MATH 144: COURSE NOTES

WILL BONEY

Contents

7. February 8

7.1. **Elementary Substructure.** There's a bit of a mismatch in what we've talked about so far.

- Our current ordering on structures, substructure, only preserves quantifier-free formulas (recall Proposition **??**).
- First-order logic and the structure of theories and definability is much, much richer.

This leads us to an unfortunate situation where we can have a definable set X in M and $M \subset N$, where the same definition gives a much smaller definable set Y in M. Indeed, we could make X go from $|M|$ to $\emptyset$ rather easily.

Example 7.1. *Work in the language of groups. Set M to be the direct sum of $\mathbb{N}$ many copies of $\mathbb{Z}_2$, N to be the direct sum of $\mathbb{N}$ many copies of $\mathbb{Z}_4$, and set $\phi(x) \equiv$ "$\forall y. y + y \neq x$". Then we can define an embedding $f : M \to N$ by $f(x) = x + x$. Then $\phi(M) = |M| - \{0\}$ and $\phi(N) = \emptyset$.*

The problem here is that our notion of substructure is wrong. Instead we have to pick a better notion of substructure, namely *elementary substructure*.

Definition 7.2. *Let M and N be L-structures. We say that M is an elementary substructure of N–written $M \prec N$–when $|M| \subset |N|$, for all $\mathbf{a} \in |M|$ and formulas $\phi(\mathbf{x})$ we have*

$$M \vDash \phi(\mathbf{a}) \text{ iff } N \vDash \phi(\mathbf{a})$$

An embedding $f : M \to N$ is an elementary embedding iff $f(M) \prec N$.

Date: February 9, 2016.

Note a simple corollary is that $M \prec N$ implies $M \subset N$. Compare this with Proposition **??** which says (after a little thinking) that substructure can be defined this way if we quantify over only *quantifier free* formulas.

A good way of thinking about the notion of elementary substructure is that truth in M is the same as truth in N, where truth is meant to be relativized to first order formulas.

Example 7.3.
- $(2\mathbb{N}, +, 0) \subset (\mathbb{N}, +, 0)$ *but* $(2\mathbb{N}, +, 0) \not\prec (\mathbb{N}, +, 0)$
  As above, we can look at the notion of evenness $(\exists y. x = y + y)$*. In* $\mathbb{N}$*, 2 is even*[1]*, but this fails in* $2\mathbb{N}$*.*
- $(\mathbb{Q}, <) \prec (\mathbb{R}, <)$
  For now, you're just going to have to trust me here. We need more machinery that we will develop (quantifier elimination), but the essential essence is that the only real questions that formulas can ask are "Is this some valid arrangement of points?," including wether or not we can stick additional points between them (existential quantifiers). The basic question of "valid arrangement" is settled just by the interpretation of the order (which is the same), and the question about extra points is settled by the denseness of both structures.

But how do we prove these?

A priori[2], checking wether M is an elmentary substructure of N is complicated. However, the following reduces the question to single existential quantifiers. The importance of existential quantifiers will return when we talk about Skolemization, Definition 7.11.

Theorem 7.4 (Tarski-Vaught Test, [**?**].2.1.2)**.** *Suppose that $M \subset N$. Then $M \prec N$ iff the following holds: for every $\phi(x, \mathbf{y})$ and $\mathbf{a} \in |M|$, if $N \vDash \exists x \phi(x, \mathbf{a})$, then there is $b \in |M|$ such that $N \vDash \phi(b, \mathbf{a})$.*

Note that the satisfaction has to occur in N!

Proof: First, suppose that $M \prec N$. Since N models an existential, so does M. Thus, there is $b \in |M|$ such that $M \vDash \phi(b, \mathbf{a})$. But this transfers to N by the definition of elementary substructure.

Second, suppose that the property holds. We prove that $M \prec N$ by induction on formulas. We have that $M \subset N$, so by Proposition **??**, all our steps except for the existential are covered. The existential step is covered by the property. †

Here's a nice application:

Proposition 7.5. *Suppose that $M \subset \mathcal{N} := (\mathbb{N}, <)$. Then $M \prec \mathcal{N}$ iff $M = \mathcal{N}$.*

A nice corollary is a property that is sometimes called coherence.

