MATH 145A HOMEWORK 5

DUE OCT. 25

(1) Prove the unproved properties of the map from $(0,1) - \mathbb{Q}$ to ${}^{\omega}\omega$ defined in class. Namely
   (a) Show that f is a surjection.
   (b) Show that f is continuous. (Hint: show that if r_n converges to r in $\mathbb{R}$, then $f(r_n)$ converges to $f(r)$ in ${}^{\omega}\omega$)

(2) Exercise 5.3.

(3) Show that, for all α, $\Sigma^0_\alpha \subset \Sigma^0_{\alpha+1} \cap \Pi^0_{\alpha+1}$.

(4) Show that my definition of the Borel hierarchy agrees with the books. Namely,
   (a) Show that the definitions of Π^0_α agree, i. e.,

$$\{A \subset {}^{\omega}\omega : A = \cap_{n<\omega} B_n \text{ where } B_n \in \cup_{\beta<\alpha}\Sigma^0_\beta\} = \{A \subset {}^{\omega}\omega : {}^{\omega}\omega - A \in \Sigma^0_\alpha\}$$

   (b) Show that the hierarchy stabilizes at ω_1, that is, $\Sigma^0_{ON} = \Sigma^0_{\omega_1} = \Sigma^0_\beta$ for any $\beta \geq \omega_1$. (Hint: use cofinality)

(5) A *filter* F on a set I is a collection of subsets of I (so $F \subset \mathcal{P}(I)$) that satisfies the following properties:
   - $I \in F$ and $\emptyset \notin F$;
   - if $X \subset Y \subset I$ and $X \in F$, then $Y \in F$
   - if $X, Y \in F$, then $X \cap Y \in F$

   Let I be an infinite set and define $F \subset \mathcal{P}(I)$ by, for all $X \subset I$,

$$X \in F \iff I - X \text{ is finite}$$

   This is called the cofinite or Frechet filter
   (a) Show that F is a filter.
   (b) Show that, for all $i \in I$, $\{i\} \notin F$.
   (c) Find a set $X_0 \subset I$ such that $X_0, I - X_0 \notin F$. (It might help to first think about $I = \mathbb{N}$)