

Parametrization Warm-Up

1. Give a parametrization for the sphere of radius a centered at the origin. How would you restrict your parameters to get the upper hemisphere? How would you restrict your parameters to get the part of the sphere with x , y and z all positive?

2. Parametrize the paraboloid $z = 4 - x^2 - y^2$ over the region...

(a) the unit square $[0, 1] \times [0, 1]$

(b) the unit disk $x^2 + y^2 \leq 1$

Let $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ be a vector field in \mathbb{R}^3

- $\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

- $\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$

3. Find $\text{div } \mathbf{F}$ for each of the following vector fields:

(i) $\mathbf{F} = \langle xy, yz, xz \rangle$

(ii) $\mathbf{F} = \langle yz, xz, xy \rangle$

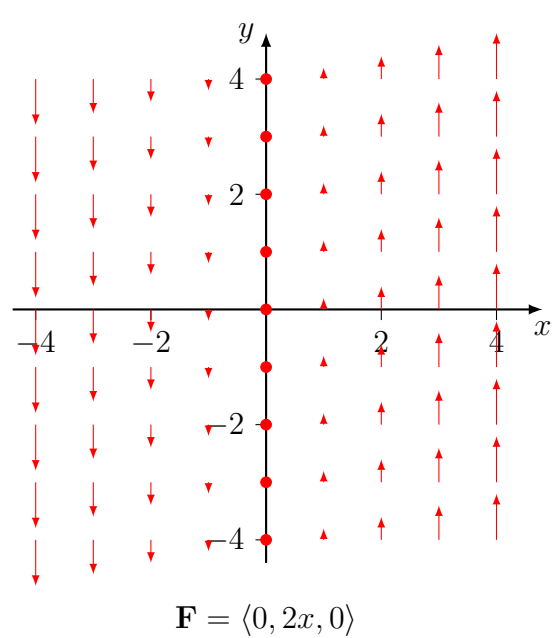
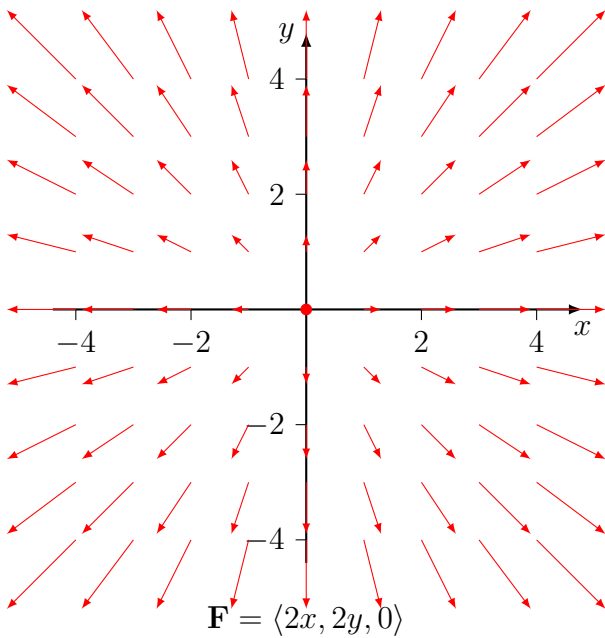
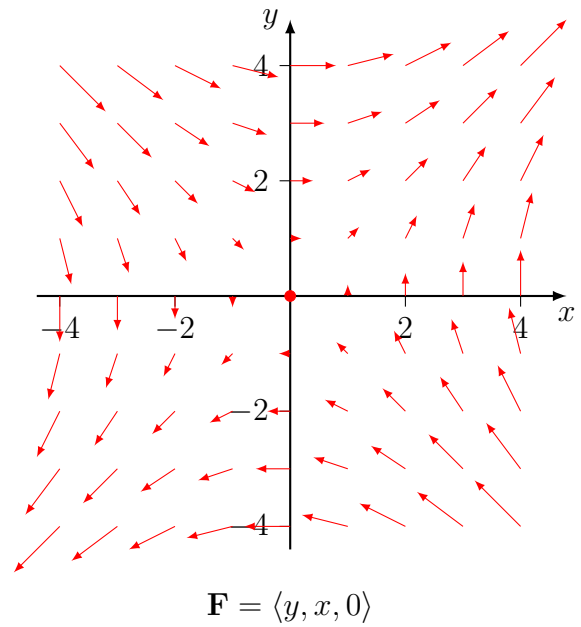
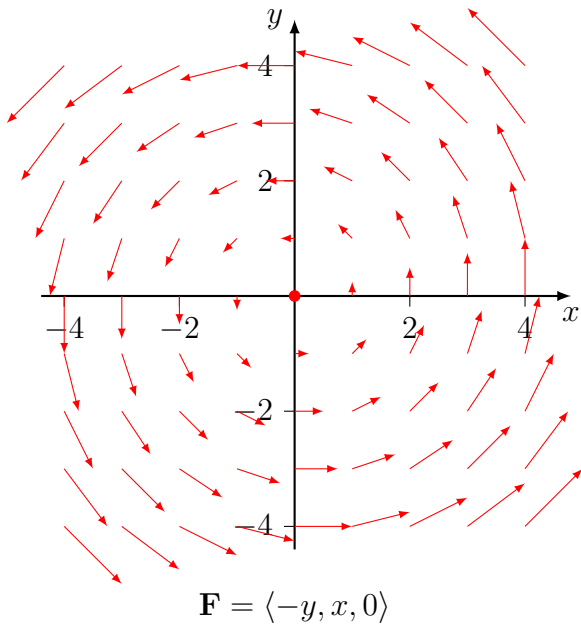
(iii) $\mathbf{F} = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle$
where $r = \sqrt{x^2 + y^2 + z^2}$

