. \tag{3.14}$$

Lemma 3.5. *Suppose that G is a polynomial that is independent of z_1. Then*

$$q\widetilde{T}_0\left((z_1^{-1} - 1)G\right) = (z_1 - 1)G + \frac{1-t}{\gamma}G. \tag{3.15}$$

$$\widetilde{T}_0\left(G\right) = t_0 G. \tag{3.16}$$

Suppose that G is a polynomial that is independent of z_N. Then

$$\widetilde{T}_N\left((z_N-1)G\right) = (z_N^{-1}-1)G + \frac{1-t}{\delta}G. \tag{3.17}$$

$$\widetilde{T}_N\left(G\right) = t_N G \tag{3.18}$$

Proof. If G is independent of z_1, then

$$\frac{1-s_0}{z_1-qz_1^{-1}}G = 0, \qquad \frac{1-s_0}{z_1-qz_1^{-1}}(z_1^{-1}-1)G = \frac{z_1^{-1}-q^{-1}z_1}{z_1-qz_1^{-1}}G = -q^{-1}G.$$

We also have

$$ac = -\frac{\alpha}{\gamma}, \quad a+c = -\frac{1-t+\alpha-\gamma}{\gamma}, \quad bd = -\frac{\beta}{\delta}, \quad b+d = -\frac{1-t+\beta-\delta}{\delta}.$$

Then

$$\widetilde{T}_0(G) = -acq^{-1}G = t_0 G$$

and

$$\begin{aligned} q\widetilde{T}_0\left((z_1^{-1}-1)G\right) &= -ac(z_1^{-1}-1)G - (z_1-a-c+acz_1^{-1})(-G) \\ &= z_1G + (ac-a-c)G = (z_1-1) + \frac{1-t}{\gamma}G. \end{aligned}$$

If G is independent of z_N, then

$$\frac{1-s_N}{z_N-z_N^{-1}}G = 0, \qquad \frac{1-s_N}{z_N-z_N^{-1}}(z_N-1)G = \frac{z_N-z_N^{-1}}{z_N-z_N^{-1}}G = G.$$

Then

$$\widetilde{T}_N(G) = -bdG = t_N G$$

and

$$\begin{aligned} \widetilde{T}_N\left((z_N-1)G\right) &= -bd(z_N-1)G + (bdz_N-b-d+z_N^{-1})G \\ &= z_N^{-1}G + (bd-b-d)G = (z_N^{-1}-1)G + \frac{1-t}{\delta}G. \end{aligned}$$

□

Our first goal is to prove the following.

Theorem 3.6. *Choose $1 \le i \le N-1$. For any words x and y in the letters $\{\circ, *, \bullet\}$, where $|x| = i-1$, we have that*

$$\widetilde{T}_i(F_{x\bullet\circ y}(\mathbf{z};t)) = F_{x\circ\bullet y}(\mathbf{z};t).$$

Proof. Let $\mu = (\mu_1, \dots, \mu_N) = x\bullet\circ y$ and $|x| = i-1$, so that the $\bullet$ and $\circ$ are in positions i and $i+1$. As before, let $V = \{i \mid \mu_i \in \{\pm 1\}\}$.

In the definition of $F_\mu(\mathbf{z};t)$, we will divide the sum over subsets of $[N]$ into four cases based on whether the subset contains both i and $i+1$, just one of them, or neither. We then get:

$$\begin{aligned} F_{x\bullet\circ y}(\mathbf{z};t) &= (z_i-1)\left(\frac{1}{z_{i+1}}-1\right)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})(y|_{\overline{S}})) \\ &\quad +(z_i-1)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{i\in S}(z_i^{\mu_i}-1)\tilde{R}((x|_{\overline{S}})\circ(y|_{\overline{S}})) \\ &\quad +\left(\frac{1}{z_{i+1}}-1\right)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})\bullet(y|_{\overline{S}})) \\ &\quad +\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})\bullet\circ(y|_{\overline{S}})). \end{aligned}$$

Using (3.11), we get

$$\widetilde{T}_i\left((z_i-1)\left(\frac{1}{z_{i+1}}-1\right)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})(y|_{\overline{S}}))\right)=$$
$$(z_{i+1}-1)\left(\frac{1}{z_i}-1\right)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})(y|_{\overline{S}})).$$

