MATH 21A

# Midterm 2 Review

1. **Some PDEs we know**

| Name | Equation (up to a constant factor) |
|---|---|
| Wave | $u_{tt} = u_{xx}$ |
| Heat | $u_t = u_x x$ |
| Laplace | $f_x x + f_y y = 0$ |
| Transport/Advection | $f_t = f_x$ |
| Eiconal | $f_x^2 + f_y^2 = 1$ |
| Burgers | $f_t + f f_x = f_x x$ |

2. **How to read a contour plot**

   - **max/min**
     Max and min appear at the centers of concentric circles
   - **saddle**
     Saddle points appear at the intersection of level curves
   - **estimating $\nabla f$**
     The direction of $\nabla f$ is perpendicular to the level curve in the direction of increase.
     The magnitude of $\nabla f$ is proportional to the density of contour lines (more contour lines means larger $|\nabla f|$).
   - $D_{\mathbf{u}} f$
     Look at the change in the contour lines in the direction of $\mathbf{u}$ (e. g., $\langle 1, -1\rangle/\sqrt{2}$ is to the down right).
     - $D_{\mathbf{u}} f > 0 \iff$ function is increasing in that direction, i. e., to the higher side of the contour
     - $D_{\mathbf{u}} f = 0 \iff$ function is constant in that direction, i. e., along the level curve
     - $D_{\mathbf{u}} f < 0 \iff$ function is decreasing in that direction, i. e., to the lower side of the contour

     For sign of partials, $f_x$ is to the right and $f_y$ is up.
   - **max on some curve**
     From Lagrange, we know local extrema of $f$ on a curve $g = c$ come at the points where a level curve of $f$ is tangent to $g = c$

3. **Common themes from T/F**

   (a) Clairaut's Theorem
   If asked if two partials are equal, count the number taken in each variable.
   $f_{xyzyxyzyyxyzy} = f_{xxxyyyyyyyzzz}$

   (b) Changing order of integration
   If asked if two integrals are equal, sketch the regions. In rectangle, make sure both the order of variables AND bounds are switched.
   $\int_0^1 \int_1^2 f(x,y)dxdy \neq \int_0^1 \int_1^2 f(x,y)dydx$

(c) Lagrange multipliers
The value of $\lambda$ doesn't matter

(d) Chain rule
If you see either $\frac{d}{dt}f(\mathbf{r}(t))$ or $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$, try replacing it with the other.

(e) Fundamental Theorem of Calculus
Use this if the function variables are in the bounds of the integral.
Given $f(x,y) = \int_y^x f(s)ds$, then $\nabla f = \langle f(x), -f(y) \rangle$.

(f) Don't evaluate integrals
If you're asked to evaluate a not-easy integral, think of the geometric meaning instead.
$\int_0^\pi \int_0^1 r dr d\theta$ is the area of the top half of a circle of radius 1.

**1.** Find extrema of $f(x,y,z) = 8x - 4z$ given $x^2 + 10y^2 + z^2 = 5$.

**2.** Find extreme values of

$$2x^2 + 3y^2 - 4x - 5$$

in the region defined by

$$x^2 + y^2 \leq 16$$

**3.** Evaluate

$$\int_0^1 \int_{3y}^3 e^{x^2} dxdy$$

**4.** Switch the order of integration

$$\int_{-2}^2 \int_{x^2}^4 f(x,y)dydx \text{ and } \int_0^1 \int_{y^3}^{y^2} f(x,y)dxdy$$

