More Examples of Linear Transformations

1. True or false: If A is an $n \times m$ matrix, then $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$ is a linear transformation.

2. Describe what the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}x \\ \frac{1}{2}y \end{bmatrix}$ does.

3. Let T be counterclockwise rotation by the angle θ in $\mathbb{R}^2$. Find the matrix of T.

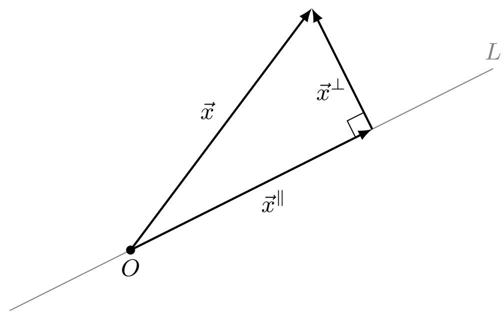
4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear given by the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Where does this send the vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$? Draw the effect this transformation has on the unit square.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear given by the matrix $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Where does this send the vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$? Draw the effect this transformation has on the unit square.

Definition. Let L be a line in $\mathbb{R}^n$ going through the origin, and let $\vec{x}$ be any vector in $\mathbb{R}^n$. Then, we can write

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

where $\vec{x}^{\parallel}$ is parallel to L and $\vec{x}^{\perp}$ is perpendicular to L.



(Intuitively, we get $\vec{x}^{\perp}$ by "dropping a perpendicular" from $\vec{x}$ to L.) The orthogonal projection of $\vec{x}$ onto L, denoted $\text{proj}_L(\vec{x})$, is defined to be $\vec{x}^{\parallel}$.

6. Find the matrix of the orthogonal projection onto the x-axis in $\mathbb{R}^2$.

7. Let L be the line $y = 3x$ in $\mathbb{R}^2$. Find the matrix of proj_L.

8. Let L be the line $y = 3x$ in $\mathbb{R}^2$. Find the matrix for the transformation that reflects a vector across L. What vectors does this transformation fix? Try and answer this algebraically and geometrically.

9. In this problem, we explore the transformation that is a rotation dilation. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates a vector $\frac{\pi}{4}$ counter clockwise and then dilates it by a factor of 3. Find the matrix for this transformation.

10. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by $\frac{\pi}{3}$ and let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear given by $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$.

(a) Write down the matrix for T.

(b) What is the matrix for the transformation that shears and then rotates?

(c) What is the matrix for the transformation that rotates and then shears?

11. Suppose $T : \mathbb{R}^m \to \mathbb{R}^p$ and $L : \mathbb{R}^p \to \mathbb{R}^n$ are linear transformations with matrices A and B, respectively.

(a) We define the composition $L \circ T : \mathbb{R}^m \to \mathbb{R}^n$ by $(L \circ T)(\vec{x}) = L(T(\vec{x}))$. Express $(L \circ T)(\vec{x})$ in terms of A, B, and $\vec{x}$.

(b)

(c) How can we find the matrix of $L \circ T$?

12. Let $A = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 \\ 5 & 1 \end{bmatrix}$. Find AB and BA (if they make sense).

13. Let $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find AB and BA (if they make sense).

14. Let $A = \begin{bmatrix} 1 & 4 \\ -3 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$. Find AB and BA (if they make sense).

15. If A is an $n \times m$ matrix, what is $I_n A$? How about AI_m?

16. The matrix of reflection over the line $y = 2x$ in $\mathbb{R}^2$ is $A = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$. Find A^2.

17. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the dilation by a factor r and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by θ. Show that for every $\vec{x} \in \mathbb{R}^2$,

$$S \circ T(\vec{x}) = T \circ S(\vec{x})$$

This is the rotation dilation.

18. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the shear given by $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by θ.

(a) Find the matrix A for T and B for S.

(b) Compute AB and BA.

(c) Why do we not describe certain transformations as "rotation shears?"