Cofinality

An important aspect of cardinals is how we can break them down into other cardinals

1) A countable union of countable sets is countable

Phrased another way,

$\aleph_1$ can't be broken into $(<\aleph_1)$-many $(<\aleph_1)$ sized sets

regular

2) $\aleph_\omega$ is the first limit ordinal

$$\aleph_\omega = \sup_{n<\omega} \aleph_n = \bigcup_{n<\omega} \aleph_n$$

$\aleph_\omega$ can be ~~written as~~ broken into ω-many ($<\aleph_\omega$)

$(<\aleph_\omega)$-sized sets

singular

Turns out to be a super important distinction

Def Let α be a limit ordinal. (silly otherwise) $\mathrm{cf}\,\alpha$ (cof α) is the min β s.t. $\exists f : \beta \to \alpha$ cofinal

$f : \beta \to \alpha$ is cofinal iff $\forall \alpha_0 < \alpha \; \exists \beta_0 < \beta$ s.t. $f(\beta_0) \geq \alpha_0$

α is regular iff $\mathrm{cf}\,\alpha = \alpha$

α is singular iff $\mathrm{cf}\,\alpha < \alpha$

α is Godzilla iff $\mathrm{cf}\,\alpha > \alpha$

Prop No ordinal is Godzilla

Prf $id : \alpha \to \alpha$ is cofinal

"cofinal" = unbounded

$\swarrow$ not $\searrow$ final

Lemma Composition of two cofinal maps is cofinal

The cofinality of an ordinal represents the "end" behavior of an ordinal

$\omega, \omega+\omega, 2^{\aleph_0}+\omega, \aleph_\omega, \omega^2, \omega^\omega, \kappa+\omega 2, \aleph_{\omega^2}$

The all have countable sequences that are unbounded

$\left[\mathrm{cf}\,\alpha = \mathrm{cf}(\beta+\alpha) = \mathrm{cf}\,\aleph_\alpha \right]$
Review/Midterm

How do we tell the difference?

Theorem κ^+ is regular

This uses choice in a huge way!

Prop If α limit ordinal,

$\Rightarrow$ ~~the cf α is regular cardinal~~

Pf(A)

1) $cf(cf\,\alpha) = cf\,\alpha$

2) $cf\,\alpha$ is a regular cardinal

Pf 1) ~~Suppose $cf\,cf\,\alpha < cf\,\alpha$~~

Get cofinal $f: cf\,\alpha \to \alpha$ and

$g: cf\,cf\,\alpha \to cf\,\alpha$

Then $f \circ g: cf\,cf\,\alpha \to \alpha$ is cofinal

So $cf\,\alpha \le cf\,cf\,\alpha \le cf\,\alpha$

2) $cf\,\alpha$ is regular

S $\kappa < cf\,\alpha$ w/ $f: \kappa \to cf\,\alpha$ a bijection.

Then f is cofinal, so ~~$cf\,\alpha \le \kappa$~~ $\times$

Find $g: cf\,\alpha \to \alpha$ cofinal

Then $g \circ f: \kappa \to \alpha$ cofinal $\times$

~~Lemma~~ ~~Prf~~

Lemma If κ is a cardinal, then

κ is regular

iff

κ cannot be written as a union of $<\kappa$-many $<\kappa$-sized sets

Prf $\Rightarrow$ Suppose κ is regular and

$\kappa = \bigcup_{i<\alpha} X_i$ with $\alpha < \kappa$ and $|X_i| < \kappa$

Claim 1 $\sup X_i < \kappa$

If it were, then we could find a bijection $f : |X_i| \to X_i$. Then $f : |X_i| \to \kappa$ is cofinal and $\mathrm{cf}\,\kappa \le |X_i| < \kappa$ $*$

Claim 2 $g : \alpha \to \kappa$ given by $g(i) = \sup X_i$ is cofinal

Given $\beta \in \kappa$, $\exists i < \alpha$ s.t. $\beta \in X_i$. Then $\beta < g(i)$.

$*$

So $\mathrm{cf}\,\kappa \le \alpha < \kappa$

$\Leftarrow$ Suppose κ is singular, so $\alpha = \mathrm{cf}\,\kappa < \kappa$ and there is $f : \alpha \to \kappa$ cofinal.