Using (3.12), we get

$$\begin{aligned} &\widetilde{T}_i\left((z_i-1)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})(y|_{\overline{S}})\right)= \\ &(z_{i+1}-1)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})\circ(y|_{\overline{S}})) \\ &-(t-1)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})\circ(y|_{\overline{S}})). \end{aligned}$$

Using (3.13), we get

$$\begin{aligned} &\widetilde{T}_i\left(\left(\frac{1}{z_{i+1}}-1\right)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})(y|_{\overline{S}}))\right)= \\ &\left(\frac{1}{z_i}-1\right)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})\bullet(y|_{\overline{S}})) \\ &-(t-1)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})\bullet(y|_{\overline{S}})). \end{aligned}$$

Using (3.14), we get

$$\widetilde{T}_i \left(\sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1) \tilde{R}(\mu|_{\overline{S}}) \right) = t \sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1) \tilde{R}(\mu|_{\overline{S}}).$$

Therefore

$$\begin{aligned}
\widetilde{T}_i F_{x\bullet\circ y}(\mathbf{z};t) &= (z_{i+1}-1)\left(\frac{1}{z_i}-1\right) \sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1) \tilde{R}((x|_{\overline{S}})(y|_{\overline{S}})) \\
&\quad +(z_{i+1}-1) \sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1) \tilde{R}((x|_{\overline{S}})\circ(y|_{\overline{S}})) \\
&\quad +\left(\frac{1}{z_i}-1\right) \sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1) \tilde{R}((x|_{\overline{S}})\bullet(y|_{\overline{S}})) \\
&\quad + \sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1)(t\tilde{R}((x|_{\overline{S}})\bullet\circ(y|_{\overline{S}})) \\
&\quad -(t-1)(\tilde{R}((x|_{\overline{S}})\bullet(y|_{\overline{S}})) + \tilde{R}((x|_{\overline{S}})\circ(y|_{\overline{S}}))).
\end{aligned}$$

Now we use (3.6) and get that

$$t\tilde{R}((x|_{\overline{S}})\bullet\circ(y|_{\overline{S}})) - (t-1)(\tilde{R}((x|_{\overline{S}})\bullet(y|_{\overline{S}})) + \tilde{R}((x|_{\overline{S}})\circ(y|_{\overline{S}}))) = \tilde{R}((x|_{\overline{S}})\circ\bullet(y|_{\overline{S}})).$$

So

$$\begin{aligned}
\widetilde{T}_i F_{x\bullet\circ y}(\mathbf{z};t) &= \left(\frac{1}{z_i}-1\right)(z_{i+1}-1) \sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1) \tilde{R}((x|_{\overline{S}})(y|_{\overline{S}})) \\
&\quad +(z_{i+1}-1) \sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1) \tilde{R}((x|_{\overline{S}})\circ(y|_{\overline{S}})) \\
&\quad +\left(\frac{1}{z_i}-1\right) \sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1) \tilde{R}((x|_{\overline{S}})\bullet(y|_{\overline{S}})) \\
&\quad + \sum_{S \subseteq V \setminus \{i,i+1\}} \prod_{j \in S} (z_j^{\mu_j} - 1) \tilde{R}((x|_{\overline{S}})\circ\bullet(y|_{\overline{S}})) \\
&= F_{x\circ\bullet y}(\mathbf{z};t),
\end{aligned}$$

as desired. □

We also need to prove the following identities.

Theorem 3.7. *Choose* $1 \le i \le N-1$. *For any words* x *and* y *in the letters* $\{\circ, *, \bullet\}$, *where* $|x| = i-1$, *we have that*

$$\widetilde{T}_i(F_{x*\circ y}(\mathbf{z};t)) = F_{x\circ * y}(\mathbf{z};t) \tag{3.19}$$

and

$$\widetilde{T}_i(F_{x\bullet * y}(\mathbf{z};t)) = F_{x*\bullet y}(\mathbf{z};t). \tag{3.20}$$

Proof. We prove (3.19). Let $\mu = (\mu_1, \ldots, \mu_N) = x{*}{\circ}y$ and $|x| = i-1$, so that the $*$ and $\circ$ are in positions i and $i+1$. As before, let $V = \{i \mid \mu_i \in \{\pm 1\}\}$.