[1][citation needed]

[2]Well, all math is an a priori truth, so maybe "at first glance" is a better phrase.

Corollary 7.6 (Coherence)**.** *Suppose M_1, M_2, and M_3 are L-structures such that*

- $M_1 \prec M_3$*;*
- $M_2 \prec M_3$*; and*
- $|M_1| \subset |M_2|$.

Then $M_1 \prec M_2$.

Proof: Let $\mathbf{a} \in |M_1|$ and $\phi(\mathbf{x})$ such that $M_1 \vDash \phi(\mathbf{a})$. By the first point, $M_3 \vDash \phi(\mathbf{a})$. By the second point, $M_2 \vDash \phi(\mathbf{a})$. Specializing to atomic ϕ and using $|M_1| \subset |M_2|$, we have $M_1 \subset M_2$. To conclude $M_1 \prec M_2$, we use that each implication is reversible (or apply the direction we have to $\neg\phi$. †

When dealing with substructure, we introduced the atomic diagram AD, Definition **??**, to capture the notion of being a superstructure. Now that we realize that elementary substructure, rather than just substructure, is the right notion, we have the corresponding object, the elementary diagram.

Definition 7.7. *Set $M \vDash T$ and $A \subset |M|$. Then the elementary diagram of A is*

$$ED_M(A) := \{\phi(c_{\mathbf{a}}) \mid \mathbf{a} \in A \text{ and } \phi \text{ is a formula such that } M \vDash \phi(\mathbf{a})\}$$

Everything we could prove about the atomic diagram, we can do in the same way with the elementary diagram with the correspondingly stronger conclusion.

Proposition 7.8. *Let $M \vDash T$ and $A \subset |M|$.*

1. *If $(N, n_a)_{a\in A} \vDash ED_M(A)$, then the map $a \mapsto n_a$ preserves formulas (this is called a partial elementary embedding).*
2. *In particular, if $(N, m)_{m\in|M|} \vDash ED_M(M)$, then the map $m \mapsto n_m$ is an elementary embedding.*

Proof: Homework. †

When we introduced types (Definition **??**), there seemed like a bit of a mismatch between the substructure claimed in types and the elementary diagram used. We can make sense of that here.

Proposition 7.9. *Suppose M is an L-structure. If $p(\mathbf{x})$ is a type over M, then there is $N \prec M$ that realizes p.*

Proof: Just like the proof of Proposition **??**. †

Corollary 7.10. *Suppose M is an L-structure and κ is a cardinal. Then there is N of size at least κ such that $M \prec N$.*

Proof: Just like the proof of Corollary **??**. †

7.2. **Skolemization.** From the Tarski-Vaught Test (Theorem 7.4), we know that existentials of single variables are the key thing separating substructure from elementary substructure. We can exploit this observation to Skolemize a theory.

Definition 7.11 (Skolemization)**.** *Fix a theory T in a language L.*

- *Define a sequence of languages L^n for $n \in \mathbb{N}$ as follows:*
  - *L^0 is the original language L*
  - *L^{n+1} takes the original language L^n and, for each existential L^n formula $\exists x \phi(x, \mathbf{y})$, adds a $\ell(\mathbf{y})$-ary function that is denoted by $F_{\exists x \phi(x, \mathbf{y})(\mathbf{z})}$.*
- *Set $L^{sk} = L^\omega = \cup_{n \in \mathbb{N}} L^n$ to be the union of the languages.*
- *Set T^{sk} to be*

$$\{\forall \mathbf{z} \left(\exists x \phi(x, \mathbf{z}) \to \phi\left(F_{\exists x' \phi(x', \mathbf{y})}(\mathbf{z}), \mathbf{z}\right)\right) \mid \exists x' \phi(x', \mathbf{y}) \text{ is an } L^{sk}\text{-formula}\}$$

L^{sk} and T^{sk} are called the Skolemizations of L and T. If $M \vDash T$, then an expansion of M to L^{sk} that models T^{sk} is called a Skolemization of M, and typically denoted M^{sk}.

The idea is to add a function for every existential formula that will pick out a witness to the existential (if one exists). Here's an example.

