Math 141b Problem Set 6

Due March 12, 2019

1. Show that there are continuum many countable models of PA.
   Recall the outline from class:

   - For each set X of primes, use compactness to find a model M_X that has an element m which is divisible by the nth prime iff $n \in X$ (we don't say anything about the nonstandard primes).
   - Given $M \vDash PA$, set

     $spec(M) = \{X \subset \mathbb{N} : X$ is a set of primes and M has an element m_X that is divisible by the n

     If M is countable, show that $spec(M)$ is countable.
   - Conclude that there are continuum many models of PA.

2. Let $\mathbb{N}_{PA}$ be the standard model of PA (so the universe is $\mathbb{N}$ and it has addition,etc. in the language) and let M be a nonstandard model of PA.

   (a) Let $\phi(\mathbf{x})$ be a Δ_0-sentence. Show that, for any $n_1, \ldots, n_k \in \mathbb{N}$, we have

   $$\mathbb{N}_{PA} \vDash \text{“}\phi(n_1, \ldots, n_k)\text{”} \iff M \vDash \text{“}\phi(n_1, \ldots, n_k)\text{”}$$

   (b) Let $\psi(\mathbf{x})$ be a Σ_1-sentence. Show that, for any $n_1, \ldots, n_k \in \mathbb{N}$, we have

   $$\mathbb{N}_{PA} \vDash \text{“}\psi(n_1, \ldots, n_k)\text{”} \implies M \vDash \text{“}\psi(n_1, \ldots, n_k)\text{”}$$

3. Ultraproducts are great! But if we want to move beyond first-order logic, we need to look at particular types of ultrafilter. Call an ultrafilter U on a set I countably complete (or ω_1-complete) iff, for all $\{X_n \in U : n \in \mathbb{N}\}$, $\bigcap_{n\in\mathbb{N}} X_n \in U$. Show one of the following (they're the same idea):

   - If U is a countably complete ultrafilter, then $\prod \mathbb{N}/U$ is isomorphic to $\mathbb{N}$.
   - We say a binary relation R is *well-founded* iff there is no infinite R-decreasing sequence (i.e., there is no x_n for $n \in \mathbb{N}$ such that $x_{n+1} R x_n$ for all n. If U is a countably complete and each $M_i = (X_i, R_i)$ is well-founded, then the ultraproduct $\prod M_i/U$ is well-founded.

   Unfortunately $ZFC+$"there is a countably complete, non-principal ultrafilter" proves $Con(ZFC)$, so it's a strong assumption.

4. We know that nonstandard models of PA must contain a copy of $\mathbb{N}$ followed by a linearly ordered set of $\mathbb{Z}$-chains from our previous work about fragments of arithmetic. Show that the the $\mathbb{Z}$-chains actually form a dense linear ordering (recall PS2, #4).