5. Find the volume of the solid bounded by $z = x$, $y = x$, $x + y = 2$, and $z = 0$.

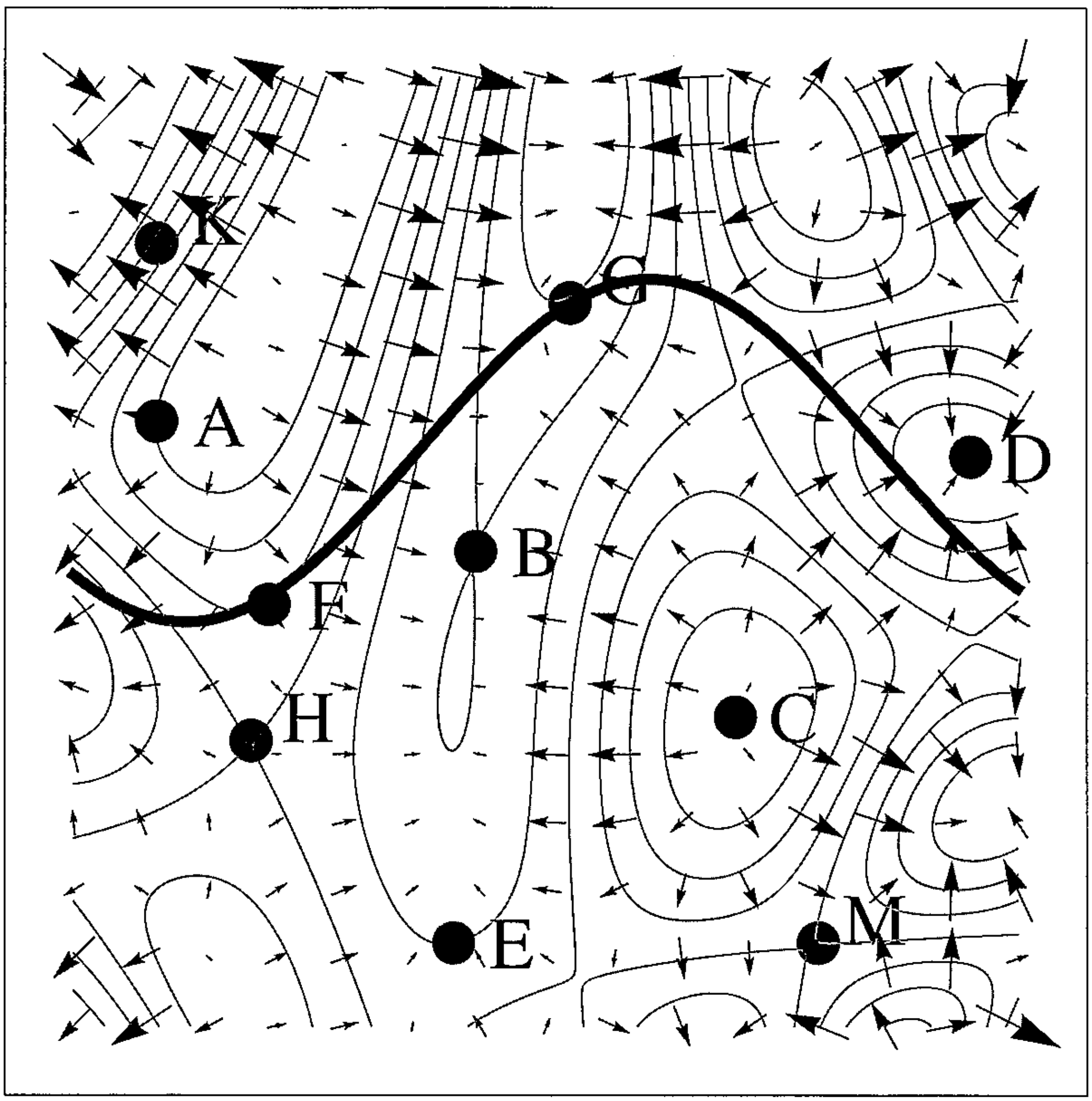
6. Find the volume of the solid bounded by $x^2 + y^2 = 1$ and $y^2 + z^2 = 1$.

7. Set up the integral to find the area of surface parameterized by $\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3, \sin^3 v \rangle$ where $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

8. Find the area of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices $(0, 0)$, $(0, 1)$, and $(2, 1)$.

(10 points) A function $f(x, y)$ of two variables has level curves as shown in the picture. We also see a constraint in the form of a curve $g(x, y) = 0$ which has the shape of the graph of the cos function. The arrows show the gradient. In this problem, each of the 10 letters $A, B, C, D, E, F, G, H, K, M$ appears exactly once.

