MATH 145A HOMEWORK 3

DUE SEP. 22

The first two problems are left overs from last week's homework.

(1) If R is a binary relation on a set X, show that the definition we gave in class for the transitive closure is equivalent to the one given in the book.

(2) Let R be a binary relation on a set X.
   (a) Suppose that for all $x \in X$, there is $y \in Y$ such that xRy or yRx. Define X from R.
   (b) Show R is well-founded iff every subset of X has a R-minimal element iff there is no sequence $\langle x_n \in X \mid n \in \mathbb{N}\rangle$ such that $x_{n+1}Rx_n$ for every $n \in \mathbb{N}$.
   (c) Show that $\in$ is a well-founded relation on the class defined by "$x = x$". (What is $ext_{\in}(x)$?)

(3) Show the following statements from class are true.
   (a) If (X, R) is a well-ordered set and $x_* \notin X$, then $(X \cup \{x_*\}, R^+)$ such that xR^+x_* for all $x \in X$ is a well-ordered set.
   (b) If $(I, <)$ is a linearly-ordered set and, for each $i \in I$, (X_i, R_i) is a well-ordered set such that $i < j$ implies
      - $X_i \subset X_j$;
      - $R_i \subset R_j$; and
      - if $x \in X_i$ and $y \in X_j - X_i$, then xR_jy,

      then
      $$\left(\bigcup_{i\in I} X_i, \bigcup_{i\in I} R_i\right)$$
      is a well-ordered set.

(4) Exercise 2.2

(5) The following exercise introduces the order topology* on the ordinals (again we ignore certain class-technicalities; this could be avoided by replacing On by some very large ordinal).
   (a) Let $(I, <)$ be a linearly ordered set and recall, for $x, y \in I$, we have $(x, y)_I := \{i \in I \mid x < i < y\}$. Set
   $$B = \{(x, y)_I \mid x < y \in I\}$$
   to be the collection of open intervals in I. Show that B is a basis for a topology on I; this is called the order topology.

(b) Describe the order topology on $(On, \in)$. In particular, for which ordinals α is $\{\alpha\}$ an open set? For which ordinals $\alpha < \beta$ is the "closed" interval $[\alpha, \beta]$ an open set?

If you haven't seen the notion of topology before, here's what you need to know/do for Problem 5:

- To show B is a basis, set
$$\tau := \{X \subset I \mid \forall x \in X, \exists Y \in B \text{ such that } x \in Y \subset X\}$$
Show the following things about τ:
  - $\emptyset, I \in \tau$
  - If $\{X_j \in \tau \mid j \in J\}$ is any collection of sets in τ, then $\cup_{j \in J} X_j \in \tau$.
  - If $X_1, \ldots, X_n \in \tau$ is a finite collection of sets, then $\cap_{1 \leq i \leq n} X_i \in \tau$.

  There are other ways to show that B is a basis, and you can use one of those as well.
- τ is called a τopology (although normally without the typographic pun) and the elements of it are called open sets. A set is closed C if it is the complement of an open set, that is, $I - C \in \tau$.
- The closed intervals are
$$[i, j]_I := \{x \in I \mid i \leq x \leq j\}$$