Complex Numbers and Stability

1. **Warm-up:**

   (a) Suppose $\vec{x}$ is an eigenvector of A with eigenvalue λ_1 and also an eigenvector of B with eigenvalue λ_2. Find an eigenvector of $A+B$ and its corresponding eigenvalue.

   (b) Suppose that $\vec{x}$ is an eigenvector of A with eigenvalue λ. Find an eigenvector of A^2 and its corresponding eigenvalue.

2. What are the eigenvalues of the rotation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$?

Some helpful formulas about complex numbers:

- $re^{i\theta} = r(\cos\theta + i\sin\theta)$
- $(re^{i\theta})^n = r^n(\cos n\theta + i\sin n\theta)$
- $\log(re^{i\theta}) = (\log r) + i\theta$
- $\overline{a+bi} = a - bi$
- $\overline{re^{i\theta}} = re^{-i\theta} = r(\cos\theta - i\sin\theta)$

Given complex vectors $\vec{a}$ and $\vec{b}$, we define the dot product of $\vec{a}$ and $\vec{b}$ as

$$\vec{a}\cdot\vec{b} = \sum \overline{a_i} b_i = \overline{\vec{a}}^T \vec{b}$$

3. What are the 5th roots of unity? That is, what are the solutions to $\lambda^5 = 1$?

4. The new dot product requires some slight changes to the properties of the dot product. Let $\vec{a}$ and $\vec{b}$ be vectors with complex values (this includes all of $\mathbb{R}^n$!). Use the definition of dot product to write each of $\vec{b} \cdot \vec{a}$ and $(z\vec{a}) \cdot \vec{b}$ and $\vec{a} \cdot (z\vec{b})$ in terms of $\vec{a} \cdot \vec{b}$.

5. We explore why we make the above change to the dot product for complex values. Set $\vec{x} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

   (a) What is $\vec{x}^T \vec{x}$?

   (b) If we did not add the conjugation to the dot product, what would the length of $\vec{x}$ be?

   (c) What is the length of $\vec{x}$?

6. Set $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ to be the counterclockwise rotation by θ. Find the eigenvalues and eigenvectors of A.

- If complex number z is a root of the polynomial $p(x)$ that has real coefficients, then $\bar{z}$ is also a root.
- If λ is an eigenvalue with eigenvector $\vec{x}$ of a matrix A that has real entries, then $\bar{\lambda}$ is also an eigenvalue of A with eigenvector $\bar{\vec{x}}$.

7. Set $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(a) Compute A^2 and A^3.

(b) Find the eigenvalues and eigenvectors of A.

(c) For a real c, find the eigenvalues and eigenvectors of $A + cI_3$.

(d) Find the eigenvalues and eigenvectors of $3A^2 + 2A + I_3$.

8. Identify the complex number $x + iy$ with the rotation dilation matrix $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$.

(a) Show that this identification agrees with addition and multiplication.

(b) What are the eigenvalues and eigenvectors of $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$?

- A discrete dynamical system with transition matrix A called **asymptotically stable** when, given any starting $\vec{x}(0)$, the trajectory $\vec{x}(t) \to 0$ as $t \to \infty$.
- A good test for A to determine if it is asymptotically stable is

9. Recall the discrete dynamical system for owls and rats. The transition matrix was $A = \begin{bmatrix} .5 & .3 \\ -0.2 & 1.2 \end{bmatrix}$. Is this system asymptoticaly stable?

10. Consider the almost shear $\begin{bmatrix} 0.999 & 1000 \\ 0 & .9 \end{bmatrix}$.

   (a) Describe the transformation geometrically.

   (b) Is this system asymptotically stable?

   (c) Consider the trajectory that starts at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What is the first t such that $|\vec{x}(t)| < \frac{1}{2}$?

11. Consider the discrete dynamical system whose transition matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Show that the trajectories of this system lie on a circle. Is this system asymptotically stable?