Finding the Eigenvalues and Eigenvectors of a Matrix

1. The Fibonacci sequence is defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Express this as a discrete dynamical system.

Let A be an $n \times n$ matrix.

- An eigenvalue λ of A is
- An eigenvector of A is
- The eigenspace E_λ is
- The characteristic polynomial of A is

2. Find the eigenvalues and the corresponding eigenvectors of $\begin{bmatrix} 3 & -2 & 6 \\ 1 & 0 & 10 \\ 0 & 0 & 7 \end{bmatrix}$. Use this to find an *eigenbasis*, a basis for $\mathbb{R}^3$ consisting of eigenvectors.

3. In this problem, we will use eigenvalues and eigenvectors to find a closed form for the Fibonacci number f_n.

   (a) Find the eigenvalues and corresponding eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

   (b) Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in terms of an eigenbasis for A.

   (c) Find an expression for f_n.

4. In a California redwood forest, the spotted owl is the primary predator of the wood rat. Suppose that the following system models the interaction between owls and rats. Here, $o(t)$ represents the owl population at time t and $r(t)$ represents the rat population at time t.

$$\begin{cases} o(t+1) = \quad 0.5o(t) + 0.3r(t) \\ r(t+1) = -0.2o(t) + 1.2r(t) \end{cases}$$

(a) Let $\vec{x}(t) = \begin{bmatrix} o(t) \\ r(t) \end{bmatrix}$. Find a matrix A so that $\vec{x}(t+1) = A\vec{x}(t)$.

(b) Find the eigenvalues and eigenvectors of A.

(c) Suppose there are currently 2000 owls and 1000 rats. How many owls and rats are there in t years?

(d) Suppose there are currently 100 owls and 200 rats. How many owls and rats are there in t years?

5. Find the eigenvalues and eigenvectors $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$.

6. Let $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, which represents a shear. Find all eigenvalues and eigenvectors of A. Is there an eigenbasis for A?

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A

- The algebraic multiplicity of λ is
- The geometric multiplicity of λ is
- $tr(A) =$
- The relationship between the eigenvalues of A and $tr(A)$ is
- The relationship between the eigenvalues of A and $det(A)$ is

7. Let V be a plane in $\mathbb{R}^3$ which contains the origin, and let A be the matrix of proj_V.

(a) Find the eigenvalues of A, and describe the eigenspaces geometrically. Is there an eigenbasis for A?

(b) Can you find the characteristic polynomial of A? Can you determine the algebraic multiplicity of each eigenvalue?

8. Find the eigenvalues of $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ \frac{9}{4} & \frac{3}{2} & \frac{9}{4} \end{bmatrix}$. There is a way to do this without finding the characteristic polynomial.

9. Let R be the matrix of rotation counter-clockwise by 90° in $\mathbb{R}^2$. What do you think the eigenvalues and eigenvectors of R are? Confirm by calculating the characteristic polynomial of R.

10. True or false: if $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis for A, then $(\vec{v}_1, \ldots, \vec{v}_n)$ is an eigenbasis for A^2.

11. Let A be a noninvertible $n \times n$ matrix. Explain why 0 must be an eigenvalue of A. Find the geometric multiplicity of the eigenvalue 0 in terms of $\text{rank}(A)$.