Inverses and Images and Kernels, Oh My!

Warm-up: Let $C = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{bmatrix}$ and recall $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find a matrix D such that $CD = DC = I_3$.
Hint: Remember the homework you just turned in.

1. Let $A = \begin{bmatrix} -1 & -4 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & -4 \\ 1 & 1 \end{bmatrix}$.

   (a) Find AB and BA. What can you say about the relation between A and B.

   (b) Show $rref(A) = I_2$.

   (c) Use Gauss-Jordan elimination on $[A|I_2]$ to get $\left[I_2|A^{-1}\right]$.

2. Show that the matrix $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ is invertible and find its inverse.

3. Is the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ invertible?

4. Recall that the matrix of a counterclockwise rotation by θ in $\mathbb{R}^2$ is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   (a) Show A is invertible and find A^{-1}.

(b) What transformation does this correspond to?

5. The matrix of the projection onto the line $y = 3x$ in $\mathbb{R}^2$ is $\frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$. Does this matrix have an inverse? Give an algebraic justification and a geometric justification.

6. Suppose A and B are invertible $n \times n$ matrices. Is AB invertible? Provide justification.

7. For each of the following matrices, find the image and the kernel.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$

(c) $C = \begin{bmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{bmatrix}$

(d) $D = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix}$

8. Let A be an $n \times n$ matrix.

   (a) Show that $im(A^2) \subset im(A)$.

   (b) Find a matrix A so that $im(A^2) \neq im(A)$.

   (c) Show that $ker(A) \subset ker(A^2)$.

   (d) Find a matrix A so that $ker(A) \neq ker(A^2)$.

9. Let L be a line in $\mathbb{R}^n$ and set $proj_L : \mathbb{R}^n \to \mathbb{R}^n$. What are the image and kernel of $proj_L$?

10. The kernel of a transformation T is the set of all vectors that T maps to $\vec{0}$. One might wonder why we only care about $\vec{0}$ and not other vectors $\vec{b}$.

(a) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, find $\vec{x}, \vec{y} \in \mathbb{R}^3$ (with $\vec{x} \neq \vec{0}$) so $A\vec{x} = 0$ and $A\vec{y} = \begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix}$. Compute $A(\vec{x} + \vec{y})$.

(b) For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, find different $\vec{x}, \vec{y} \in \mathbb{R}^3$ so $A\vec{x} = A\vec{y} = \begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix}$. Compute $A(\vec{x} - \vec{y})$.

(c) Let A be an $n \times n$ matrix and $\vec{x_0}, \vec{b} \in \mathbb{R}^n$. Use the above to argue that, for $\vec{x} \in \mathbb{R}^n$, we have that $A\vec{x} = \vec{b}$ exactly when there is some $\vec{y} \in ker(A)$ such that $\vec{x} = \vec{x_0} + \vec{y}$.