Math 141b Problem Set 1

Due Feb. 7, 2019

For simplicity, we will assume all our language τ are at most countable.

1. In class, we used the Łoś Test to determine that T_S is complete:

   Łoś Test. *If a theory T with no finite models is categorical in an infinite cardinal, then it is complete.*

   Prove this. Two things from 141a that will help are:

   (a) Recall that a theory T is categorical in a cardinal κ iff every two models of T of size κ are isomorphic. Being isomorphic implies that two structures satisfy the same theory.

   (b) **The Löwehnheim-Skolem Theorem:** If M is an infinite structure in a language τ and κ is an infinite cardinal, then there is a structure N of size κ such that $M \equiv N$ (this relation is *elementary equivalence* and means that the two structures satisfy the same first-order theory).

2. Using our characterization of models of T_S, show that T_S is categorical in every uncountable cardinal.

3. Prove the following test for quantifier elimination by induction on formulas:

   Theorem. *Let T be a (first-order) theory in a language τ. Assume that for every formula $\phi(y_1, \ldots, y_n)$ of the form*

$$\exists x \bigwedge_{1 \leq i < n} \phi_i(x, y_1, \ldots, y_n)$$

   with each ϕ_i atomic or negated atomic (and all free variables listed), there is a quantifier-free formula $\psi(y_1, \ldots, y_n)$ such that

$$T \vDash \forall y_1, \ldots, y_n \left(\phi(y_1, \ldots, y_n) \leftrightarrow \psi(y_1, \ldots, y_n)\right)$$

   Show that T has quantifier elimination.

4. Using that T_S eliminates quantifiers, show that $A \subset \mathbb{N}$ is definable in $\mathbb{N}_S$ iff A is finite or its complement is finite (cofinite).

Review

Let's review the inductive definitions of formulas:

1. **Language** A language τ (also called vocabulary or similarity type) is a collection of function symbols, relations symbols, and constant symbols (the relation and function symbols come with their arity attached). In natural numbers with successor, the language τ_S consists of a unary function symbol S and a constant symbol 0 (and no relation symbols).

2. **Structure** A structure M (also called a model) in a language τ consists of

   - an underlying set or universe (often denoted using the same symbol); and
   - interpretations of the symbols of the language inside the universe.

   In τ_S, a structure M consists of a set X along with a unary function $F : X \to X$ and a distinguished constant $a \in X$. To connect them to the language, we will write $S^M = F$ and $0^M = a$.

3. **Terms** The terms of a language τ are the expressions that the language can form. The set of terms is defined inductively by

   (a) a variable (such as $x, y, \dots$) is a term;

   (b) a constant symbol is a term; and

   (c) if F is an n-ary function symbol in τ and $t_1, \dots, t_n$ are terms, then $F(t_1, \dots, t_n)$ is a term.

   In τ_S, some work can show that all terms are of the form '$S^n 0$' or '$S^n x$' for some variable x.

4. **Formulas** Formulas are the things that the language can express. The set of formulas is defined inductively by

   (a) if R is an n-ary relation or equality and $t_1, \dots, t_n$ are terms, then $R(t_1, \dots, t_n)$ is a formula;

   (b) if $\phi(x_1, \dots, x_n)$ is a formula, then so is $\neg\phi(x_1, \dots, x_n)$;

   (c) if $\phi(x_1, \dots, x_n)$ and $\psi(x_1, \dots, x_n)$ are formulas, then so are $\phi \wedge \psi(x_1, \dots, x_n)$ and $\phi \vee \psi(x_1, \dots, x_n)$; and

   (d) if $\phi(y, x_1, \dots, x_n)$ is a formula, then so are $\forall y \phi(y, x_1, \dots, x_n)$ and $\exists y \phi(y, x_1, \dots, x_n)$.

   A free variable in a formula ϕ is a variable that is used but never quantified over. If a formula has no free variables, then it is a sentence. All the formulas listed in T_S are sentences.

The formulas listed in the first item are called the atomic formulas. A formula built without quantification (the fourth item) is called a quantifier-free formula.

5. **Evaluating Truth** Structures are semantic objects and formulas are syntactic objects, and these are brought together by evaluating formulas in structures. Let M be a τ-structure, $\phi(x_1, \ldots, x_n)$ a τ-formula, and $m_1, \ldots, m_n \in M$. We want to evaluate if the formula ϕ holds in M when substituting $m_1, \ldots, m_n$ for the free variables $x_1, \ldots, x_n$. If this is the case, we write

$$M \vDash \phi(m_1, \ldots, m_n)$$

The formal definition of this is an inductive definition going through the inductive definitions of terms and formula; in the end, it is precisely the procedure you might guess. For instance, in the structure $(\mathbb{N}, S, 0)$ with the formula

$$\text{“}\exists z(SSSx = z \rightarrow (S0 = y))\text{”}$$

and the tuple $7, 1$, we have $\mathbb{N}_S \vDash \phi(7, 3)$ iff there is a $z \in \mathbb{N}$ such $SSS7 = 10 = z$ implies $S0 = 1 = y$. This is of course true by taking $z = 4$.