Set $X_i = f(i)+1$ [$f(i) \in \kappa$, so $f(i) \subseteq \kappa$]

Claim 1 $|X_i| < \kappa$

κ is a cardinal, so $f(i)$ must have smaller size

Claim 2 $\kappa = \bigcup_{i<\alpha} X_i$

Given $\beta < \kappa$, $\exists i < \alpha$ s.t. $f(i) \ge \beta$, so $\beta \in f(i)+1 = X_i$

$\top$

Pcf of Thm κ^+ is a cardinal

Suppose we could write $\kappa^+ = \bigcup_{i<\alpha} X_i$ w/ $|X_i| < \kappa^+$
$|X_i| \leq \kappa$

WLOG $|X_i| = \kappa$ $(X_i' = X_i \cup \kappa)$

Then $|\kappa^+| = |\bigcup_{i<\alpha} X_i| \leq |\alpha| \cdot \sup_{i<\alpha} |X_i| \leq \kappa \cdot \kappa = \kappa$ $\times$

Cor (PHP) If $f: \kappa^+ \to \kappa$, then $\exists \beta < \kappa$ s.t. $f^{-1}\{\beta\} = \{\alpha < \kappa^+ \mid f(\alpha) = \beta\}$ has size κ^+

So successor cardinals are regular.

Limits?

Thm If α is limit, $\mathrm{cf}\, \aleph_\alpha = \mathrm{cf}\, \alpha$

One direction is not so bad, but other helps to prove some things about cf

Definition says all functions, but enough to look at increasing

Lemma Suppose α is limit ordinal.
Then there is a cofinal and increasing $f: \mathrm{cf}\,\alpha \to \alpha$
$\alpha < \beta \to f(\alpha) < f(\beta)$

Pcf Let $g: \mathrm{cf}\,\alpha \to \alpha$ be cofinal

Want $f(\beta) = \max\{g(\beta), \sup_{\gamma<\beta}(f(\gamma)+1)\}$

Recursion on $(\mathrm{cf}\,\alpha, \in)$
$G: \mathrm{cf}\,\alpha \times V \to V$ by

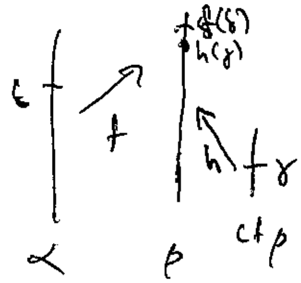
$$G(\beta, x) = \begin{cases} \max \text{---} & \text{if } x \text{ is a fn from } \mathrm{cf}\,\alpha \text{ to } \alpha \\ 0 & \text{o/w} \end{cases}$$

So cofinality measures the <u>first</u> place there's an increasing unbounded map. Conversely it's the <u>only</u> place there is one

Super surprising

<u>Prop</u> Let α, β be limit ordinals and $f: \alpha \to \beta$ cofinal and non decreasing

Then $\operatorname{cf}\alpha = \operatorname{cf}\beta$



<u>Pf]</u> Let $g: \operatorname{cf}\alpha \to \alpha$ and $h: \operatorname{cf}\beta \to \beta$ cofinals
By previous, can assume increasing.

1) $f \circ g: \operatorname{cf}\alpha \to \beta$ is cofinal So $\operatorname{cf}\beta \le \operatorname{cf}\alpha$

2) Define $k: \operatorname{cf}\beta \to \alpha$ by
$$k(\gamma) = \min\{\delta < \alpha \mid f(\delta) > h(\gamma)\}$$

Let $\delta < \alpha$. h is cofinal in β, so
get $\gamma_0 < \operatorname{cf}\beta$, so $h(\gamma_0) > f(\delta) + 1$

f is cofinal, so there is $\delta' < \alpha$ s.t. $f(\delta') > h(\gamma_0)$

So $k(\gamma_0)$ is well-defined

Since f is non decreasing
$$k(\gamma_0) \geq \delta$$

Back to theorem: α limit $\to cf\aleph_\alpha = cf\alpha$

<u>Prf of Thm</u> If α limit,

$$\aleph_\alpha = \bigcup_{i<\alpha} \aleph_i$$

~~[illegible]~~ Setting $f(i) = \aleph_i$ means $f: \alpha \to \aleph_\alpha$ is cofinal and non decreasing

~~So cf~~

So $cf\aleph_\alpha = cf\alpha$

Okay, so lots of limit cardinals are singular.

$$\aleph_\omega, \aleph_{\omega 2}, \aleph_{\omega^2}, \aleph_{\omega_1}, \dots$$

If $\aleph_\alpha$ were regular limit, would need

$$\cancel{\kappa = cf\kappa = cf\aleph_\kappa}$$

$$\aleph_\alpha = cf\aleph_\alpha = cf\alpha$$

<u>Prop</u> $\alpha \le \aleph_\alpha$

<u>Prf</u> Ordinal induction

$0 < \aleph_0$

$\beta \le \aleph_\beta \to \beta+1 < \aleph_{\beta+1}$

<u>limit α</u>

So
$$\aleph_\alpha = cf\,\aleph_\alpha = cf\,\alpha \leq \alpha \leq \aleph_\alpha$$

Need $\aleph_\alpha = \alpha$. Can we do this?

