Linear Spaces

Key ideas when looking at $\mathbb{R}^{\mathbf{n}}$ and its subspaces:

- Span of vectors $\vec{v}_1, \ldots, \vec{v}_k$
- Linear independence of vectors $\vec{v}_1, \ldots, \vec{v}_k$
- Basis of a subspace
- Dimension of a subspace
- Linear transformations $\mathbb{R}^m \to \mathbb{R}^n$

Definition 4.1.1. A linear space V is a set endowed with addition and scalar multiplication such that the following eight rules are satisfied for all $\vec{f}, \vec{g}, \vec{h} \in V$ and all c, c_1, c_2 in $\mathbb{R}$:

1. $(\vec{f} + \vec{g}) + \vec{h} = \vec{f} + (\vec{g} + \vec{h})$.
2. $\vec{f} + \vec{g} = \vec{g} + \vec{f}$.
3. There exists a neutral element $\vec{0}$ in V such that $\vec{f} + \vec{0} = \vec{f}$ for all $f \in V$. We usually call $\vec{0}$ the zero element. (*Note:* The scalar 0 multiplied by any element of V is equal to $\vec{0}$.)
4. For each $\vec{f} \in V$, there exists $\vec{g} \in V$ such that $\vec{f} + \vec{g} = \vec{0}$. We write $\vec{g}$ as $-\vec{f}$. (*Note:* The scalar -1 multiplied by $\vec{f}$ is $-\vec{f}$.)
5. $c(f + g) = cf + cg$.
6. $(c_1 + c_2)f = c_1 f + c_2 f$.
7. $c_1(c_2 f) = (c_1 c_2) f$.
8. $1f = f$.

1. In each part, decide whether the set is a linear space. If it is, find the zero element (also known as the neutral element).

   (a) $M(n, m)$, the set of $n \times m$ matrices.

(b) The set of 2×2 matrices with positive entries.

(c) The set V of polynomials of the form $ax^2 + bx + c$ where $a \geq c$.

(d) C^∞, the set of smooth functions from $\mathbb{R}$ to $\mathbb{R}$. (A smooth function is a function that can be differentiated as many times as you like; in other words, the n-th derivative $f^{(n)}$ is differentiable for every n.)

2. Which of the following are subspaces of C^∞?

(a) The set U of all smooth functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(5) = 1$.

(b) The set V of all smooth functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(7) = f'(4)$.

(c) The set W of all smooth functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(4) \geq 0$.

(d) P, the set of all polynomials.

3. Consider P_3, the linear space of polynomials of degree ≤ 3.

   (a) Are the elements $1, x, x^2, x^3$ of P_3 linearly independent?

   (b) Do $1, x, x^2, x^3$ span P_3?

   (c) Do $1, x, x^2, x^3$ form a basis of P_3? What is the dimension of P_3?

4. Find a basis of $M(2, 2)$. What is the dimension of $M(2, 2)$?

5. (a) Let b and c be constants, and let V be the set of smooth solutions of the differential equation $f'' + bf' + cf = 0$. Show that V is a subspace of C^∞.

   (b) Let S be the subspace of C^∞ consisting of all smooth solutions of $f'' - 6f' + 9f = 0$. What would a basis for S look like? (Later, we will return to the problem of finding a basis for this linear subspace.)

6. Find the dimension of P_n, the linear space of polynomials of degree at most n. What can you say about the dimension of P, the set of all polynomials?

7. Let V be the subspace of P_3 consisting of all polynomials $f(t)$ in P_3 with the property that $f(1) = f(2)$. Find a basis of V.

8. True or false: $\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$ is a basis of $M(2,2)$.

9. Let X be the set of all 2×2 matrices A that commute with $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. (Remember that we say A commutes with B if $AB = BA$.)

   (a) Show that X is a linear space.

   (b) Find a basis of X.