More Determinants

- Problems 1-3 are determinant problems we will cover as a class
- Problems 4-10 are for you to practice computing determinants
- Problems 11-15 are to practice using the properties of determinants

1. Find the determinant of an $n \times n$ triangular matrix (entry in the i row and j column is 0 when $j < i$)

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

2. Let A be the matrix $\begin{bmatrix} 1 & -1 & 3 \\ 4 & 5 & 7 \\ 0 & 2 & 0 \end{bmatrix}$.

   (a) Find the determinant by expanding along the first row.

   (b) Find the determinant by expanding along the third row.

3. Find the determinant of $\begin{bmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{bmatrix}$ by row-reducing to a triangular matrix.

4. For what values of λ is the matrix $\begin{bmatrix} 3-\lambda & -2 & 6 \\ 1 & -\lambda & 10 \\ 0 & 0 & 7-\lambda \end{bmatrix}$ invertible? The symbol λ is the Greek letter "lambda." We will return to this style of problem after spring break to compute *eigenvalues*.

5. Calculate $\det \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 7 & 1 & 0 \\ 1 & 2 & 4 & 1 \\ 5 & 10 & 2 & 5 \end{bmatrix}$.

6. Find $\det \begin{bmatrix} 6 & 0 & 1 & 0 & 0 \\ 9 & 3 & 2 & 3 & 7 \\ 8 & 0 & 3 & 2 & 9 \\ 0 & 0 & 4 & 0 & 0 \\ 5 & 0 & 5 & 0 & 1 \end{bmatrix}$.

7. Set $A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 3 & -5 \end{bmatrix}$.

(a) Find $\det A$.

(b) Find $\det B$.

(c) Find $\det AB$.

(d) What is the relationship between the previous three answers? Can you generalize to any product of 2×2 matrices?

8. (a) Find the determinant of the upper triangular matrix $\begin{bmatrix} 2 & 7 & 10 & 0 & 3 \\ 0 & -7 & -10 & -9 & 7 \\ 0 & 0 & 8 & 3 & -10 \\ 0 & 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$.

(b) Find the determinant of the lower triangular matrix $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 7 & -7 & 0 & 0 & 0 \\ 10 & -10 & 8 & 0 & 0 \\ 0 & -9 & 3 & -3 & 0 \\ 3 & 7 & -10 & -5 & 5 \end{bmatrix}$.

(c) Generalize: For any square matrix A, how does the determinant of A^T relate to $\det A$?

(d) Find the determinant of the matrix $\begin{bmatrix} 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 8 & 3 & -10 \\ 2 & 7 & 10 & 0 & 3 \\ 0 & -7 & -10 & -9 & 7 \\ 0 & 0 & 0 & -3 & -5 \end{bmatrix}$ with minimal computation.

9. Set $A = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 7 & 1 & 0 \\ 1 & 2 & 4 & 1 \\ 5 & 10 & 2 & 5 \end{bmatrix}$. For which values of the scalar λ does the equation $A\vec{x} = \lambda\vec{x}$ have a unique solution x? *Hint: What is the relation with this problem and number #4?*

10. Let A and B be square matrices and form the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Find the determinant of $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ in terms of $\det A$ and $\det B$,

Let A and B be $n \times n$ matrices and c be a scalar.

(a) $\det(A^T) =$

(b) $\det(AB) =$

(c) $\det(A^{-1}) =$

(d) $\det(A^k) =$

(e) $\det(cA) =$

(f) $|\det(A)| =$

11. If a matrix has two equal rows, what can you say about its determinant?

12. We say that a matrix A is *skew-symmetric* if $A^T = -A$.

    (a) If A is a 5×5 skew-symmetric matrix, what can you say about $\det A$?

    (b) If A is a 4×4 skew-symmetric matrix, can you draw the same conclusion as in the previous part?

13. True or false: The function $f : \mathbb{R}^4 \to \mathbb{R}$ defined by $f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \det \begin{bmatrix} x_1 & 1 & 3 & 7 \\ x_2 & 4 & 2 & 5 \\ x_3 & -3 & 6 & -2 \\ x_4 & 1 & -4 & 3 \end{bmatrix}$ is a linear transformation.

14. The absolute value of the determinant of an $n \times n$ matrix A corresponds to the n-dimensional volume of a parallelpiped in $\mathbb{R}^n$. Use this to geometrically argue that the determinant of a non-invertible matrix is 0. *Hint: Look at dimensions 2 and 3 and try to generalize from there.*

15. We have said that matrices A and B are similar if there is an invertible matrix S such that $A = SBS^{-1}$. If A and B are similar, what is the relationship between their matrices?