MATH 145A SET THEORY MIDTERM

You have 83 minutes to complete this test. No calculators or notes are allowed.

For each of the parts, select two (and only two) problems to do. **Make it clear which problems you're choosing, especially if you turn in partial results on several problems.**

Make sure that your name is on your exam booklet (you can ignore the rest) and that any additional pages you use *all* have your name on them.

Note: There are three parts and three pages to the exam.

(1) **Part I**: Do two of the following problems.
Problems are worth 15 points each.
  (a) Do both of the following:
    (i) Define what it means for α to be an ordinal and give the well-ordering on ON, the class of ordinals. (8 points)
    (ii) Define what it means for κ to be a cardinal and give a definition of what it means for the size of X to be less than the size of Y. (7 points)
  (b) Do one of the following.
    (i) State and prove Cantor's Theorem. This is the theorem that relates $|X|$ and $|\mathcal{P}(X)|$.
    (ii) Describe Russell's Paradox with proof. This is the paradox that lead to our introduction of ZFC.
  (c) Do both of the following:
    (i) State the definition of a well-founded relation. (6 points)
    (ii) Let X be a set. Show that $(X, \in \cap (X \times X))$ is a well-founded. Here, $\in \cap (X \times X) = \{(x_0, x_1) \in X \times X \mid x_0 \in x_1\}$ is the restriction of $\in$ to X. (9 points)

(2) **Part II**: Do two of the following problems.
Problems are worth 15 points each
  (a) The following exercise is about ordinal arithmetic. Do both of the following.
    (i) Show that $\omega \cdot 2 = \omega + \omega$. (5 points)
    (ii) Show that, for all ordinals α, β, γ,

$$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$$

(10 points)

(b) Let α, β be limit ordinals. Do both of the following.
   (i) $\text{cf } \alpha = \text{cf } (\beta + \alpha)$ (8 points)
   (ii) $\text{cf } \alpha = \text{cf } \aleph_\alpha$ (7 points)

   You can use results about cofinality proved in class **except** for (b) above (i. e., you can't simply say we proved (b) in class, but can cite the results we used to prove (b)).

(c) Do both of the following.
   (i) Prove $|\mathbb{Q}| = |\mathbb{N}| = |\mathbb{Z}|$ (5 points)
   (ii) For $n < \omega$, show that $\aleph_n^{\aleph_0} = \max\{2^{\aleph_0}, \aleph_n\}$. (10 points)

(3) **Part III**: Do two of the following problems.
Problems are worth 20 points each

(a) The goal of this problem is to prove Mostowski's Collapsing Theorem using Transfinite Recursion.

   Theorem 1 (Mostowski)**.** *If E is a well-founded and extensional relation on a set P, there there is a transitive set M and an isomorphism $\pi : (P, E) \cong (M, \in)$.*

   We say that a binary relation R on X is *extensional* iff for every $x, y \in X$, we have

   $$x = y \iff \forall z \in X\,(zRx \iff zRy)$$

   You can prove this theorem however you want, but here's a nice outline:
   (i) We wish to define π on P so that $\pi(x) = \{\pi(z) \mid zEx\}$. Find a function G so that the output of well-founded recursion is π (and show that this is the case).
   (ii) Show that π is injective and $xEy \iff \pi(x) \in \pi(y)$.
   (iii) Set $M = \pi"P$. Show that π is surjective and that M is transitive.
   (iv) Conclude the theorem.

   In fact this isomorphism is unique, but you don't need to show this. If you do show this, 5 bonus points.

(b) Do all of the following
   (i) For $r, s \in \mathbb{R}$, define the relation $\sim$ by

   $$r \sim s \iff r - s \in \mathbb{Q}$$

   Show that this is an equivalence relation on $\mathbb{R}$. (You can do this with numbers rather than sets ;)) (6 points)
   (ii) Let X, Y be sets and $f : X \to Y$ a function. Define a relation R on X by

   $$x_0 R x_i \iff f(x_0) = f(x_1)$$

Show that this is an equivalence relation on X. (6 points)

(iii) Find a set A and a function $g : \mathbb{R} \to A$ so that you can represent the equivalence relation from (3(b)i) as in (3(b)ii), that is, so $g(r) = g(s) \iff r \sim s$. Hint: You don't need anything specific about $\mathbb{R}$ or $\mathbb{Q}$ here. (8 points)

(c) Let κ be a cardinal and $f : \kappa^+ \to \kappa$. Show that there is some $\alpha < \kappa$ such that $f^{-1}\{\alpha\}$ has size κ^+, where

$$f^{-1}\{\alpha\} = \{\beta < \kappa^+ \mid f(\beta) = \alpha\}$$

(d) Do one of the following. In each, give the statement of the axioms you chose.

(i) The Axiom Schema of Replacement is equivalent to the Axioma Schemas of Comprehension and Collection.

(ii) The Axiom of Foundation is equivalent to the non-existence of infinite, decreasing $\in$-chains.

(iii) The Axiom of Choice is equivalent to the Well-Ordering Principle.