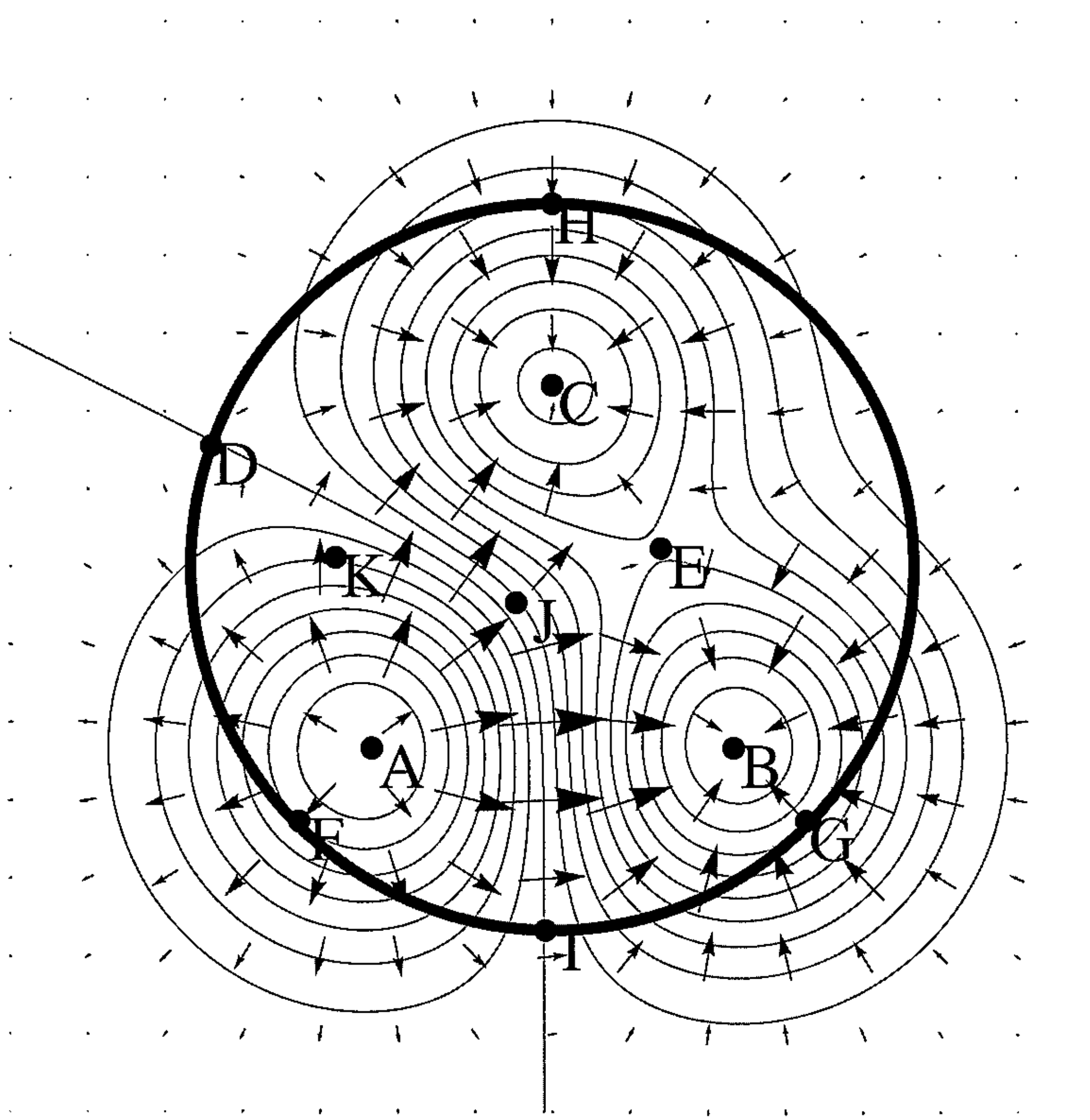
| Enter A-P | Description |
|---|---|
| | a local maximum of $f(x, y)$. |
| | a local minimum of $f(x, y)$. |
| | a saddle point of $f(x, y)$ where $f_{xx} < 0$. |
| | a saddle point of $f(x, y)$ where $f_{xx} > 0$. |
| | a saddle point of $f(x, y)$ where $f_{xx}$ is close to zero |
| | a point, where $f_x = 0$ and $f_y \neq 0$ |
| | a point, where $f_y = 0$ and $f_x \neq 0$ |
| | the point, where $\|\nabla f\|$ is largest |
| | a local maximum of $f(x, y)$ under the constraint $g(x, y) = 0$. |
| | a local minimum of $f(x, y)$ under the constraint $g(x, y) = 0$. |