We divide the sum in the definition of $F_\mu(\mathbf{z};t)$ over subsets of $[N]$ into two cases based on whether the subset contains $i+1$ or not. We then get:

$$\begin{aligned} F_{x*\circ y}(\mathbf{z};t) \;&=\; \left(z_{i+1}^{-1} - 1\right) \sum_{S \subseteq V\setminus\{i+1\}} \prod_{j\in S} (z_j^{\mu_j} - 1)\tilde{R}((x|_{\overline{S}}){*}(y|_{\overline{S}})) \\ &\quad + \sum_{S \subseteq V\setminus\{i+1\}} \prod_{j\in S} (z_j^{\mu_j} - 1)\tilde{R}((x|_{\overline{S}}){*}{\circ}(y|_{\overline{S}})). \end{aligned}$$

Applying (3.13) and (3.14) to the two sums in $\widetilde{T}_i(F_\mu(\mathbf{z};t))$, we get

$$\begin{aligned} \widetilde{T}_i(F_{x*\circ y}(\mathbf{z};t)) &= \left(z_i^{-1} - 1\right) \sum_{S \subseteq V\setminus\{i+1\}} \prod_{j\in S} (z_j^{\mu_j} - 1)\tilde{R}((x|_{\overline{S}}){*}(y|_{\overline{S}})) \\ &\quad - (t-1) \sum_{S \subseteq V\setminus\{i+1\}} \prod_{j\in S} (z_j^{\mu_j} - 1)\tilde{R}((x|_{\overline{S}}){*}(y|_{\overline{S}})) \\ &\quad + t \sum_{S \subseteq V\setminus\{i+1\}} \prod_{j\in S} (z_j^{\mu_j} - 1)\tilde{R}((x|_{\overline{S}}){*}{\circ}(y|_{\overline{S}})) \\ &= \left(z_i^{-1} - 1\right) \sum_{S \subseteq V\setminus\{i+1\}} \prod_{j\in S} (z_j^{\mu_j} - 1)\tilde{R}((x|_{\overline{S}}){*}(y|_{\overline{S}})) \\ &\quad + \sum_{S \subseteq V\setminus\{i+1\}} \prod_{j\in S} (z_j^{\mu_j} - 1)(t\tilde{R}((x|_{\overline{S}}){*}{\circ}(y|_{\overline{S}})) - (t-1)\tilde{R}((x|_{\overline{S}}){*}(y|_{\overline{S}}))). \end{aligned}$$

Using (3.7), we get

$$\begin{aligned} \widetilde{T}_i(F_{x*\circ y}(\mathbf{z};t)) = \quad &\left(z_i^{-1} - 1\right) \sum_{S \subseteq V\setminus\{i+1\}} \prod_{j\in S} (z_j^{\mu_j} - 1)\tilde{R}((x|_{\overline{S}}){*}(y|_{\overline{S}})) \\ &+ \sum_{S \subseteq V\setminus\{i+1\}} \prod_{j\in S} (z_j^{\mu_j} - 1)\tilde{R}((x|_{\overline{S}}){\circ}{*}(y|_{\overline{S}})), \end{aligned}$$

which is equal to $F_{x\circ * y}(\mathbf{z};t)$ when written as a sum of the two cases depending on whether or not i is in S.

(3.20) is proved in the same manner. □

Theorem 3.8. *For any word x in the letters $\{\circ, *, \bullet\}$ of length $|x| = N-1$, we have that*

$$q\widetilde{T}_0(F_{\circ x}(\mathbf{z};t)) = F_{\bullet x}(\mathbf{z};t).$$

Proof. Let $\mu = (\mu_1, \ldots, \mu_N) = \circ x$ and $|x| = N-1$, so that $\circ$ is in position 1. As before, let $V = \{i \mid \mu_i \in \{\pm 1\}\}$.

