MATH 145A HOMEWORK 2

DUE SEP. 13

(1) Exercise 1.1
(2) Exercise 1.2
(3) Exercise 1.5
(4) Recall that the Axiom of Dependent Choice is the statement that:

Let R be a relation on X such that, for every $x \in X$, there is $y \in Y$ such that xRy. Then there is a sequence $\{x_n \in X \mid n \in \mathbb{N}\}$ such that $x_n R x_{n+1}$ for all $n \in \mathbb{N}$.

   (a) Show the Axiom of Choice implies the Axiom of Dependent Choice.
   (b) Show the Axiom of Choice implies the product of nonempty sets is nonempty.
   (c) Use the above to show the following fact from analysis:
   Let $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$ such that if $\langle x_n \rangle_n$ converges to a, then $\langle f(x_n) \rangle_n$ converges to $f(a)$. Then f is continuous at a.
   Bonus points for just using dependent choice.

(5) If R is a relation on a set X, show that the definition we gave in class for the transitive closure is equivalent to the one given in the book.
(6) Let R be a relation on a set X.
   (a) Define X from R.
   (b) Show R is a well-ordering iff every subset of X has a R-minimal element iff there is no sequence $\langle x_n \in X \mid n \in \mathbb{N} \rangle$ such that $x_{n+1} \in x_n$ for every $n \in \mathbb{N}$.
   (c) Show that $\in$ is a well-founded relation on the class defined by "$x = x$". (What is $ext_{\in}(x)$?)
(7) The following is an exercise on equivalence relations. Depending on your experience with them, do one of the following:
   (a) Let X be a set and $\sim$ an equivalence relation on X; that is,
   - for all $x \in X$, $x \sim x$;
   - for all $x, y \in X$, $x \sim y$ implies $y \sim x$; and
   - for all $x, y, z \in X$, $x \sim y$ and $y \sim z$ imply $x \sim z$.

   For $x \in X$, set the equivalence class of x to be $[x]_\sim := \{y \in X \mid y \sim x\}$. Show that for every $x, y \in X$, either $[x]_\sim = [y]_\sim$ or $[x]_\sim$ and $[y]_\sim$ are disjoint.

(b) Given that X is a set and $\sim$ is (an equivalence relation that is a) set, use the axioms of ZFC show that $[x]_\sim$ is a set for each $x \in X$ and that the collection of equivalence classes $X/\sim := \{[x]_\sim \mid x \in X\}$ is a set.