Introduction to Linear Transformations

1. The matrix A has reduced row echelon form

$$rref(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

What is $rank(A)$? If A is the coefficient matrix of a *consistent* system, what is the dimension of the solution space for it?

2. The matrix B has reduced row echelon form

$$rref(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

What is $rank(B)$? If B is the coefficient matrix of a *consistent* system, what is the dimension of the solution space for it?

3. Compute the following vectors:

(a) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & 2 \\ 5 & 0 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}$

4. Can you compute $\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$?

5. In each of the following, decide whether the given function is a linear transformation. If not, why not. If so, try to come up with a matrix that gives it (we will talk about a general method for this later).

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} xyz \\ 0 \end{bmatrix}$.

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = 2\vec{x}$.

   (c) $T : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T(\vec{x}) = \vec{0}$.

(d) $T : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T(\vec{x}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \sin x \\ \cos y \\ \tan z \end{bmatrix}$.

(f) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + 5 \\ 2x_1 + x_2 - 3 \end{bmatrix}$.

6. Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ satisfies $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(k\vec{x}) = kT(\vec{x})$. Is T a linear transformation?

7. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^5$ is a linear transformation, and all we know about it is that $T\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\2\\0\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\4\\6\end{bmatrix}\right) = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$. What is $T\left(\begin{bmatrix}1\\2\\5\end{bmatrix}\right)$?

8. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and the only information we have about it is that $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\-1\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\end{bmatrix}$, and $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\5\end{bmatrix}$. Find a formula for $T(\vec{x})$.

9. You can check that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1 + x_2\\2x_1 - x_2\\3x_1 + 5x_2\end{bmatrix}$ is a linear transformation. What is its matrix?

10. Find the matrix associated with the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 \\ x_2 \end{bmatrix}$.

11. Find the matrix associated with the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} b - c \\ d + a - c \end{bmatrix}$.

12. Suppose that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ stretches $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and fixes $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Describe what this transformation does, both algebraically and geometrically.

13. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \vec{x}$$

Find a matrix B such that if we define $S(\vec{x}) = B\vec{x}$, then $S\left(T(\vec{x})\right) = \vec{x}$ for every $\vec{x} \in \mathbb{R}^2$.

14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 6 & 0 \\ 5 & -1 \end{bmatrix}$$

Find a matrix B such that if we define $S(\vec{x}) = B\vec{x}$, then $S\left(T(\vec{x})\right) = \vec{x}$ for every $\vec{x} \in \mathbb{R}^2$.

15. Write down a 2×2 matrix with rank 1. This matrix doesn't have an inverse. What goes wrong if you try to find it's inverse?

16. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation, and all you know about it is that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(a) Do you have enough information to find $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$? If so, find it; if not, explain why not.

(b) Do you have enough information to find $T\left(\begin{bmatrix} -2 \\ -4 \end{bmatrix}\right)$? If so, find it; if not, explain why not.