The Divergence Theorem

1. Parameterize the boundary of each of the following with positive orientation.

(a) The solid $x^2 + 4y^2 + 9z^2 \leq 36$.

(b) The solid $x^2 + y^2 \leq z \leq 9$.

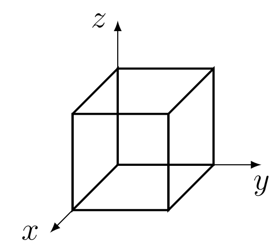
(c) The solid consisting of all points (x, y, z) inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 3$.

Divergence (or Gauss') Theorem:

Let E be a simple solid region and let S be the surface of E with positive orientation. Let $\mathbf{F}$ be a vector field with continuous partial derivatives. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

2. Let E be the solid unit cube with opposing corners at the origin and $(1, 1, 1)$ and faces parallel to the coordinate planes. Let S be the boundary surface of E, oriented with the outward-pointing normal. If $\mathbf{F} = \langle 2xy, 3ye^z, x\sin(z)\rangle$, find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem. (That is, find this flux integral by computing a triple integral instead.)



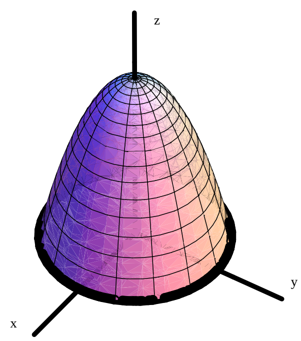
3. Let $\mathbf{F}(x, y, z) = \langle x^2, 2y, e^z\rangle$. Let $\mathcal{S}$ be the surface of the cube whose vertices are $(\pm1, \pm1, \pm1)$, oriented with outward normals. Evaluate the flux integral $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

4. Let $\mathbf{F}(x,y,z) = \langle x^3, z^2, 3y^2z \rangle$. Let $\mathcal{S}$ be the surface $z = x^2 + y^2$, $z \leq 4$ together with the surface $z = 8 - (x^2 + y^2)$, $z \geq 4$. Evaluate the flux integral $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ if $\mathcal{S}$ is oriented with outward normals.

5. A friend tells you that if S is a closed surface (that is, a surface without a boundary curve), then by Stokes' theorem we ought to have $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$ for any appropriate $\mathbf{F}$ (since there is no boundary curve C). Is this true? Can you justify it using the Divergence theorem?

Hint: Remember that weird argument about $\nabla \cdot (\nabla \times \mathbf{F})$?

6. Let E be the solid bounded by the xy-plane and the paraboloid $z = 4 - x^2 - y^2$. Let S be the boundary of E (that is, piece of the paraboloid and a disk in the xy-plane), oriented with the outward-pointing normal. If $\mathbf{F} = \langle xz\sin(yz) + x^3, \cos(yz), 3zy^2 - e^{x^2+y^2}\rangle$, find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem.

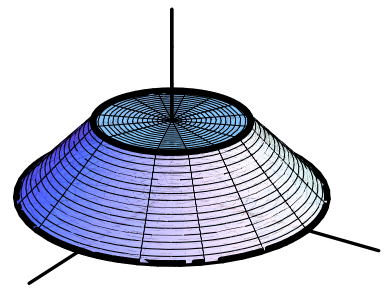


7. Here's a modification of the previous problem that adds a little clever trickery. Let's split up the surface S into a union $S = S_1 \cup S_2$ of the piece S_1 of the paraboloid and the flat disk S_2, with both pieces oriented as in the previous problem. If the vector field $\mathbf{F}$ is the same as in the previous problem, find $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$, the flux of $\mathbf{F}$ through S_1.

8. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = \left\langle z^3 \sin e^y, z^3 e^{x^2 \sin z}, y^2 + z \right\rangle$, and let $\mathcal{S}$ be the bottom half of the sphere $x^2 + y^2 + z^2 = 4$, oriented with normals pointing upward. Find $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$. (Hint: This is easier than it might seem at first)

9. Come up with some vector fields $\mathbf{F}(x, y, z)$ such that $\operatorname{div} \mathbf{F} = 1$. Use this to compute the volume of a cone with height h and radius r.

10. Let E be the part of the solid $x^2+y^2 \leq (2-z)^2$ with $0 \leq z \leq 1$. Calculate the volume of E using the divervgence theorem. Notice that S, the boundary of E, typically needs to be broken into three pieces, so it would be ideal for $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)$ to be simple (zero, for example) on one or two of these surfaces.



The Divergence Theorem – Answers and Solutions

1. We don't want to do the tedious work of parameterizing six surfaces (the six faces of the cube) in order to compute the flux integral directly. Instead we'll use the divergence theorem. Notice that

$$\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}\Big(2xy\Big) + \frac{\partial}{\partial y}\Big(3ye^z\Big) + \frac{\partial}{\partial z}\Big(x\sin(z)\Big) = 2y + 3e^z + x\cos(z).$$

Thus the divergence theorem says that

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV &= \int_0^1\int_0^1\int_0^1 \left(2y + 3e^z + x\cos(z)\right)\,dx\,dy\,dz \\ &= \int_0^1\int_0^1 \left(2y + 3e^z + \frac{1}{2}\cos(z)\right)\,dy\,dz \\ &= \int_0^1 \left(1 + 3e^z + \frac{1}{2}\cos(z)\right)\,dz \\ &= 1 + 3\,(e-1) + \frac{1}{2}\sin(1).\end{aligned}$$

That looks like it would have been unpleasant to compute via six flux integrals.

