DEFINING DEFINABILITY

Here is a collection of formulas that we will use to define the formula

$$x = Def^M(M)$$

(1) TERM(x, M, W)

$$\exists y \in W(omega(y) \wedge (\exists i \in y(x = \ulcorner x_i \urcorner) \vee \exists a \in M(x = \ulcorner a \urcorner)))$$

We will use ω as a parameter in what follows, but it will always be shorthand for $\exists y \in W(omega(y) \wedge \dots)$

(2) ATOM(x, M, W)

$$\exists y, z \in W(\text{TERM}(y, M, W) \wedge \text{TERM}(z, M, W) \wedge (x = \ulcorner y = z \urcorner \vee x = \ulcorner y \in z \urcorner))$$

(3) FSEQ$(F, \ulcorner \phi \urcorner, M, W)$

$$\begin{aligned} &\exists n \in \omega(\text{“}F \text{ is a function with domain } n+1\text{”} \wedge \\ &\qquad \forall m \in n+1(\text{ATOM}(F(m), M, W) \\ &\qquad \vee \exists k \in m, F(m) = \ulcorner \neg F(k) \urcorner \\ &\qquad \vee \exists k, \ell \in m, F(m) = \ulcorner F(k) \vee F(\ell) \urcorner \\ &\qquad \vee \exists i \in \omega, k \in m, F(m) = \ulcorner \exists x_i F(k) \urcorner) \\ &\qquad\qquad \wedge F(n) = \ulcorner \phi \urcorner) \end{aligned}$$

(4) VLEN(n, F, M, W)

$$\begin{aligned} &\exists i \in \text{dom }(F)(\text{ATOM}(F(i), M, W) \wedge (F(i)(1) = \ulcorner x_{n-1} \urcorner \vee F(i)(2) = \ulcorner x_{n-2} \urcorner)) \\ &\wedge \forall m \in \omega(\exists i \in \text{dom }(F)(\text{ATOM}(F(i), M, W) \wedge (F(i)(1) = \ulcorner x_{n-1} \urcorner \vee F(i)(2) = \ulcorner x_{n-2} \urcorner)) \to m \in n) \end{aligned}$$

(5) $\mathrm{SSEQ}(S, F, \ulcorner\phi\urcorner, M, W)$

$$\exists n, r \in \omega(\text{“}F, S \text{ are functions with domain } n{+}1\text{”} \wedge \mathrm{FSEQ}(F, \ulcorner\phi\urcorner, M, W) \wedge \mathrm{VLEN}(r, F, M, W)$$
$$\wedge \forall m \in n+1(\mathrm{ATOM}(F(m), M, W) \to \forall i, j \in r \forall a, b \in M\{*\}_{(i,j,a,b)} \wedge$$
$$\forall k \in m(F(m) = \ulcorner\neg F(k)\urcorner \to S(m) = M^r - S(k)) \wedge$$
$$\forall k, \ell \in m(F(m) = \ulcorner F(k) \vee F(\ell)\urcorner \to S(m) = S(k) \cup S(\ell)) \wedge$$
$$\forall k \in m \forall i \in r(F(m) = \ulcorner\exists x_i \phi\urcorner \to$$
$$S(m) = \{s \in M^r : \exists \bar{s} \in S(k) \forall i' < r+1(i' \neq i \to \bar{s}(i') = s(i))\})))$$

Here, $\{*\}_{(i,j,a,b)}$ stands for the following

$$(F(m) = \ulcorner x_i \in x_j\urcorner \to S(m) = \{s \in M^r : s(i) \in s(j)\})$$
$$\wedge (F(m) = \ulcorner x_i \in a\urcorner \to S(m) = \{s \in M^r : s(i) \in a\})$$
$$\wedge (F(m) = \ulcorner a \in x_i\urcorner \to S(m) = \{s \in M^r : a \in s(i)\})$$
$$\wedge (F(m) = \ulcorner a \in b\urcorner \to S(m) = \{s \in M^r : a \in b\})$$
$$\wedge (F(m) = \ulcorner x_i = x_j\urcorner \to S(m) = \{s \in M^r : s(i) = s(j)\})$$
$$\wedge (F(m) = \ulcorner x_i = a\urcorner \to S(m) = \{s \in M^r : s(i) = a\})$$
$$\wedge (F(m) = \ulcorner a = x_j\urcorner \to S(m) = \{s \in M^r : a = s(j)\})$$
$$\wedge (F(m) = \ulcorner a = b\urcorner \to S(m) = \{s \in M^r : a = b\})$$

(6) $\mathrm{SAT}(M, \ulcorner\phi\urcorner, s, W)$

$$\exists S, F \in W, \exists n \in \omega(\mathrm{SSEQ}(S, F, \ulcorner\phi\urcorner, M, W) \wedge F(n) = \ulcorner\phi\urcorner \wedge s \in S(n))$$

(7) $\mathrm{FVAR}(\ulcorner\phi\urcorner, x, M, W)$
Homework ;)

(8) $\mathrm{MASTER}(W, M)$

$$\exists f \exists x \in W(omega(x) \wedge \text{“}f \text{ is a function with domain } \omega\text{”} \wedge \forall n \in x(f(n) \subset W) \wedge$$
$$(M \subset f(0) \wedge \forall i,j \in x \forall a,b \in M(\text{“Gödel codes for terms and atomics”} \in f(0)) \wedge \omega \subset f(0))$$
$$\wedge\, \forall n+1 \in x$$
$$(\forall x,y \in f(n), \{x,y\} \in f(n+1)))$$

(9) $\mathrm{DEF}(x, M)$

$$\exists W(\mathrm{MASTER}(W,M) \wedge$$
$$[\forall z \in x \exists \ulcorner \phi \urcorner \in W \exists i \in \omega(\mathrm{FVAR}(\ulcorner \phi \urcorner, \{i\}, M, W) \wedge \forall y \in M$$
$$(y \in z \iff \exists r \in \omega \exists s \in M^r(\mathrm{SAT}(M, \ulcorner \phi \urcorner, s, W) \wedge s(i) = z)))] \wedge$$
$$[\forall \ulcorner \phi \urcorner \in W \forall i \in \omega(\mathrm{FVAR}(\ulcorner \phi \urcorner, \{i\}, M, W) \rightarrow \exists z \in x \forall y \in M$$
$$(y \in z \iff \exists r \in \omega \exists s \in M^r(\mathrm{SAT}(M, \ulcorner \phi \urcorner, s, W) \wedge s(i) = z)))])$$