Yes! Define $\alpha_n \mid n<\omega$

$\alpha_0 = \aleph_0$ Set $\alpha_* = \sup \alpha_n$

$\alpha_{n+1} = \aleph_{\alpha_n}$

Claim $\alpha_* = \aleph_{\alpha_*}$

Know $\alpha_* \leq \aleph_{\alpha_*}$

$$\aleph_{\alpha_*} = \sup_{\beta<\alpha_*} \aleph_\beta = \sup_{n<\omega} \aleph_{\alpha_n} = \sup_{n<\omega} \alpha_{n+1} = \alpha_*$$

Is this regular?

No! $cf\,\alpha_* = \omega$

Def
- κ is weakly inaccessible iff it is regular limit
- κ is strong limit iff $\lambda<\kappa \to 2^\lambda < \kappa$
- κ is (strongly) inaccessible iff it is regular strong limit

Prop 1) Strong limits are limits
2) Strong inaccessibles are weak inaccessibles
3) ZFC cannot prove weakly inaccessible exist

König's Thm

Cardinal arithmetic becomes trivial except for exponentiation. Cofinality helps w/ this

Thm 1) $\kappa^{cf\,\kappa} > \kappa$

2) If $cf\,\kappa > \lambda$, $\kappa^\lambda = \sup_{\alpha<\kappa} \alpha^\lambda + \underbrace{\kappa}$ ← omitted in class

[redacted]

Pf) 1) Let $f : cf\,\kappa \to \kappa$ be cofinal

If $\kappa^{cf\,\kappa} = \kappa$, then let $\{g_\alpha \mid \alpha < \kappa\}$ enumerate ${}^{cf\,\kappa}\kappa$

Define $g : cf\,\kappa \to \kappa$ by

$g(\beta) = \min \kappa \sim \{g_\alpha(\beta) \mid \alpha < f(\beta)\}$

Note ~~$|g_\alpha(\beta)| < \kappa$ and $f(\beta) <$~~ $|\{g_\alpha(\beta) \mid \alpha < f(\beta)\}| \le f(\beta) < \kappa$

So makes sense

For each $\alpha < \kappa$, let β $f(\beta) > \alpha$

Then $g(\beta) \neq g_\alpha(\beta)$, so $g \neq g_\alpha$ $\times$

2) Clearly $\geq$ holds

$\kappa^\lambda = |{}^\lambda\kappa|$

For $f : \lambda \to \kappa$, $\sup_{\alpha<\lambda} f(\alpha) < \kappa$ since $\lambda < cf\,\kappa$

So set $\beta_f = \sup f''\lambda$

Claim ${}^\lambda\kappa = \bigcup_{\alpha<\lambda} {}^\lambda\alpha$

$\supseteq$ clear

$\subseteq$ for $f \in {}^\lambda\kappa$, $f \in {}^\lambda\beta_f$

So $|{}^\lambda\kappa| = |\bigcup_{\alpha<\lambda} {}^\lambda\alpha| \leq \sup_{\alpha<\lambda} \alpha^\lambda$

Two functions from ON to Card

We've experienced the $\aleph$ function
aleph

$\alpha \in ON \mapsto \aleph_\alpha \in CARD$

This is a (definable) class bijection

$$\aleph_\alpha = \begin{cases} \omega & \text{if } \alpha = 0 \\ \aleph_\beta^+ & \text{if } \alpha = \beta+1 \\ \sup_{\beta<\alpha} \aleph_\alpha & \text{if } \alpha \text{ limit} \end{cases}$$

$\alpha < \beta \to \aleph_\alpha < \aleph_\beta$

We can also define the $\beth$ function
beth

$$\beth_\alpha = \begin{cases} \omega & \text{if } \alpha = 0 \\ 2^{\beth_\beta} & \text{if } \alpha = \beta+1 \\ \sup_{\beta<\alpha} \beth_\beta & \text{if } \alpha \text{ limit} \end{cases}$$

