Fourier Series

1. **Pre-warm-up:** Think back about how nice orthonormal bases were. Also, if you've seen them, fondly recall how Taylor series allowed you to write any smooth function as an "infinite degree" polynomial.

2. **Warm-up:** What are the eigenvalues and associated eigenfunctions of D^2 on C^∞_{per}?

Warning!

- From now on, C^∞_{per} is only *real-valued*, smooth, 2π-periodic functions.
- "Smooth" is not always going to be true, but close enough (piecewise smooth if you're curious).
- Might take a non-periodic $f(x)$ and pretend it's 2π-periodic by just looking at it's values on the interval $[-\pi, \pi]$.
  - (Formally, this means replacing $f(x)$ by $f_{per}(x)$ which is defined to be

    $$f_{per}(x) = f(x - 2n\pi)$$

    when x is in the interval $[(2n-1)\pi, (2n+1)\pi]$ for integer n.)

We define the *inner product* on C^{∞}_{per} to be

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$

(a) The length of a function f is $|f| = \sqrt{\langle f, f\rangle}$.

(b) The angle θ between two functions f, g satisfies $\cos\theta = \frac{\langle f,g\rangle}{|f||g|}$.

(c) f and g are orthogonal when $\langle f, g\rangle = 0$.

Given a function $f(x)$ in C^{∞}_{per}, we can write it as a Fourier series

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

where

(a) $a_0 = \langle f, \frac{1}{\sqrt{2}}\rangle = \frac{1}{\pi\sqrt{2}}\int_{-\pi}^{\pi} f(x)dx$

(b) $a_n = \langle f, \cos nx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$

(c) $b_n = \langle f, \sin nx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$

3. Find the Fourier series of $\sin^2 x$.

4. Find the Fourier series of $f(x)$ where

$$f(x) = \begin{cases} 0 & x \text{ in } [\frac{\pi}{2}, \pi] \\ 1 & x \text{ in } [0, \frac{\pi}{2}] \\ 0 & x \text{ in } [\frac{-\pi}{2}, 0] \\ -1 & x \text{ in } [-\pi, \frac{-\pi}{2}] \end{cases}$$

(a)
- $f(x)$ is called *even* if $f(x) = f(-x)$. If $f(x)$ is even, then $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$.
- If $f(x)$ is even, then the Fourier series for $f(x)$ is a cos-series, that is,

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos nx$$

(b)
- $f(x)$ is called *odd* if $-f(x) = f(-x)$. If $f(x)$ is odd, then $\int_{-L}^{L} f(x)dx = 0$.
- If $f(x)$ is odd, then the Fourier series for $f(x)$ is a sin-series, that is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

5. Find the Fourier series of $f(x) = x$. What happens when we evaluate this at $x = \frac{\pi}{2}$?

Parseval's Identity: If $f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$, then

$$|f|^2 = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

6. Find $\int_{-\pi}^{\pi} \sin^4 x dx$ using Parseval's identity.

7. Compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.