NORMALIZING SUSLIN TREES

Theorem 0.1. *If there is a Suslin tree, then there is a normal Suslin tree.*

Proof: Let T be a Suslin tree. We do a different order than in class.

Stage 1: Extendability property

Set $T^x = \{y \in T : x <_T y\}$ and $T^1 = \{x \in T : |T^x| > \omega\}$ and $<_1$ is the restriction of $<_T$ to T^1. We have that $x \in T^1$ implies $pred_{<_T}(x) \subset T^1$. This means that $ht_T(x) = ht_{T^1}(x)$. We want to show that T^1 has the extendability property: let $x \in T^1_\alpha$ and $\beta > \alpha$. We know that $T^x \cap T_\beta$ is non-empty and uncountable. Then we can partition

$$T^x - T_{\leq\beta} = \cup_{y\in T^x\cap T_\beta} T^y$$

So some T^y must be uncountable and $y \in T^1_\beta$. This is a subtree of T so is a Suslin tree.

Stage 2: Predecessors determine limit levels

We add a node at the top of every chain. Inductively by limit level α, for each $x \in T^1_\alpha$, add a node a_x between $pred(x)$ and x. Formally, set

$$\begin{aligned}
T^{2,0} &= T^1 \\
T^{2,<\alpha} &= \cup_{\text{limit } \beta<\alpha} T^{2,\beta} \\
T^{2,\alpha} &= T^{2,<\alpha} \cup \{x_A : \exists y \in T^{2,<\alpha}_\alpha \text{ and } A = pred(x)\} \\
<_{2,\alpha} &= <_{2,<\alpha} \cup \{(y, x_A), (x_A, z) : \exists x \in T^{2,<\alpha}_\alpha, A = pred(x)\, (y \in A \text{ or } z \leq_{T^{2,<\alpha}} x)\} \\
T^2 &= \cup_{\alpha<\omega_1} T^{2,\alpha}
\end{aligned}$$

Then T^2 remains a Suslin tree by arguments we gave in class; similarly, T^2 maintains the extendability property. If $x \in T^2_\alpha$ for limit α, then $x \in T^{2,\alpha}_\alpha$. Thus, we have $x_{pred(x)}$ by construction. Then if $x, y \in T^{2,\alpha}_\alpha$ with $pred(x) = pred(y)$, we have

$$x = x_{pred(x)} = x_{pred(y)} = y$$

Stage 3: Every point splits at the next level

Claim: Given $y \in T^2$ and $\alpha > ht_{T^2}(x)$, there is $\beta > \alpha$ and $x \in T^2_\beta$ and $y_1 \neq y_2 \in T^2_{\beta+1}$ such that $y <_{T^2} x <_{T^2} y_\ell$ for $\ell = 1, 2$.

Call such x *branching points.*

Proof: If y and α are counter-examples, then $T^y \cap T_{>\alpha}$ is an uncountable, linear-ordered set; to show this, we need to have ensured that predecessors determine limit levels (which is not the order from class). This contradicts T^2 not having any branches.

Now set T^3 to be the collection of branching points of T^2 and $<_3$ the restriction of $<_2$ to T^3. The argument from class shows that this is a Suslin tree with the extendability property and so every node splits at the next level. We need to argue that predecessors determine limit levels. Moving from T^2 to T^3 we have thrown away lots of nodes, so the height of a node might shrink. Suppose that $x \neq y \in T^3_\alpha$. Then set $z = \max(pred_{T^3}(x) \cap pred_{T^3}(y)) \in T^3$. If $pred_{T^3}(x) = pred_{T^3}(y)$, then this common predecessor set has a max, so α is a successor ordinal. Thus, if $x \neq y$ have limit height, then they must have distinct predecessors.
Stage 4: Add a root.

If there's not a root in T^3, then add one to get T^4. †

I also want to finish the proof that the line built from the Suslin tree has the Suslin property. Recall T was our Suslin tree and $<^*$ was the linear ordering we put on it. Given $x \in T$, set $I_x = \{y \in T : x <_T y\}$.
If $a <^* b$ are from T, then we can find $x \in T$ such that $I_x \subset (a, b)_*$. To do this, let $a' <_T a$ and $b' <_T b$ be immediate successors of $z = pred(a) \cap pred(b)$. Thus, $a' <^z b'$. Since $<^z$ is dense, there is an immediate successor (in T) x of z such that $a' <^z x <^z b'$. Then $I_x \subset (a, b)_*$. Moreover, if I_x and I_y are disjoint, then x and y are incomparable.
Now suppose that X is a collection of disjoint open intervals in $(T, <^*)$. For each $J \in X$, pick $x_J \in T$ such that $I_{x_J} \subset J$. These are pairwise disjoint since X is. Then $\{x_J : J \in X\}$ is an antichain, so must be countable. Thus, X is countable.