Changing Coordinates for Fun and Profit

1. The vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \in \mathbb{R}^4$ are linearly independent. Find a basis for $\mathbb{R}^4$ containing them.

2. Let $A = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$.

   (a) Find $\dim \operatorname{im}(A)$.

   (b) Use the Rank-Nullity Theorem to compute $\dim \ker(A)$.

   (c) Directly compute $\dim \ker(A)$ to verify the above answer.

3. (a) Can you find a collection of vectors $\mathcal{B} \subset \mathbb{R}^n$ containing $\vec{0}$ that is linearly independent.

(b) What is $\dim\{\vec{0}\}$?

4. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$.

(a) Find the matrix S that maps $\mathcal{B}$-coordinates to the vector in $\mathbb{R}^2$ they identify.

(b) Find the matrix S^{-1} that maps $\vec{x} \in \mathbb{R}^2$ to its $\mathcal{B}$-coordinates.

(c) If a point $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is on the line $y = 3x$, what are its $\mathcal{B}$-coordinates?

(d) Find the matrix of the orthogonal projection onto $y = 3x$ in $\mathbb{R}^2$.

(e) What about the vectors in $\mathcal{B}$ were crucial to make this work.

5. Let $V = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right)$. Find the matrix that represents reflection across the plane V in $\mathbb{R}^4$.

6. Let V be the line through the origin containing $\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$. Find the matrix that represents rotation around V that is clockwise when V is pointing up.