Definition 7.12 (Dense Linear Order Without Endpoints)**.** *Recall $L_{ord} = \{<\}$. The theory of* dense linear orders without endpoints*, abbreviated DLOWOE or just DLO, is axiomatized by the the following sentences:*

(1) $\forall x \neg (x < x)$
(2) $\forall x, y, z \, (x < y \wedge y < z \to x < z)$
(3) $\forall x, y \, (x < y \to \exists z (x < y < z))$
(4) $\forall x \exists y (x < y)$
(5) $\forall x \exists y (y < x)$

It is clear that $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$ model this theory, while $(\mathbb{Z}, <)$ and $([0, 1), <)$ do not.

Example 7.13. *The existentials in the theory do two things:*

- *Denseness: for each x, y, there is a point between them. So*

$$F_{\exists z (y' < z < x')}(x, y) = \begin{cases} \text{is some element between } y \text{ and } x & \text{if } y < x \\ \text{is arbitrary} & \text{if } y \geq x \end{cases}$$

- *No end points: for each x, there is a point below and above it. So*

$$F_{\exists z (x' < z)}(x) = \text{ some point above } x$$
$$F_{\exists z (z < x')}(x) = \text{ some point below } x$$

Note there's a bunch of freedom in the above. A natural skolemization of $(\mathbb{Q}, <)$ would be $(\mathbb{Q}, <, \frac{x+y}{2}, x + 1, x - 1)$, but we could also skolemize it as $(\mathbb{Q}, <, \frac{3x+2y}{5}, \max\{x^2, 1\}, \min\{-x^2, -1\})$.

Later, when we talk about quantifier elimination, we will see that the existentials above are the only ones that matter in $DLO(WOE)$.

Everything has a skolemization, but note that the expansion requires making a choice. Thus, we can't do something like define a uniform skolemization from $(\text{Mod}T, \prec)$ to $(\text{Mod}(T^{sk} \cup T^*), \subset)$. However, we could do this if T has *definable* Skolem functions.

We will prove the main result about skolemizations next time, but first we prove the main application.

Theorem 7.14 (The Löwenheim-Skolem Theorem)**.** *Let M be an infinite L-structure.*

1. *If $A \subset |M|$, then there is $M_0 \prec M$ that contains A such that $\|M_0\| = |A| + |L|$.*
2. *If $\kappa \geq \|M\| + L$, then there is $N \succ M$ of size κ.*

Proof: For (1), first expand L and M to Skolemizations L^{sk} and M^{sk}. Now set M_0^+ to be the closure of A under the functions of M^{sk}. Then $\|M_0^{sk}\| = |A| \times |L^{sk}| = |A| + |L|$ by Theorem 8.1.(X). Also, $M_0^{sk} \restriction L \prec M$ by Theorem 8.1.(X).

For (2), first apply Corollary 7.10 to find $N^+ \succ M$ of size $\geq \kappa$. Set $A \subset N^+$ to contain $|M|$ and be of size κ. By (1), there is $N \prec N^+$ of size κ containing M. By Corollary 7.6, we have $M \prec N$, as desired. †

8. February 10

The following is the key fact about skolemizations is that every model can be extended to a skolemization, and that skolemizations reduce everything to a universal theory.

Theorem 8.1. *Let L be a language and T be a theory.*

1. *Every $M \vDash T$ has an expansion M^{sk} to T^{sk}. Moreover, if $N \prec M$ has a given skolemization N^{sk}, then there is a skolemization that $\subset_{L^{sk}}$ extends N^{sk}.*
2. *If M^{sk} and N^{sk} both model T^{sk} with $M^{sk} \subset_{L^{sk}} N^{sk}$, then $M^{sk} \restriction L \prec_L N^{sk} \restriction L$.*
3. *There is a universal L^{sk}-theory T^* such that $Mod(T^* \cup T^{sk})$ is precisely the skolemizations of models of T.*

Proof: For the first item, expand M for each L^n in turn as follows: given $\exists x \phi(x, \mathbf{y})$ in L^n, define the function $F_{\exists x \phi(x,\mathbf{y})}$ by

$$F_{\exists x \phi(x,\mathbf{y})}(\mathbf{a}) = \begin{cases} \text{arbitrary} & M^n \vDash \neg\,(\exists x \phi(x, \mathbf{y})) \\ \text{arb. element of } \phi(M^n, \mathbf{a}) & M^n \vDash \exists x \phi(x, \mathbf{y}) \end{cases}$$

This will model T^{sk} by construction. For the moreover, if $N \prec M$ already has a skolemization, then supplement the above expansion by setting, for $\mathbf{a} \in |N|$,

$F^{M^{n+1}}_{\exists x\phi(x,\mathbf{y})}(\mathbf{a}) = F^{N^{sk}}_{\exists x\phi(x,\mathbf{y})}(\mathbf{a})$.

