Symmetric Matrices and the Spectral Theorem

1. Which rotation-dilations have trajectories that are circles?

- A matrix A is **symmetric** when
- A matrix A is **anti-symmetric** or skew-symmetric when

2. Let A be a symmetric matrix with real entries.

   (a) What is the relation between $(A\vec{x}) \cdot \vec{y}$ and $\vec{x} \cdot (A\vec{y})$?

   (b) Show that the eigenvalues of A are real.

   (c) Show that the eigenspaces of distinct eigenvalues are orthogonal.

(d) Argue that A is diagonlizable.

(e) Why does the previous argument not apply to the shear $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$?

3. Suppose that A is diagonalizable in an orthonormal basis; that is, there is an orthogonal matrix S and diagonal matrix D such that $A = SDS^{-1}$. Show that A is symmetric.

The last two problems have proved the **Spectral Theorem**: a matrix with real values is diagonalizable in an orthonormal basis exactly when it is symmetric.

4. True or false: if A is a 17×13 matrix, then $A^T A$ is diagonalizable over $\mathbb{R}$.

5. Let $A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}$.

   (a) Is A diagonalizable over $\mathbb{R}$? If so, find a diagonal matrix that A is similar to.

   (b) Describe the linear transformation $T(\vec{x}) = A\vec{x}$ geometrically.

6. Diagonalize the symmetric matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ with an orthogonal matrix.

7. Diagonalize the symmetric matrix $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ with an orthogonal matrix.