MATH 145A HOMEWORK 9

DUE NOV. 22

(1) Prove the Downward Löwenheim-Skolem-Tarski Theorem: Let τ be a language and M be a τ-structure. If A is a subset of the underlying set/universe, then there is a τ-structure N with underlying set Y such that
   (a) $N \prec M$
   (b) $A \subset Y$
   (c) $|Y| \leq |\tau| + |A|$

   As a hint, use the functions f_ϕ from the proof of reflection.

(2) Revenge of the Midterm: The goal of this problem is to prove Mostowski's Collapsing Theorem using Transfinite Recursion.

   Theorem 0.1 (Mostowski)**.** *If E is a well-founded and extensional relation on a set P, there there is a unique transitive set M and an isomorphism $\pi : (P, E) \cong (M, \in)$.*

   We say that a binary relation R on X is *extensional* iff for every $x, y \in X$, we have
   $$x = y \iff \forall z \in X\,(zRx \iff zRy)$$
   You can prove this theorem however you want, but here's a nice outline:
   (a) We wish to define π on P so that $\pi(x) = \{\pi(z) \mid zEx\}$. Find a function G so that the output of well-founded recursion is π (and show that this is the case).
   (b) Show that π is injective and $xEy \iff \pi(x) \in \pi(y)$.
   (c) Set $M = \pi" P$. Show that π is surjective and that M is transitive.
   (d) Conclude the theorem.

(3) Exercise 6.1
(4) Exercise 6.2
(5) Exercise 6.3
(6) Exercise 6.7