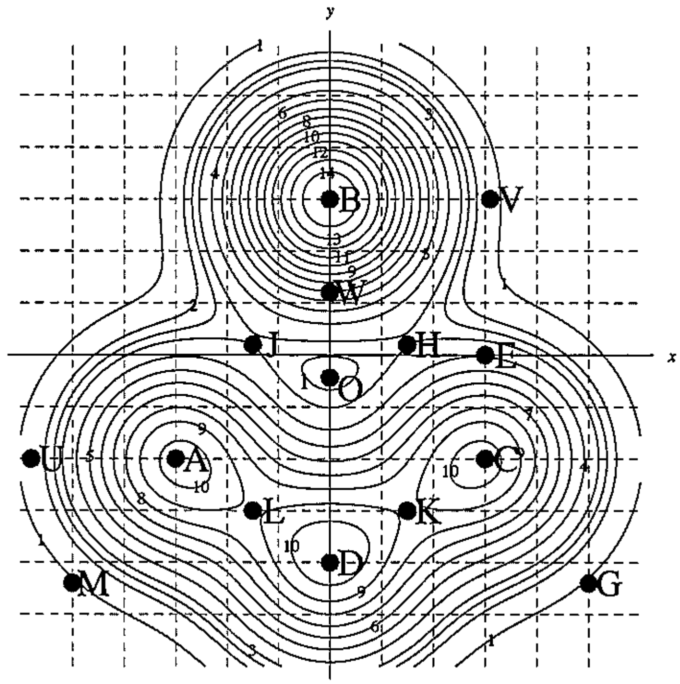


| Problem 3) (10 points) |
|---|

(10 points) Let's label some points in the following contour map of a function $f(x,y)$ indicating the height of a region. The arrows indicate the gradient $\nabla f(x,y)$ at the point. Each of the 11 selected points appears each exactly once.

| Enter A-K | description |
|---|---|
| | a local minimum of $f(x,y)$ inside the circle |
| | a saddle point of $f(x,y)$ inside the circle |
| | a point, where $f_x \neq 0$ and $f_y = 0$ |
| | a point, where $f_x = 0$ and $f_y > 0$ |
| | a point, where $f_x = 0$ and $f_y < 0$ |
| | a point on the circle, where $D_{\vec{v}} f = 0$ with $\vec{v} = \langle 2, -1 \rangle / \sqrt{5}$. |
| | the lowest point on the circle |
| | the highest point on the circle |
| | the local but not global maximum inside or on the circle |
| | the global maximum inside or on the circle |
| | the steepest point inside the circle |






a) (6 points) Enter one label into each of the boxes.

| | |
|---|---|
| | At which point is the length of the gradient maximal? |
| | At which point is the global maximum? |
| | At which point is $f_x > 0, f_y = 0$? |
| | At which point is $D_{\langle 1,1\rangle/\sqrt{2}} f = 0$, $D_{\langle 1,-1\rangle/\sqrt{2}} f < 0$? |
| | At which point is $f$ maximal under the constraint $g(x,y) = y = 0$? |
| | At which point does $f$ have a local minimum? |

b) (4 points) Note that the zero vector is considered both parallel and perpendicular to any other vector.

