Math 21a

*curl*ing Up With Green's Theorem

Curl:

- If $\mathbf{F}(x,y) = \langle P, Q\rangle$, then

$$\operatorname{curl}\mathbf{F} = Q_x - P_y$$

- If $\mathbf{F}(x,y,z) = \langle P, Q, R\rangle$, then

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle$$

1. We are going to compute some curls of the following vector fields:

(a) $\mathbf{F} = \langle xz, xyz, -y^2\rangle$

(b) $\langle -y, x, 0\rangle$

(c) $\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\rangle$

(d) $\langle \frac{-y}{(x^2+y^2)^{3/2}}, \frac{x}{(x^2+y^2)^{3/2}}, 0\rangle$

2. Suppose that $f(x, y, z)$ is a function that has all derivatives continuous. Verify that $\operatorname{curl} \nabla f = \mathbf{0}$ by computing the left hand side and applying Clairaut's Theorem.

3. Compute the curl of the following vector fields.

(a) $\langle xyz, 0, -x^2y \rangle$

(b) $\langle \ln x, \ln(xy), \ln(xyz) \rangle$

Green's Theorem: Suppose C is a positively oriented, piecewise-smooth, simple closed curve in the plane that bounds a region D. If P and Q have continuous derivatives (in an open set containing the region D) and $\mathbf{F}(x,y) = \langle P, Q \rangle$, then

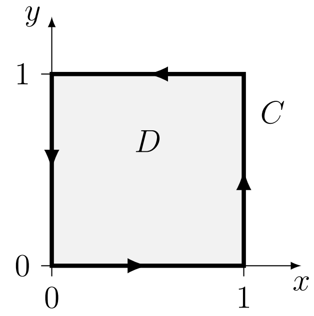
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \operatorname{curl} \mathbf{F} dA = \iint_D (Q_x - P_y)\, dA.$$

For each of the following regions D, associated boundary curves C, and line integrals…

(a) Compute the given line integral by applying Green's theorem and computing a double integral.

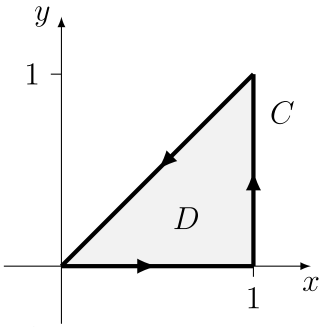
(b) Compute the given line integral directly by parameterizing the path C.

4.



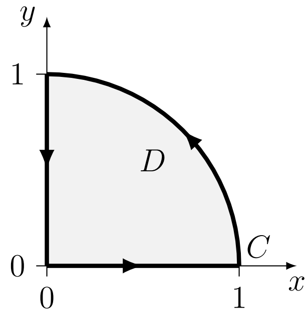
$$\int_C \left\langle xy, x^2 - y^2 \right\rangle \cdot d\mathbf{r}$$

5.



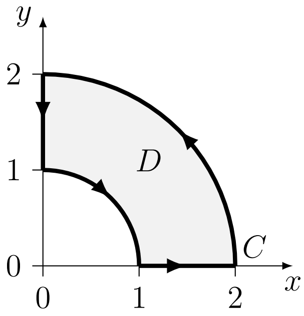
$$\int_C \left\langle x^3, -xy^2 \right\rangle \cdot d\mathbf{r}$$

6.



$$\int_C \langle x, -y \rangle \cdot d\mathbf{r}$$

7.



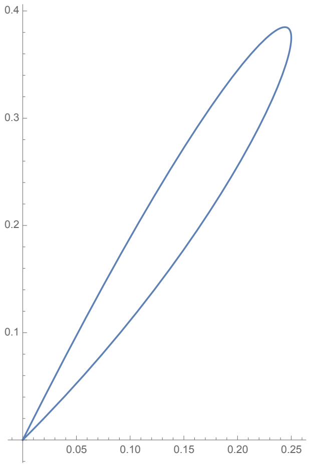
$$\int_C \langle xy^2, -x^2y \rangle \cdot d\mathbf{r}$$

8. (a) Find a vector field $\mathbf{F}(x, y)$ so $\operatorname{curl}\mathbf{F} = 1$. While you're at it, find a couple such $\mathbf{F}$.

(b) Use each vector field you found in the first part to write the area of a region R as an integral over it's positively oriented boundary C.

9. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

10. Find the area of the region enclosed by the curve $\mathbf{r}(t) = \langle t - t^2, t - t^3 \rangle$, $0 \leq t \leq 1$.



11. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle 3y - e^{\sin x}, 7x + \sqrt{y^4 + 1} \rangle$ and C is the circle of radius 3 at the origin oriented clockwise.

12. Compute $\int_C \langle \sqrt{x} + y^3, x^2 + \sqrt{y} \rangle \cdot d\mathbf{r}$ where C is the curve $y = \sin x$ from $(0, 0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0, 0)$. (Be careful of the orientation!)