<u>CH</u> $\aleph_1 = \beth_1$

<u>GCH</u> $\aleph_\alpha = \beth_\alpha \ \forall \alpha \in ON$

<u>Prop</u> GCH iff $\forall \kappa,\ 2^\kappa = \kappa^+$

<u>Prf</u> $\Rightarrow$ $\kappa = \aleph_\alpha = \beth_\alpha$

Then $\kappa^+ = \aleph_{\alpha+1} = \beth_{\alpha+1} = 2^{\beth_\alpha} = 2^\kappa$

$\Leftarrow$ Let α min so $\aleph_\alpha \neq \beth_\alpha$
α can't be limit
$\alpha = \beta+1$,

No inaccessibles

Recall
$$V_\alpha = \begin{cases} \emptyset & \alpha = 0 \\ P(V_\beta) & \alpha = \beta+1 \\ \bigcup_{\beta<\alpha} V_\beta & \alpha \text{ limit} \end{cases}$$

How much of ZFC does V_α satisfy?

~~Thm 1) If $\alpha > \omega$,~~

What does this mean? $(V_\alpha, \in)$ is a structure and we ask if it satisfies the axioms with quantifiers relativized to V_α

Prop 1) If $\alpha > \omega$, V_α satisfies Infinity

2) If α limit, then V_α satisfies Pairing, Union, Powerset

3) If $\alpha > 0$, V_α satisfies Empty Set

4) V_α satisfies Extensionality, Foundation

USE RANK

~~5) If α~~

Have everything except Replacement

Prop If κ is strongly inaccessible, then V_κ satisfies Replacement

Pcf) Claim 1 ~~$|V_\alpha| = \beth$~~ $|V_{\omega+\alpha}| = \beth_\alpha$

$\alpha = 0 \quad |V_\omega| = \aleph_0$

$\alpha = \beta+1 \quad |V_{\omega+\beta+1}| = |\mathcal{P}(V_{\omega+\beta})| = 2^{|V_\beta|} = 2^{\beth_\beta} = \beth_{\beta+1}$

α limit $\quad |V_{\omega+\alpha}| = \bigcup_{\beta<\alpha} |V_{\omega+\beta}| = \sup_{\beta<\alpha} \beth_{\beta} = \beth_\alpha$

Claim 2 $\kappa = \beth_\kappa$ (κ strong inaccessible)

$\kappa \leq \beth_\kappa$

If $\alpha < \kappa$, $\beth_\alpha < \kappa$

So $\beth_\kappa \leq \kappa$

So if $\alpha < \kappa$, $|V_\alpha| \leq \beth_\alpha < \beth_\kappa = \kappa$

Let $a \in V_\kappa$ s.t. $V_\kappa \models$ "$\varphi(x,y)$ defines a function w/domain a"

e.g., relativize quantifiers to V_κ

This means φ^{V_κ} defines a function w/domain a and codomain V_κ

Replacement says $\exists f : a \to V_\kappa$

Need to show $f \in V_\kappa$

$a \in V_{\alpha+1}$ for $\alpha < \kappa$, so $|a| < \kappa$. Get bij $g : a \to \lambda$

Define $h : \lambda \to \kappa$ by

$h(\alpha) = \text{rank } f(g^{-1}(\alpha))$

$\lambda < \mathrm{cf}\,\kappa = \kappa$, so can't be cofinal

Set $\beta = \sup h''\lambda < \kappa$

Then $f \in V_{\alpha+\beta+\omega} \subseteq V_\kappa$

and $V_\kappa \models$ inf shows replacement holds $\dagger$

So what?

<u>Thm</u> (Gödel's Incompleteness Theorem)

No "sufficiently powerful" recursive set of axioms can prove its own consistency

No formal defin, but ZFC can express all of math, so should be plenty

A nice description, which we gave for ZFC

So $ZFC \nvdash \mathrm{Con}(ZFC)$

Okay, that's weird...

More to the point

Thm ZFC cannot prove the existence of strongly inaccessibles

Prf] Suppose It could.

We proved (in ZFC) existence of strogly inaccessible implies there is a model of ZFC

Having a model of a set of sentences implies their consistency

"Provability" is closed under modus ponens

Then ZFC proves Con(ZFC) $\times$

Theorem (with a debt)

ZFC cannot prove the existence of regular limit cardinals

Prf] Using Gödel's Constructible Universe L,

~~ZFC +~~ Con($\cancel{Z}$FC + $\exists$ regular limit) $\to$ Con(ZFC + $\exists$ strongly inaccessible)