For the second item, we use the Tarski-Vaught Test, Theorem 7.4. We have that $M^{sk} \restriction L \subset_L N^{sk} \restriction L$. Suppose we have a formula $\exists x\phi(x, \mathbf{y})$ in L and $\mathbf{a} \in |M^{sk}|$. Then $N^{sk} \restriction L \vDash \exists x\phi(x, \mathbf{a})$ implies that

$$N^{sk} \restriction L \vDash \phi\left(F^{N^{sk}}_{\exists x\phi(x,\mathbf{y})}(\mathbf{a}), \mathbf{a}\right)$$

and $F^{N^{sk}}_{\exists x\phi(x,\mathbf{y})}(\mathbf{a}) = F^{M^{sk}}_{\exists x\phi(x,\mathbf{y})}(\mathbf{a}) \in |M^{sk} \restriction L|$ because $\mathbf{a} \in |M|$. This is exactly what we need for the Tarski-Vaught Test, so we can conclude $M^{sk} \restriction L \prec N^{sk} \restriction L$.

Given an L-formula $\phi(\mathbf{x})$, we construct a universal L^{sk}-equivalent $\phi^{sk}(\mathbf{x})$ as follows: first, we can write each sentence so any uses of negations appear before any uses of quantifiers. Then construct ϕ^{sk} following the same process as ϕ, *except* any use of the existential $\psi(x, \mathbf{y}) \mapsto \exists x\psi(x, \mathbf{y})$ is replaced by the rule

$$\psi(x, \mathbf{y}) \mapsto \psi\left(F_{\exists x\psi(x,\mathbf{y})}(\mathbf{y}), \mathbf{y}\right)$$

It is easy to see that

$$T^{sk} \vDash \forall \mathbf{x}\left(\phi(\mathbf{x}) \leftrightarrow \phi^{sk}(\mathbf{x})\right)$$

Then set $T^* := \{\phi^{sk} \mid \phi \in T\}$. †

A final note: I mentioned that compactness is the main theorem of first order logic, and the following amazing theorem, due to Per Lindström, says that I am half right in a very precise way.

Theorem 8.2. *If $\mathcal{L}$ is a logic such that*

1. *$\mathcal{L}$ is at least as expressible as first-order logic;*
2. *every $\mathcal{L}$-sentence in a countable language has a countable model[3]; and*
3. *$\mathcal{L}$ satisfies the Compactness Theorem,*

then $\mathcal{L}$ is first-order logic.

Put another way, first-order logic is the strongest that satisfies the Compactness Theorem and the downward part of the Löwenheim-Skolem Theorem.

This theorem makes reference to an abstract notion of a logic, which we won't define formally. Informally, it turns out to be precisely what you expect after working with different logics: a logic consists of a way of associating to each *language* L a set of *sentences* and a *satisfaction relation* between L-structures and L-sentences in a coherent way. Note there is no mention of formulas, but formulas can be seen as sentences in a language that adds constants for the variables.

Here's a surprising application of the Löwenheim-Skolem Theorm called Skolem's paradox.

[3]This is called the Löwenheim-Skolem property after the Löwenheim-Skolem Theorem 7.14

Example 8.3. *Suppose that* ZFC + $Con($ZFC$)$ *is consistent (it is). Then we have a model* $(M, \in)$ *of set theory, which, as any set theorist will tell you, is a model of all of mathematics. By the Löwenheim-Skolem Theorem 7.14, there is a countable elmentary submodel* $\mathcal{M}^* := (M^*, \in) \prec (M, \in)$*. Cantor famously proved that there are uncountable sets, and he did so within* ZFC *(as all math is done), so* $\mathcal{M}^*$ *thinks there are uncountable sets. In particular, there is some* $x \in M^*$ *that* $\mathcal{M}^* \vDash$ *"*x *is uncountable"*. *However,* $\mathcal{M}^*$ *can only give me countably many elements that are in* x*, so it's wrong (we could use the Mostowski collapse to make this argument a little cleaner by constructing an isomorphic copy of* $\mathcal{M}^*$ *such that each element only consists of things in the universe). It can be spectactularly wrong because* $\mathcal{M}^*$ *thinks there are really really big sets, but they are all just countable.*

Hilary Putnam [**?**] used this as inspiration for an argument that one cannot even formulate the skepticist's hypothesis that I am just a brain in a vat (in order to refute it).

