Another approach to measure

Most of our attempts to generate measure were built on length

But we could be less ambitious.

Let $r_0 \in \mathbb{R}$

Define $\mu_{r_0} : P(\mathbb{R}) \to \{0,1\}$ by

$$\mu_{r_0}(A) = \begin{cases} 1 & r_0 \in A \\ 0 & r_0 \notin A \end{cases}$$

This is a measure except translation invariance

Since it's $\{0,1\}$-valued, we could represent it another way

$U_{r_0} \subseteq P(\mathbb{R})$ by

$$U_{r_0} = \{A \subset \mathbb{R} : \mu_{r_0}(A) = 1\}$$

What are the properties to these sets have?

Filters

Def $F \subset P(I)$ is a filter iff

- $\emptyset \notin F$, $I \in F$
- If $X \subseteq Y \subset I$ and $X \in F$, then $Y \in F$
- If $X, Y \in F$, then $X \cap Y \in F$

Given a filter F, we can find a $\underbrace{\text{measure}}_{}$ finitely additive measure μ_F on

$$\{ X \subset I : X \in F \text{ or } I - X \in F \}$$ by

$$\mu_F(X) = \begin{cases} 1 & X \in F \\ 0 & I - X \in F \end{cases}$$

When is this a finitely additive measure on all of I?

<u>Def</u> • $\mathcal{U} \subset \mathcal{P}(I)$ is an <u>ultrafilter</u> iff

$$\forall X \in I \,(X \in \mathcal{U} \text{ or } I - X \in \mathcal{U})$$

• A filter or ultrafilter F is called <u>principal</u> iff

$\exists\, i_0 \in I$ s.t. ~~$X \in \mathcal{U}$ iff $i_0 \in X$~~ $i_0 \in X \to X \in F$

Note we've gone very general, so translation invariance no longer makes sense (HW: Frechet is filter)

<u>Prop</u> A filter F is ~~maximal~~ an ultrafilter iff it is maximal, e.g., for all filters $F' \supseteq F$, $F = F'$.

<u>Lemma</u> Any $F_0 \subset I$ ~~with~~ satisfying

$$\forall X_1, \dots, X_n \in F_0, \bigcap_{i \le n} X_i \neq \emptyset$$

Can be extended to a filter

In your homework, you'll prove the existence of nonprincipal ultrafilters. What can you do with them?

Ultralimits

Fix an ultrafilter $\mathcal{U}$ on ω.

For a sequence $\langle a_n \in \mathbb{R} \mid n < \omega \rangle$ we say the ultralimit $\lim_{\mathcal{U}} a_n$ of this sequence is some $L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0\ \exists X_\varepsilon \in \mathcal{U}$ s.t.

$$n \in X_\varepsilon \rightarrow |L - a_n| < \varepsilon$$

Prop ~~For every bounded a_n~~ Every bd sequence has an ultralimit.

Lemma If $\mathcal{U}$ is an ultrafilter on I and for $X \in \mathcal{U}$ $X = X_1 \cup \cdots \cup X_n$, then some $X_i \in \mathcal{U}$

Prf] Suppose not. Then $X - X_i \in \mathcal{U}$ so

$$\bigcap_i (X - X_i) = \emptyset \in \mathcal{U}. \quad \times$$

Prf of Prop] Build $b_n \leq b_{n+1} < c_{n+1} \leq c_n$ and $X_{n+1} \subset X_n \in \mathcal{U}$ by induction so

1) $c_n - b_n < \frac{1}{2^n}$

2) $X_n = \{k < \omega : a_k \in [b_n, c_n]\} \in \mathcal{U}$

This is enough By completeness, we have

$b_n \to L \leftarrow c_n$, $L \in [b_n, c_n]$ $\forall n < \omega$

Claim $\lim_{\mathcal{U}} a_n = L$

Prf) Let $\varepsilon > 0$. Find $n < \omega$ so $\frac{1}{2^n} < \varepsilon$

Claim X_n works as X_ε

$k \in X_n \to a_k \in [b_n, c_n]$

$|a_k - L| < |c_n - b_n| < \frac{1}{2^n}$

Construction $\langle a_k \mid k < \omega \rangle$ is bounded so $\exists$ B s.t.

$-B < a_k < B$ for all $n < \omega$.

Partition $[-B, B]$ into $\leq 4B$ ~~pieces~~ intervals of

length $< \frac{1}{2}$ $\{I_i \mid i \leq 4B\}$

Then $\omega = \bigcup X_0^i$

where $X_0^i = \{k < \omega \mid a_k \in I_i\}$

One is in $\mathcal{U}$ i_0

$X_0 = X_0^{i_0}$

$[b_0, c_0] = I_{i_0}$

Induction step is easier

Question 1) What about an unbounded sequence?
2) What is the ultralimit of a principal ultrafilter

Ultraproducts, briefly

What else can we do with ultrafilters?

A lot!
Including my favorite thing, ultraproducts!

Properly model theory, so will just do a bit:

Let $\mathcal{U}$ be a nonprincipal ultrafilter on ω.

Nonstandard analysis, etc

Let $\prod \mathbb{R} = {}^{\omega}\mathbb{R} = \{f : \omega \to \mathbb{R}\}$

For $f, g \in \prod \mathbb{R}$, set
$$f =_{u} g \text{ iff } \{n < \omega : f(n) = g(n)\} \in \mathcal{U}$$

Set ${}^*\mathbb{R} = \prod \mathbb{R} / u$

Turn ${}^*\mathbb{R}$ into a field by
$$[f]_u + [g]_u = [f+g]_u$$
$$[f]_u \cdot [g]_u = [f \cdot g]_u$$
$$[f]_u < [g]_u \text{ iff } \{n < \omega : f(n) < g(n)\} \in \mathcal{U}$$

${}^*\mathbb{R}$ is ordered field, $\forall x \neq 0 \, \exists y \, (xy = 1)$

~~Can~~ show this is well-defined

Let $j : \mathbb{R} \to {}^*\mathbb{R}$ by
$$j(r) = [n \mapsto r]_u$$

This preserves all other ordered field
structure and so much more

~~What is the~~
Is there a difference?

Yes!

Look at $[n \mapsto \frac{1}{n}]_{\mathcal{U}}$

In HW, show this is an infinitesimal

Not restricted to $\mathbb{R}$, ordered fields, etc.

↳ Compactness