We divide the sum over subsets of $[N]$ in the definition of $F_\mu(\mathbf{z};t)$ into two cases based on whether the subset contains 1 or not. We get:

$$\begin{aligned}F_{\circ x}(\mathbf{z};t) &= \left(z_1^{-1}-1\right)\sum_{S\subseteq V\setminus\{1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\\ &\quad+\sum_{S\subseteq V\setminus\{1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(\circ(x|_{\overline{S}})).\end{aligned}$$

Thus

$$\begin{aligned}q\widetilde{T}_0(F_{\circ x}(\mathbf{z};t)) &= (z_1-1)\sum_{S\subseteq V\setminus\{1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\\ &\quad+\frac{1-t}{\gamma}\sum_{S\subseteq V\setminus\{1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\\ &\quad+\frac{\alpha}{\gamma}\sum_{S\subseteq V\setminus\{1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(\circ(x|_{\overline{S}})).\end{aligned}$$

Now we use (3.9) to get that

$$\tilde{R}(\bullet(x|\overline{S}))=\frac{1}{\gamma}\left(\alpha\tilde{R}(\circ(x|_{\overline{S}}))+(1-t)\tilde{R}(x|_{\overline{S}})\right).$$

Then

$$\begin{aligned}q\widetilde{T}_0(F_{\circ x}(\mathbf{z};t)) &= (z_1-1)\sum_{S\subseteq V\setminus\{1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\\ &\quad+\sum_{S\subseteq V\setminus\{1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\left(\frac{1-t}{\gamma}+\frac{\alpha}{\gamma}\right)\\ &= F_{\bullet x}(\mathbf{z};t).\end{aligned}$$

□

Theorem 3.9. *For any word x in the letters $\{\circ,*,\bullet\}$ of length $|x|=N-1$, we have that*

$$\widetilde{T}_N(F_{x\bullet}(\mathbf{z};t))=F_{x\circ}(\mathbf{z};t).$$

Proof. Let $\mu=(\mu_1,\ldots,\mu_N)=x\bullet$ and $|x|=N-1$, so that $\bullet$ is in position N. As before, let $V=\{i\mid\mu_i\in\{\pm1\}\}$.

We divide the sum over subsets of $[N]$ in the definition of $F_\mu(\mathbf{z};t)$ into two cases based on whether the subset contains N or not. We get:

$$\begin{aligned}F_{x\bullet}(\mathbf{z};t) &= (z_N-1)\sum_{S\subseteq V\setminus\{N\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\\ &\quad+\sum_{S\subseteq V\setminus\{N\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})\bullet).\end{aligned}$$

Thus

$$\begin{aligned}\widetilde{T}_N(F_{x\bullet}(\mathbf{z};t)) &= \left(z_N^{-1}-1\right)\sum_{S\subseteq V\setminus\{N\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\\ &\quad+\frac{1-t}{\delta}\sum_{S\subseteq V\setminus\{N\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\\ &\quad+\frac{\beta}{\delta}\sum_{S\subseteq V\setminus\{N\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}}).\end{aligned}$$

Now we use (3.10) to get that

$$\tilde{R}(x|_{\overline{S}})=\frac{1}{\delta}\left(\beta\tilde{R}((x|_{\overline{S}})\bullet)+(1-t)\tilde{R}(x|_{\overline{S}})\right).$$

Then

$$\begin{aligned}\widetilde{T}_N(F_{x\bullet}(\mathbf{z};t)) &= \left(z_N^{-1}-1\right)\sum_{S\subseteq V\setminus\{N\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\\ &\quad+\sum_{S\subseteq V\setminus\{N\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}(x|_{\overline{S}})\left(\frac{1-t}{\delta}+\frac{\beta}{\delta}\right)\\ &= F_{x\circ}(\mathbf{z};t).\end{aligned}$$

□

We also prove the following identity.

Theorem 3.10. *Choose words x and y in the letters $\{\circ,*,\bullet\}$ where $|x|=i-1$ and additionally choose $a\in\{\circ,*,\bullet\}$. Then*

$$\widetilde{T}_i(F_{xaay}(\mathbf{z};t))=tF_{xaay}(\mathbf{z};t). \tag{3.21}$$

Proof. Let $\mu=(\mu_1,\ldots,\mu_N)=xaay$. If $a=*$, the equality trivially holds by (3.14). Otherwise, we will show $F_{xaay}(\mathbf{z};t)$ is symmetric in variables z_i,z_{i+1}. Let $u=\mu_i=\mu_{i+1}$. Then

$$\begin{aligned}F_{xaay}(\mathbf{z};t) &= (z_i^u-1)(z_{i+1}^u-1)\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})(y|_{\overline{S}}))\\ &\quad+((z_i^u-1)+(z_{i+1}^u-1))\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})a(y|_{\overline{S}}))\\ &\quad+\sum_{S\subseteq V\setminus\{i,i+1\}}\prod_{j\in S}(z_j^{\mu_j}-1)\tilde{R}((x|_{\overline{S}})aa(y|_{\overline{S}})).\end{aligned}$$