8.1. **Complete Theories.** Recall that a theory T is complete if, for every sentence ϕ (in $L(T)$), either $T \vDash \phi$ or $T \vdash \neg\phi$. Now that we've pinpointed elementary substructure as the relation between structure that we like, the following naturally pushes us towards complete theories.

Proposition 8.4. *If* $M \prec N$*, then* $M \equiv N$*.*

Proof: Sentences are just formulas with no free variables. †

Remark 8.5. *There is a formal but unimportant (at least for us) distinction between complete theories* T *and theories* T *for which, given any sentence* ϕ*, either* $\phi \in T$ *or* $\neg\phi \in T$*. Since we are always dealing with structures, there is no harm in closing theories under entailment. In practice, when dealing with complete theories, we will often give a short(est) list, highlighting the important properties, and then show that they are complete.*

Complete theories abound. Trivially, one could take a structure M and form $\mathrm{Th}(M)$, which must be a complete theory. And all theories can be found in this way. But that doesn't really reflect mathematical practice: typically, you give a list of axioms and want to know if they are a complete theory. The following theorem is very useful for this.

Definition 8.6. *A theory* T *is said to be* κ*-categorical iff any two models of size* κ *are isomorphic.*

Recall Morley's Theorem **??** from way back which says that, for countable theories, categoricity in any particular uncountable cardinal is equivalent to categoricity in a

Theorem 8.7 (Łoś' Test)**.** *Let* T *be a theory without finite models. If* T *is* κ*-categorical for some* $\kappa \geq L(T)$*, then* T *is complete.*

Proof: Let $M, N \vDash T$. Using the Löwenheim-Skolem Theorem 7.14, we can find M^* and N^*, each of size κ and related to M and N by elementary substructure, respectively. By κ-categoricity, $M^* \cong N^*$. Thus, by Proposition **??**, $M^* \equiv N^*$. By Proposition 8.4,

$$M \equiv M^* \equiv N^* \equiv N$$

Since every two models of T are elementarily equivalent, T is complete. †

Corollary 8.8. *If T is categorical in some cardinal above the size of the language, then $T \cup \{\exists \geq n x(x = x) \mid n \in \mathbb{N}\}$ is complete.*

Now we're ready to prove a bunch of stuff is complete!!!

Definition 8.9. *The theory of torsion-free, divisible abelian groups $TFDAG$ consists of the following (in L_{grp}):*

(1) T_{grp}
(2) $\forall x, y(x + y = y + x)$
(3) *for each* $n \in \mathbb{N}$, $\forall x \exists y(n \cdot y = x)$
(4) *for each* $n \in \mathbb{N}$, $\forall x(x \neq 0 \to n \cdot x \neq 0)$

Proposition 8.10. *The theory of torsion-free divisible abelian is κ-categorical for all uncountable κ. In particular, $TFDAG$ is complete.*

Proof: This theory is effectively the theory of $\mathbb{Q}$-vector spaces: given $v \in G \vDash TFDAG$, we can find $\frac{m}{n} v$ by computing $m \cdot v$ and then finding the unique $w \in G$ such that $n \cdot w = m \cdot v$; the uniqueness uses the torsion-freeness.

It is well-known that vector fields are characterized (up to isomorphism) by their dimension. Given uncountable $G_1, G_2 \vDash TFDAG$, their dimension is equal to their size. Thus, $TFDAG$ is categorical in every uncountable cardinal.