2. This is even worse than the first problem. We couldn't possibly compute the flux integrals over the two pieces of the boundary (the paraboloid piece and the disk in the xy-plane) – the vector field $\mathbf{F}$ is simply too complicated. Instead, we'll use the divergence theorem. Notice that

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}\Big(xz\sin(yz) + x^3\Big) + \frac{\partial}{\partial y}\Big(\cos(yz)\Big) + \frac{\partial}{\partial z}\Big(3zy^2 - e^{x^2+y^2}\Big) \\ &= \Big(z\sin(yz) + 3x^2\Big) + \Big(-z\sin(yz)\Big) + \Big(3y^2\Big) = 3x^2 + 3y^2.\end{aligned}$$

Thus we have from the divergence theorem

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV \\ &= \iint_D \int_0^{4-x^2-y^2} \left(3x^2 + 3y^2\right)\,dz\,dA,\end{aligned}$$

where D is the disk $x^2 + y^2 \leq 4$ in the xy-plane. Thus we'll use polar coordinates for this double integral, or cylindrical coordinates for the triple integral:

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} \left(3r^2\right)\,r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 \left(12r^3 - 3r^5\right)\,dr\,d\theta \\ &= \int_0^{2\pi} \left[3r^4 - \frac{1}{2}r^6\right]_0^2\,d\theta \\ &= \int_0^{2\pi} (48 - 32)\,d\theta = 32\pi.\end{aligned}$$

That wasn't bad at all.

3. The difficulty here is that S_1 is *not* the boundary surface of a simple solid region, so we can't blindly apply the divergence theorem. The standard trick is to add another piece S_2 to the boundary so that $S = S_1 \cup S_2$ bounds a region E. This was already done for us in the previous problem, and in fact we found that

$$32\pi = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S_1} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

Thus to find the requested flux we can either integrate over S_1 (icky!) or S_2 (much nicer). We parameterize S_2 with polar coordinates, so $\mathbf{r}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 0\rangle$ (with $0 \le \theta \le 2\pi$ and $0 \le r \le 2$). Then

$$\mathbf{r}_\theta \times \mathbf{r}_r = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin(\theta) & r\cos(\theta) & 0 \\ \cos(\theta) & \sin(\theta) & 0 \end{vmatrix} = \langle 0, 0, -r\rangle.$$

Note that this vector points downward, with is the appropriate orientation for S_2 (and why we chose $\mathbf{r}_\theta \times \mathbf{r}_r$ rather than $\mathbf{r}_\theta \times \mathbf{r}_\theta$). Thus the flux through S_2 is

$$\begin{aligned} \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^2 \mathbf{F}(\mathbf{r}(r,\theta)) \cdot \langle 0, 0, -r\rangle \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 \langle r^3\cos^3(\theta), 1, -e^{r^2}\rangle \cdot \langle 0, 0, -r\rangle \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 re^{r^2} \, dr \, d\theta = \int_0^{2\pi} \left. \frac{e^{r^2}}{2} \right|_0^2 d\theta \\ &= \int_0^{2\pi} \frac{e^4 - 1}{2} \, d\theta = 2\pi \, \frac{e^4 - 1}{2} = \pi \left(e^4 - 1\right). \end{aligned}$$

Thus the flux through S_2 is $32\pi - \pi\,(e^4 - 1) = \pi\,(33 - e^4)$.

4. This is true.

We can justify it with the divergence theorem by recalling that $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$. We proved this in the Curl and Divergence worksheet and Stewart does it in that section as well. The point of the second hint is that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, since $\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$, so you can remember that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ by analogy.

In any case, if S is a closed surface, then we can let E be the solid enclosed by S. Then the divergence theorem says that

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\operatorname{curl} \mathbf{F}) \, dV = \iiint_E 0 \, dV = 0.$$

5. (a) If $\operatorname{div} \mathbf{F} > 0$ at P, then $\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} > 0$ as well, at least when a is small enough. Thus we would expect flux out of P.

(b) If $\operatorname{div} \mathbf{F} < 0$ at P, a similar argument to part (a) says that we would expect more flux into P.

(c) When $\operatorname{div} \mathbf{F} = 0$ at P, we expect that $\iint_{S_a} \mathbf{F} \cdot d\mathbf{S} \approx 0$. This means that the same amount of flow in toward P as out from P.