Thus $F_{xaay}(\mathbf{z};t)$ is symmetric in z_i,z_{i+1}, so $\frac{1-s_i}{z_i-z_{i+1}}F_{xaay}(\mathbf{z};t)=0$, from which (3.21) follows. □

Finally we are ready to put together our previous results to prove Theorem 3.1.

Proof. First note that Theorem 3.8 proves (2.14). It follows from (1.3) that for any μ with $\mu_i = *$, $\tilde{R}(\mu)$ is independent of z_i. Hence (3.16) implies (2.15) and (3.18) implies (2.18). Finally we have that Theorem 3.10 proves (2.16), Theorems 3.6 and 3.7 proves (2.17), and Theorem 3.9 proves (2.19). □

Now we can put all these ingredients together to complete the proof of the main theorem.

Proof of Theorem 1.10. By Theorem 3.1, the $\{F_\mu(\mathbf{z};t)\}_\mu$ are a qKZ family and that $Y_i F_{\boldsymbol{\delta}} = y_i(\boldsymbol{\delta}) F_{\boldsymbol{\delta}}$ for $i = 1, \dots, N$. Together with Definition 2.3 and the fact that the leading coefficient of $F_{\boldsymbol{\delta}}$ is $\mathbf{z}^{\boldsymbol{\delta}}$ (see Remark 1.12), this implies that $F_{\boldsymbol{\delta}}$ equals the nonsymmetric Koornwinder polynomial $E_{\boldsymbol{\delta}}$. The theorem now follows from Proposition 2.5. □

Proof of Theorem 1.15. The proof of Theorem 1.15 follows from the Theorem 1.10.

$$\begin{aligned}
K_\lambda(\mathbf{z};t) &= \sum_{\mu \in W_0(\lambda)} F_\mu(\mathbf{z};t) \\
&= \sum_{\mu \in W_0(\lambda)} \sum_{S \subseteq [N]} \tilde{R}(\mu|_{\overline{S}}) \prod_{i \in S} (z_i^{\mu_i} - 1) \\
&= \sum_{k=0}^{N-r} \sum_{S \in \binom{[N]}{k}} \sum_{\nu \in W_0(1^{N-k}, 0^r)} \tilde{R}(\nu) \cdot \prod_{i \in S} \big((z_i - 1) + (z_i^{-1} - 1) \big) \\
&= \sum_{k=0}^{N-r} \tilde{Z}_{N-k,r} \cdot e_k(z_1 + z_1^{-1} - 2, \dots, z_N + z_N^{-1} - 2).
\end{aligned}$$

where W_0 is the set of signed permutations of the appropriate cardinality, so e.g. the sum over $\nu \in W_0(1^{N-k}, 0^r)$ indicates that we are summing over all signed permutations of the word $(1^{N-k}, 0^r)$. □

4. Conclusion

In this work we have given a combinatorial formula for the Koornwinder polynomials K_λ associated to partitions $\lambda = (1^{N-r}, 0^r)$ by using the combinatorics of the open boundary two-species ASEP. This work naturally leads to several open problems which we hope to pursue in future work.

A first problem is to find a vertex model analogue of our rhombic staircase tableaux, corresponding to particle models with open boundaries and their associated type $\tilde{C}$ special functions. We note that there has been a great deal of recent work connecting particle models on the ring and (type A) special functions to *vertex models*, see [BW19, ABW21, ANP]. In particular, [CdGW15, Section 6] gave a lattice model interpretation of their matrix product formula for Macdonald polynomials. Subsequently, [ANP] gave a queue vertex model formula for Macdonald polynomials, which encodes the multiline queues of Martin [Mar20], and gives new proofs of some formulas for the stationary distribution of the multispecies ASEP on a ring using the Yang–Baxter equation.