By Łoś' Theorem, it is complete. †

Definition 8.11. *Algebraically closed fields of characteristic p (ACF_p) are axiomatized by the following;*

(1) T_{field}
(2) *for every* $n \in \mathbb{N}$, $\forall a_n, \ldots, a_0 \exists x(a_n x^n + \cdots + a_0 = 0)$
(3)
- *(if $p > 0$)* $\forall x(p \cdot x = 0)$
- *(if $p = 0$) for each* $n \in \mathbb{N}$, $\forall x(x \neq 0 \to n \cdot x \neq 0)$

Proposition 8.12. *ACF_p is categorical in all uncountable cardinals. In particular, ACF_p is complete.*

Proof: We lean heavily on field theory for this proof. The outline was given on the first day: Let K_1, K_2 be uncountable algebraically closed fields of the same characteristic p. Since they are uncountable, they have the same transcendence

degree, i. e., there are X_1, X_2 of elements that are mutually transcendental, of the same size, and the algebraic closure of X_ℓ is K_ℓ.

Since they have the same characteristic, the map that sends 0 to 0 and 1 to 1 generates an isomorphism between their base fields. Then, one polynomial $p(x)$ at a time, we can send roots to roots and lift this map to the algebraic closure of the base fields.

Finally, given a map $f : K_1' \to K_2'$ between algebraically closed fields and e_1, e_2 that are transcendental over them, we can start with the map $f \cup \{(e_1, e_2)\}$ and extend it to their algebraic closures in precisely the same way. Thus, by choice/Zorn's Lemma, we have an isomorphism from K_1 to K_2. †

9. February 12

Last time we showed that each ACF_p is complete by showing it's uncountably categorical

This has some nice consequences that we've talked about.

Recall the following from the first day.

Theorem 9.1 (Minor Lefschetz Principle)**.** *Let ϕ be a statement in the language of rings. The following are equivalent.*

1. *ϕ is true in every algebraically closed field of characteristic 0.*
2. *ϕ is true in $\mathbb{C}$.*
3. *ϕ is true in some algebraically closed field of characteristic p for arbitrarily large primes p.*
4. *ϕ is true in all algebraically closed fields of characteristic p for sufficiently large primes p.*

Proof: Homework. †

Theorem 9.2 (Ax-Grothendieck, special case)**.** *Suppose $f(x)$ is a polynomial function from $\mathbb{C}^n$ to $\mathbb{C}^n$. If f is injective, then it is bijective.*

Proof: We follow the outline for the first day. But first, we make things a little more precise. One important note is that polynomial functions are definable. Definable functions turn out to be exactly the polynomials, so there's not an easy way to extend this.

1. Given a finite field k, the theorem is true: any injective function from k^n to k^n is a bijection because k^n is finite (and of the same size as k^n).
2. We want to extend this to algebraically closed fields of positive characteristic. Suppose for contradiction that $\mathbb{F}_p$ is an algebraically closed field of characteristic p and polynomial $f : \mathbb{F}_p^n \to \mathbb{F}_p^n$ is an injection, but not a surjection. Since it's not a surjection, there is $x \in \mathbb{F}_p^n$ that is not in the image of f. Let $X \subset \mathbb{F}$ be each element of x along with the coefficients in f.

Crucially, X is finite, so (because p is positive) there is a *finite* field $k \subset \mathbb{F}_p$ that contains X. in particular, take k to be the subfield generated by X. Then $f \restriction k^n : k^n \to k^n$; note that this includes the observation that k^n is closed under applications of f because it is generated by the coefficients of it. Then

(a) $f \restriction k^n$ is an injection: this is a straightforward proof, but note that the statement is a universal formula (with the parameters are the coefficients of f) and that $k^n \subset \mathbb{F}_p^n$ as L_{ring} structures.
(b) $f \restriction k^n$ is not a surjection: x is not in the range of f, so it's not in the range of $f \restriction k^n$.

So this contradicts the previous point.

(3) This is the actual model theoretic part. We use the definability of f and the Minor Lefscetz Principle. We are going to work in $\mathbb{C}$ in the language of rings. For each n and d, we can construct a sentence $\phi_{n,d}$ that says that, for each n-tuple of polynomials of degree bounded by d, if it is injective then it is bijective. From the previous part, we have shown $\phi_{n,d}$ is true in every algebraically closed field of positive characteristic. By the Minor Lefschetz Principle (Theorem 9.1), $\phi_{n,d}$ is true in $\mathbb{C}$.

