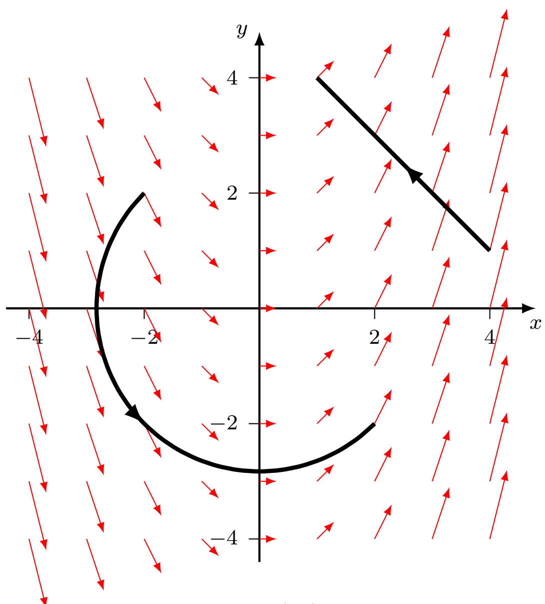
Line Integrals

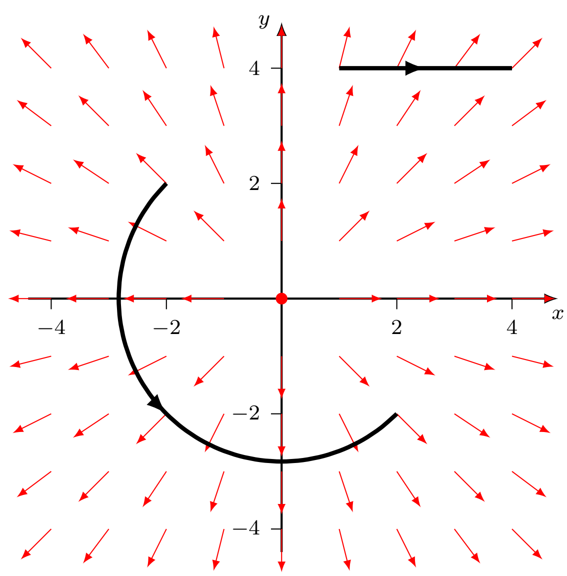
Vector Line Integrals: Suppose C is a smooth curve parameterized by $\mathbf{r}(t)$ $(a \leq t \leq b)$, and $\mathbf{F}$ is a continuous vector field defined on C. Then the *line integral of* $\mathbf{F}$ *along* C *is*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

1. The following is a nice conceptual argument:



Field (II)



Field (III)

For each curve C drawn on the vector field, determine if possible whether $\int_C \mathbf{F} \cdot d\mathbf{r}$ would be positive, negative, or zero.

Fundamental Theorem of Line Integrals: If $\mathbf{F}$ is a gradient field with $\mathbf{F} = \nabla f$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

2. Let $\mathbf{F}(x, y) = \langle 3 + 2xy, x^2 - 3y^2 \rangle$.

- Show this is a gradient field by finding the potential.

- Use this to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is given by $\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t \rangle$ for $0 \leq t \leq \pi$.

- What is the line integral of the straight line from $(0, 1)$ to $(0, e^\pi)$?

3. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the following pairs of vector fields $\mathbf{F}$ and curves C. Note that, in each case, $\mathbf{F}$ is a gradient field.

(a) $\mathbf{F} = \langle -x, y \rangle$ and C is the quarter-circle centered at the origin starting at $(0, -1)$ and proceeding counterclockwise to $(1, 0)$

(b) $\mathbf{F} = \langle y, x \rangle$ and C is the line segment starting at $(0, -1)$ and ending at $(1, 0)$

(c) $\mathbf{F} = \langle \cos x + 2xy, x^2 \rangle$ and C is the box that goes $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(0, 1)$ back to $(0, 0)$.

(d) $\mathbf{F} = \langle ye^x, e^x \rangle$ and C is the craziest path you can imagine from $(2, 3)$ to $(-1, -1)$.

4. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle e^x \cos y, -e^x \sin y \rangle$ and C is the curve $\mathbf{r}(t) = \langle t^3 - \frac{1}{t^3}, e^{\cos \pi t} \rangle$ for $1 \leq t \leq 2$.

5. Let $\mathbf{F} = \nabla f$ where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that satisfy

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 1 \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

The following are equivalent for a vector field $\mathbf{F}$:

- **Gradient Field:** $\mathbf{F} = \nabla f$ for some f.
- **Conservative:** All line integrals are path independent, i. e., only depend on the end points.
- **Closed Loop Property:** The line integral over any closed path is 0.

If $\mathbf{F} = \langle P, Q \rangle$ is devined on a simply connected, open region, the following is also equivalent to the above:

- **Irrotational:** $curl\mathbf{F} = Q_x - P_y = 0$.

6. Are the follow regions open, connected, simply connected?

(a) $\{(x, y) : 0 < y < 3\}$

(b) $\{(x, y) : 1 \leq x^2 + y^2 \leq 4, y \geq 0\}$

(c) $\{(x, y) : (x, y) \neq (2, 3)\}$

(d) $\{(x, y) : 1 < |x| < 2\}$

(e) $\{(x, y) : x^2 \leq y^2\}$

(f) $\{(x, y) : x^2 + y^2 > 1\}$

7. * Let $\mathbf{F}(x,y) = \langle \frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$.

(a) Show that this vector field is irrotational.

(b) Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 and C_2 are the upper and lower halves of the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$.

(c) Why is this not a contradiction?