MATH 145A HOMEWORK 5

DUE NOV. 1

Note there are two pages to this homework.

(1) Use Zorn's Lemma to show that every filter extends to a maximal filter. Conclude that every set has a nonprincipal ultrafilter on it.
(2) Show that U is a nonprincipal ultrafilter on I iff it extends the Frechet filter
(3) Find a sequence $\langle a_n \in \mathbb{R} : n < \omega \rangle$ and a nonprincipal ultrafilters U_1 and U_2 on ω such that
$$\lim_{U_1} a_n \neq \lim_{U_2} a_n$$
(4) Regarding ultraproducts and ${}^*\mathbb{R}$, do (a) or (b) below:
    (a) Show that $=_U$ is an equivalence relation and that addition on ${}^*\mathbb{R}$ is well defined.
    (b) Show that $x := [n \mapsto \frac{1}{n}]_U$ is not $j(r)$ for any $r \in \mathbb{R}$ and, moreover, that it satisfies $j(0) < x < j(\frac{1}{n})$ for every $n < \omega$.
(5) Let λ be a cardinal of uncountable cofinality. We say that $C \subset \lambda$ is a *club* iff the following hold:
    - **cl**osed: if $X \subset C$ is a set with no last element that is *not* cofinal in λ, then $\sup X \in C$
    - **u**n**b**ounded: C is cofinal in λ, i. e., if $\alpha < \lambda$, then there is $\beta \in C$ such that $\beta > \alpha$.

    Do the following:
    (a) Show that the intersection of two club subsets of λ is club.
    (b) Define
    $$F = \{X \subset \lambda : \exists \text{ club } C \subset \lambda \text{ such that } C \subset X\}$$
    Show that F is a filter. It is called the club filter.
    (c) For some bonus points, suppose that $\alpha < \operatorname{cf} \lambda$. Show that the intersection of α many sets from F is in F and/or show that F is not an ultrafilter.
(6) We say that $(X, <)$ is a dense linear order without endpoints iff
    (a) $<$ is a linear order on X
    (b) for all $x < y$ from X, there is $z \in X$ such that $x < z < y$
    (c) for all $x \in X$, there are $y, z \in X$ such that $y < x < z$

    For an ordinal $\alpha \neq 0$, define the ordered structure $(\alpha \times \mathbb{Q}, <_\alpha)$, where $<_\alpha$ is the lexicographic ordering so $(\beta, p) <_\alpha (\gamma, q)$ iff $\beta < \gamma$ or ($\beta = \gamma$ and

$p < q$). Show that $(\alpha \times \mathbb{Q}, <_\alpha)$ is a dense linear order without endpoints. What is its size?