†

Proposition 9.3. *DLOWOE is $\aleph_0$-categorical. In particular, it is complete.*

This proof is longer than the other ones, but it's probably worth it. Actually, I'm going to prove a related result, and have y'all do the above on homework.

We introduce the *back-and-forth method*, which turns out to be interesting later. The method takes two structures $\mathcal{A}$ and $\mathcal{B}$ and looks at the class of partial embeddings between them (often with finite domain and range). Obviously, the empty set is such an embedding. We would like to be closed under the following extensions:

- If f is a partial embedding in the class and $a \in A$, then f can be extended to a partial embedding in the class with a added to the domain (the forth)
- If f is a partial embedding in the class and $b \in B$, then f can be extended to a partial embedding in the class with b added to the range (the back)

This is a crucial tool in model theory, in part emphasized by the existence of a (sadly now defunct) model theory blog called "Forking, Forcing, and back&Forthing." One of the really cool things about back and forth systems is the following:

Fact 9.4. *There is a back and forth system between M and N iff there is a forcing extension of the universe where they are isomorphic.*

(This is [**?**, Exercise 2.5.34])

This involves some set-theory, but the model-theoretic proof of this is the fact that if M and N are countable and have a back and forth system between them, then they are isomorphic. We will exhibit this with the *theory of random graphs*.

Definition 9.5. *The theory of the random graph (RG) consists of*

(1) T_{graph}
(2) *for each* n *and* m,

$$\forall x_1, \ldots, x_n \forall y_1, \ldots, y_m \exists z \left(\bigwedge_{1 \leq i \leq n, 1 \leq j \leq m} x_i \neq y_j \to \bigwedge_{1 \leq i \leq n} zEx_i \wedge \bigwedge_{1 \leq i \leq m} \neg(zEy_i) \right)$$

Proposition 9.6. *RG is* $\aleph_0$*-categorical. Thus, RG is complete.*

Proof: Let G and H be countable random graphs. Let $\mathcal{F}$ be the set of finite partial L_{graph}-embeddings between them. We want to show the following:

(1) $\mathcal{F} \neq \emptyset$
(2) If $f \in \mathcal{F}$ and $g \in G$, then there is $f' \in \mathcal{F}$ that extends f and has g in the domain.
(3) If $f \in \mathcal{F}$ and $h \in H$, then there is $f' \in \mathcal{F}$ that extends f and has h in the range.

We will prove this momentarily, but first we show that these properties of $\mathcal{F}$ suffice to complete the proof.
Claim: If $\mathcal{F}$ has these properties, then $G \cong H$.
Enumerate their universes $\{g_n \mid n \in \mathbb{N}\}$ and $\{h_n \mid n \in \mathbb{N}\}$, respectively. We construct an increasing sequence of partial embeddings $\{f_n \in \mathcal{F} \mid n \in \mathbb{N}\}$ such that g_n is in the domain of f_{2n} and h_n is in the range of f_{2n+1}, and we do so by induction on $n \in \mathbb{N}$.

For $n = 0$, let $f \in \mathcal{F}$ be arbitrary. By (2), find some $f' \in \mathcal{F}$ extending f with g_0 in its domain; set this to be f_0.

For $n = 2k$ with $k > 0$, we have $f_{n-1} \in \mathcal{F}$. By (2), find some $f' \in \mathcal{F}$ extending f_{n-1} with g_k in its domain; set this to be f_n.

For $n = 2k + 1$, we have $f_{n-1} \in \mathcal{F}$. By (3), find some $f' \in \mathcal{F}$ extending f_{n-1} with h_k in its domain; set this to be f_n.

Now set $f_\omega = \cup_{n \in \mathbb{N}} f_n$. This function is a L_{graph}-embedding whose domain is G and range is H. Thus, $f : G \cong H$.

Now we must show $\mathcal{F}$ has these properties.

For (1), the empty function $\emptyset \in \mathcal{F}$.