6. (a) We're aiming to use the divergence theorem, so we compute the divergence of $\mathbf{F}$:

$$\begin{aligned}\operatorname{div} \mathbf{F} &= \frac{\partial}{\partial x}\left(x - 3y^2 z\right) + \frac{\partial}{\partial y}\left(e^{xz} - y\right) + \frac{\partial}{\partial z}\left(2z - \cos(xy)\right) \\ &= 1 - 1 + 2 = 2.\end{aligned}$$

Thus the divergence theorem implies that

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E 2\, dV = 2\mathrm{Vol}(E) = \frac{8}{3}\pi a^3.$$

(b) What this problem is getting at is that if we have a vector field $\mathbf{F}$ with $\operatorname{div} \mathbf{F} = 1$, then we can compute the volume of E via the flux over the boundary S:

$$\mathrm{Vol}(E) = \iiint_E 1\, dV = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}.$$

We'll compute this flux with the given $\mathbf{F} = \frac{1}{3}\langle x, y, z\rangle$.
We parameterize the sphere of radius a with

$$\mathbf{r}(\phi, \theta) = \langle a\sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos(\phi)\rangle,$$

from which we find that

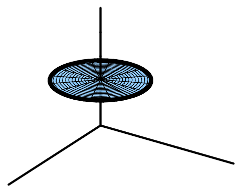
$$\begin{aligned}\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos(\phi)\cos(\theta) & a\cos(\phi)\sin(\theta) & -a\sin(\phi) \\ -a\sin(\phi)\sin(\theta) & a\sin(\phi)\cos(\theta) & 0 \end{vmatrix} \\ &= a^2\sin(\phi)\left\langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\right\rangle \\ &= a\sin(\phi)\left\langle a\sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos(\phi)\right\rangle \\ &= a\sin(\phi)\,\mathbf{r}(\phi, \theta)\end{aligned}$$

is the outward-pointing normal. Since $\mathbf{F} = \frac{1}{3}\mathbf{r}$ and $|\mathbf{r}| = a$, we get $\mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \frac{a\sin(\phi)}{3}|\mathbf{r}|^2 = \frac{a^3\sin(\phi)}{3}$. Thus the flux is

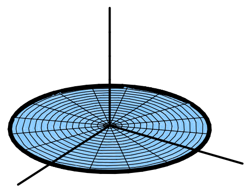
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} \frac{a^3\sin(\phi)}{3}\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{2a^3}{3}\, d\theta = \frac{4\pi a^3}{3}.$$

As expected, this flux is the volume of E.

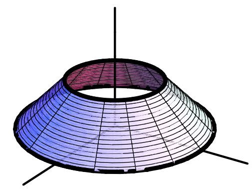
7. As in Problem 6(b), we will compute the volume by computing the flux. This time (as suggested) we'll use $\mathbf{F} = \frac{1}{2}\langle x, y, 0\rangle$, so $\operatorname{div}\mathbf{F} = 1$. We split S into three pieces, the top, bottom, and sides:



S_1, the top



S_2, the bottom



S_3, the sides

Notice that S is the union $S_1 \cup S_2 \cup S_3$, so

$$\text{Vol}(E) = \iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_3} \mathbf{F}\cdot d\mathbf{S}.$$

Two of these three new flux integrals are really simple. Notice that the normals for S_1 and S_2 are either $\mathbf{n} = \mathbf{k} = \langle 0,0,1\rangle$ or $\mathbf{n} = -\mathbf{k}$. Because of the clever way we chose $\mathbf{F}$, however, we thus get $\mathbf{F}\cdot d\mathbf{S} = 0$! So the flux integrals over S_1 and S_2 are zero.

On S_3 we need to actually compute the flux integrals. This involves parameterizing the surface, given to us as $x^2 + y^2 = (2-z)^2$. Thus, for fixed z, this is a circle of radius $2 - z$. This means we have the parameterization

$$\mathbf{r}(z,\theta) = \langle (2-z)\cos(\theta), (2-z)\sin(\theta), z\rangle,$$

and so

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(2-z)\sin(\theta) & (2-z)\cos(\theta) & 0 \\ -\cos(\theta) & -\sin(\theta) & 1 \end{vmatrix} = \big\langle (2-z)\cos(\theta), (2-z)\sin(\theta), (2-z)\big\rangle$$

is the appropriately oriented normal. Since $\mathbf{F}(\mathbf{r}(z,\theta)) = \frac{1}{2}\langle (2-z)\cos(\theta), (2-z)\sin(\theta), 0\rangle$, we get the flux integral

$$\iint_{S_3} \mathbf{F}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \frac{1}{2}(2-z)^2\,dz\,d\theta = \int_0^{2\pi}\left[-\frac{1}{6}(2-z)^3\right]_0^1 d\theta = \int_0^{2\pi}\frac{7}{6}\,d\theta = \frac{7\pi}{3}.$$

Finally, we have found the volume of E: $0 + 0 + \frac{7\pi}{3} = \frac{7\pi}{3}$. (Notice that this agrees with the usual formula for the volume of a cone: $V = \frac{1}{3}\pi r^2 h$. The "full" cone has $r = h = 2$, while the removed cone has $r = h = 1$, so the truncated cone has volume $\frac{1}{3}\pi 2^3 - \frac{1}{3}\pi 1^3 = \frac{7\pi}{3}$.)