(iv) $\mathbf{F} = \text{grad } f$, where f is a function with continuous second derivatives

4. Is there a vector field \mathbf{F} such that $\text{curl } \mathbf{F} = \langle xyz, -y^2z, yz^2 \rangle$?

5. Show that any vector field of the form $\mathbf{F}(x, y, z) = \langle f(y, z), g(x, z), h(x, y) \rangle$ is incompressible.

6. Here are sketches of a few vector fields $\mathbf{F} = \langle P(x, y), Q(x, y), 0 \rangle$ (they're drawn in the plane, but they're defined in all of space). Can you tell which one has zero curl? Zero divergence?



7. Find a parameterization of the sphere of radius a that is positively oriented (pointing out) and one that is negatively oriented (pointed in).

8. Take the parameterization of the hemisphere of radius 2 with $z \geq 0$ that points down. Find a parameterization of the boundary circle that gives a positive orientation.

9. Given a parameterization $\mathbf{r}(u, v)$ over a domain R , what is a parameterization of the same surface that has the opposite orientation?

Suppose \mathbf{F} is a continuous vector field on an oriented surface S with unit normal vector \mathbf{n} . The surface integral of \mathbf{F} over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$$

for a parametrically defined surface.

10. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle y, -x, z \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant with inward orientation.

11. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x, y, 2z \rangle$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the unit square $[0, 1] \times [0, 1]$ with the *downward* orientation.

12. Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle x, y, 2x + 2y \rangle$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the unit disk (centered at the origin) with downward orientation.

13. Use Stokes' Theorem to evaluate $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$ for the vector field $\langle xyz, 0, -x^2y \rangle$ where S is the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$ oriented downwards. (Recall you found the curl of this vector field to be $\langle -x^2, 3xy, -xz \rangle$ on the last worksheet)

14. * Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle -z, x, y \rangle$ and S is the full unit hemisphere (including the base!) on and above the xy -plane (so $x^2 + y^2 + z^2 = 1$ plus a disk) with the outward orientation.