The Gram-Schmidt Process and Orthogonal Matrices

- Problems 1-3 cover the Gram-Schmidt process and finding QR-decomposition of matrices
- Problems 4-6 cover the transpose of matrices
- Problem 7 covers orthogonal transformations

For a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ of V in $\mathbb{R}^n$, we find an orthonormal basis $\mathcal{B}^* = \{\vec{u}_1, \ldots, \vec{u}_k\}$ for V by the following process:

1. Step 1:
   - $\vec{w}_1 =$
   - $\vec{u}_1 =$
   - $V_1 =$
2. Step 2:
   - $\vec{w}_2 =$
   - $\vec{u}_2 =$
   - $V_2 =$
3. Step i:
   - (from previous step) $V_{i-1} =$
   - $\vec{w}_i =$
   - $\vec{u}_i =$
   - $V_i =$

1. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}$; these three vectors are linearly independent. Let V be the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

   Find an orthonormal basis of V.

2. Using the calculations from **??**, find the QR decomposition of $\begin{bmatrix} 1 & 3 & 1 \\ 1 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$.

3. Find the QR-decomposition of $\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

4. Suppose A is an $n \times m$ matrix and B is an $m \times p$ matrix. Decide whether each of the following expressions makes sense; if so, what size is the matrix?

   (a) $(AB)^T$ (b) $A^T B^T$ (c) $B^T A^T$

5. If A is an $n \times m$ matrix, what is the relationship between $\text{rank}(A^T)$ and $\text{rank}(A)$?

6. Suppose M is an invertible matrix. Is M^T necessarily invertible? If so, what can you say about its inverse? If not, why not?

7. Which of the following matrices are orthogonal? For each, justify your answer; see if you can come up with algebraic and geometric reasons for each.

(a) The horizontal shear $\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

(b) The dilation $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$

(d) The rotation by θ $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

(e) The projection $\begin{bmatrix} \frac{1}{7} & -\frac{8}{21} & \frac{10}{21} \\ -\frac{6}{7} & \frac{13}{21} & \frac{10}{21} \\ -\frac{3}{7} & -\frac{4}{21} & \frac{26}{21} \end{bmatrix}$

(f) The reflection $\begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & -\frac{3}{5} \end{bmatrix}$

(g) Any reflection