A second problem is to extend our results to give a combinatorial formula for all Koornwinder polynomials. Towards that end, we now briefly sketch how the analogous problem was solved for type A Macdonald polynomials, and what are the difficulties in the Koornwinder case.

Let us refer to the multispecies ASEP on a ring involving particles $\{0, 1, \ldots, m\}$ as the *rank* m multispecies ASEP on a ring. If $m = 1$, the multispecies ASEP on a ring is rather trivial – its stationary distribution is uniform – but for $m > 1$, the stationary distribution becomes more interesting. To get some insight on what happens for $m > 1$, recall that the multispecies ASEP on a ring is connected to the type A Macdonald polynomials. More specifically, the partition function of the rank m multispecies ASEP on a ring is the specialization of a related Macdonald polynomial (with largest part m) at $x_1 = \cdots = x_N = 1$ and $q = 1$ [CdGW15]. Moreover, when $q = 1$, Macdonald polynomials admit the following factorization [Mac95, Chapter VI, Equation (4.14)(vi)]:

$$P_\lambda(\mathbf{x}; 1, t) = \prod_{i \geq 1} P_{(1^{\lambda'_i}, 0^{N-\lambda'_i})}(\mathbf{x}; 1, t) = \prod_{i \geq 1} P_{(1^{\lambda'_i})}(\mathbf{x}; 1, t) = \prod_{i \geq 1} e_{\lambda'_i}(\mathbf{x}), \tag{4.1}$$

where e_μ is the elementary symmetric polynomial. This means that the partition function of the rank m mASEP on a ring is just the product of some elementary symmetric polynomials. Martin's multiline queue formula for the stationary distribution of the mASEP [Mar20] reflects this structure: each multiline queue is built by stacking rows of balls on top of each other, where each row of balls can be thought of as describing a rank 1 ASEP. Once one has the multiline queues with the t statistic, adding the parameters q and x_i to obtain a formula for Macdonald polynomials turns out to be miraculously simple [CMW22].

Returning to the ASEP with open boundaries, let us refer to the multispecies ASEP with particles $\{0, \pm 1, \pm 2, \ldots, \pm m\}$ as the *rank* m open boundary ASEP. Unlike the ASEP on a ring, the rank 1 open boundary ASEP has a very non-trivial stationary distribution. Even in the case of the rank 1 open ASEP with no second class particles, there was a tremendous amount of work on the ASEP [DEHP93, USW04, BE04, DS05, BCE+06, CW07] before a combinatorial formula for the stationary distribution (with all parameters $\alpha, \beta, \gamma, \delta, q$ general) was given in [CW11]; this formula was subsequently extended to the rank 1 case with second class particles in [CMW17], using rhombic staircase tableaux. Similar to the case of the ASEP on a ring, the partition function of the rank m open boundary mASEP is the specialization of a related Koornwinder polynomial (with largest part m) at $x_1 = \cdots = x_N = 1$ and $q = 1$ [CGdGW16]. Moreover, just as Macdonald polynomials factor at $q = 1$, the Koornwinder polynomials admit the factorization from Proposition 1.16:

$$K_\lambda(\mathbf{z}; a, b, c, d; q, t) = \prod_{i \geq 1} K_{(1^{\lambda'_i}, 0^{N-\lambda'_i})}(\mathbf{z}; a, b, c, d; q, t) = \prod_{i \geq 1} K_{1^{\lambda'_i}}(\mathbf{z}; a, b, c, d; q, t). \tag{4.2}$$

In this paper we have given formulas (Theorem 1.10 and Theorem 1.15) in terms of rhombic staircase tableaux for the "rank 1" Koornwinder polynomials K_λ appearing on the right-hand side of (4.2). Using the factorization (4.2), we also gave a combinatorial formula for arbitrary Koornwinder polynomials at $q = 1$ (Corollary 1.17). Thus, one might hope to figure out how to insert the parameter q into Corollary 1.17, so as to give a formula for general Koornwinder polynomials. In particular, by analogy with the Macdonald case, one might hope that there might be a combinatorial formula for arbitrary Koornwinder polynomials where the new combinatorial object is somehow built by stacking rhombic staircase tableaux on top of each other. However, as rhombic staircase tableaux are already quite complicated, and our formulas in Theorem 1.10 and Theorem 1.15 are not that simple, it is so far unclear to us how to define the correct object.

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