For (2), let $f \in \mathcal{F}$ and $g \in G$, partition dom f into X and Y where $X = \{g' \in \text{dom } f \mid gEg'\}$ and $Y = \{g' \in \text{dom } f \mid \neg(gEg')\}$. Set $n = |X|$ and $m = |Y|$. One of the axioms of RG is

$$\forall x_1, \ldots, x_n \forall y_1, \ldots, y_m \exists z \left(\bigwedge_{1 \leq i \leq n, 1 \leq j \leq m} x_i \neq y_j \to \bigwedge_{1 \leq i \leq n} zEx_i \wedge \bigwedge_{1 \leq i \leq m} \neg(zEy_i) \right)$$

Since H models this axioms, we know that

$$H \vDash \exists z \left(\bigwedge_{g' \in X} zEf(g') \wedge \bigwedge_{g' \in Y} \neg (zEf(g')) \right)$$

Let $h \in H$ witness this. Then $f \cup \{(g, h)\}$ is a partial embedding with finite domain that extends f with g in the domain, as desired.

For (3), use the same argument as (2), working with f^{-1} as a partial embedding from H to G if you really want to be pedantic. †

Similar methods work for $DLOWOE$, as you'll see on your homework.

The name random graph deserves a little explanation. In combinatorics[4], the key method of probabilistic combinatorics is to randomly construct graphs by randomly adding an edge between two vertices from a fixed set with a fixed probability. Then you can ask what the odds are that a graph constructed this way has certain properties. For instance, (I think) the existence of high girth, high chromatic number graphs is proven nonconstructively by showing that, with a particular probability scheme, more than half the graphs have high girth and more than half the graphs have high chromatic number; thus, there must be one with both.

If you fix[5] a probability p and "construct" the random graph on $\aleph_0$ many vertices, then you almost surely get a model of RG. Moreover, we can get a $0-1$ law for first order sentences: for a first order sentence ϕ, set

$$p_N(\phi) = \frac{\text{number of graphs on } n \text{ vertices that model } \phi}{\text{number of graphs on } n \text{ vertices}}$$

Theorem 9.7 ([?].2.4.4)**.** *For any first order sentence ϕ in L_{graph}, $\lim_{N\to\infty} p_N(\phi)$ is 0 or 1. Moreover, $RG \vDash \{\phi \mid \lim_{N\to\infty} p_N(\phi) = 1\}$.*

Proof: If $RG \vDash \phi$, then there is some finite subset of it that does so. This consists of T_{graph} and up to N of the existence axioms. Call the conjunction of these sentences ψ_N. Then $\psi_N \vDash \phi$ and, by the above argument that the random graph process almost surely produces a model of RG, we know[6] that $\lim_{n\to\infty} p_n(\psi_N) = 1$. By the implication $p_n(\phi) \geq p_n(\psi_N)$, so $\lim_{n\to\infty} p_n(\phi) = 1$.

If $RG \nvDash \phi$, then by completeness $RG \vDash \neg\phi$. So $\lim_{n\to\infty} p_n(\phi) = 1 - \lim_{n\to\infty} p_n(\neg\phi) = 0$. †

This might make you think that all complete theories are categorical somewhere. This is very much not the case. Here are two examples.

[4]This story is from the perspective of a non-combinatorialist who, nonetheless, had several combinatorial friends in grad school. Which is to say, it could be wrong.

[5]The theory of probabilistic combinatorics is much richer than this description; in particular, one can add

[6]This is pretty handwavy. See [**?**, Lemma 2.4.3] for a detailed proof.

Definition 9.8. *The theory of real closed fields RCF consists of the following:*

(1) T_{field}
(2) $\forall x \exists y \, (y^2 = x \vee y^2 = -x)$
(3) *for* odd $n \in \mathbb{N}$, $\forall a_n, \ldots, a_0 \exists x (a_n = 0 \vee a_n x^n + \ldots a_0 = 0)$
(4) *for* $n \in \mathbb{N}$, $\forall x_1, \ldots, x_n (x_1^2 + \ldots x_n^2 \neq 0)$

Proposition 9.9. *RCF is complete, but not categorical in any cardinal.*

This will be proved later.

We've now established the standard model theoretic framework: for a complete, consistent theory T, we work with the structures Mod$T$ and the elementary embeddings between them. We want to study this category a bit, but first we want to take a bit of a detour: set theory. This will allow us to talk about elementary chains and, while we're there, discuss ultraproducts, prove the compactness theorem, and do nonstandard analysis.

E-mail address: wboney@math.harvard.edu