| | parallel | perp | |
|---|---|---|---|
| The gradient $\nabla f$ is always | | | to the surface $f = c$. |
| For a Lagrange minimum, $\nabla g$ is | | | to $\nabla f$. |
| If $(0,0)$ is a min. of $f$ then $\nabla f(0,0)$ is | | | to $\langle 1,0\rangle$. |
| If $(0,0)$ is max. of $f$ and $g = z - f(x,y)$ then $\nabla g$ is | | | to $\langle 0,0,1\rangle$. |

Problem 1) True/False questions (20 points), no justifications needed

| | | | |
|---|---|---|---|
| 1) | T | F | If $\vec{r}(t)$ is a space curve satisfying $\vec{r}\,'(0) = 0$ and $f(x,y,z)$ is a function of three variables then $\frac{d}{dt} f(\vec{r}(t)) = 0$ at $t = 0$. |
| 2) | T | F | The integral $\int\int_R 1\ dxdy$ is the area of the region $R$ in the $xy$-plane. |
| 3) | T | F | If $f(x,y)$ is a linear function in $x, y$, then $D_{\vec{u}} f(x,y)$ is independent of $\vec{u}$. |
| 4) | T | F | If $f(x,y)$ is a linear function in $x, y$, then $D_{\vec{u}} f(x,y)$ is independent of $(x,y)$. |
| 5) | T | F | If $(0,0)$ is a saddle point of $f(x,y)$ it is possible that $(0,0)$ is a minimum of $f(x,y)$ under the constraint $x = y$. |
| 6) | T | F | The equation $f_{xy}(x,y) = 0$ is an example of a partial differential equation. |
| 7) | T | F | The linearization of $f(x,y) = 4 + x^3 + y^3$ at $(x_0, y_0) = (0,0)$ is $L(x,y) = 4 + 3x^2 + 3y^2$. |
| 8) | T | F | Assume $(1,1)$ is a saddle point of $f(x,y)$. Then $D_{\vec{v}} f(1,1)$ takes both positive and negative values as $\vec{v}$ varies over all directions. |
| 9) | T | F | The integral $\int_{\pi/2}^{\pi} \int_0^2 r\ dr d\theta$ is equal to $\pi$. |
| 10) | T | F | If $\lvert\nabla f(0,0)\rvert = 1$, then there is a direction in which the slope of the graph of $f$ at $(0,0)$ is 1. |
| 11) | T | F | The vector $\nabla f(a,b)$ is a vector in space orthogonal to surface defined by $z = f(x,y)$ at the point $(a,b)$. |
| 12) | T | F | If $f(x,y,z) = 1$ defines $y$ as a function of $x$ and $z$, then $\partial y(x,z)/\partial x = -f_x(x,y,z)/f_y(x,y,z)$. |
| 13) | T | F | In a constrained optimization problem it is possible that the Lagrange multiplier $\lambda$ is 0. |
| 14) | T | F | The area $\int\int_R \lvert\vec{r}_u \times \vec{r}_v\rvert\ dudv$ of a surface is independent of the parametrization. |
| 15) | T | F | The function $f(x,y) = x^6 + y^6 - x^5$ has a global minimum in the plane. |
| 16) | T | F | The area of a graph $z = f(x,y)$ where $(x,y)$ is in a region $R$ is the integral $\int\int_R \lvert f_x \times f_y\rvert\ dxdy$. |
| 17) | T | F | The gradient of a function $f(x,y)$ of two variables can be written as $\langle D_{\vec{i}} f(x,y), D_{\vec{j}} f(x,y)\rangle$, where $\vec{i} = \langle 1,0\rangle$ and $\vec{j} = \langle 0,1\rangle$. |
| 18) | T | F | The length of the gradient of $f$ at a critical point is positive if the discriminant $D(x,y) = f_{xx} f_{yy} - f_{xy}^2$ is strictly positive. |
| 19) | T | F | If $f(0,0) = 0$ and $f(1,0) = 2$ then there is a point on the line segment between $(0,0)$ and $(1,0)$, where the gradient has length at least 2. |
| 20) | T | F | The tangent plane of the surface $-x^2 - y^2 + z^2 = 1$ at $(0,0,1)$ intersects the surface at exactly one point. |

| Problem 1) True/False questions (20 points), no justifications needed |
|---|

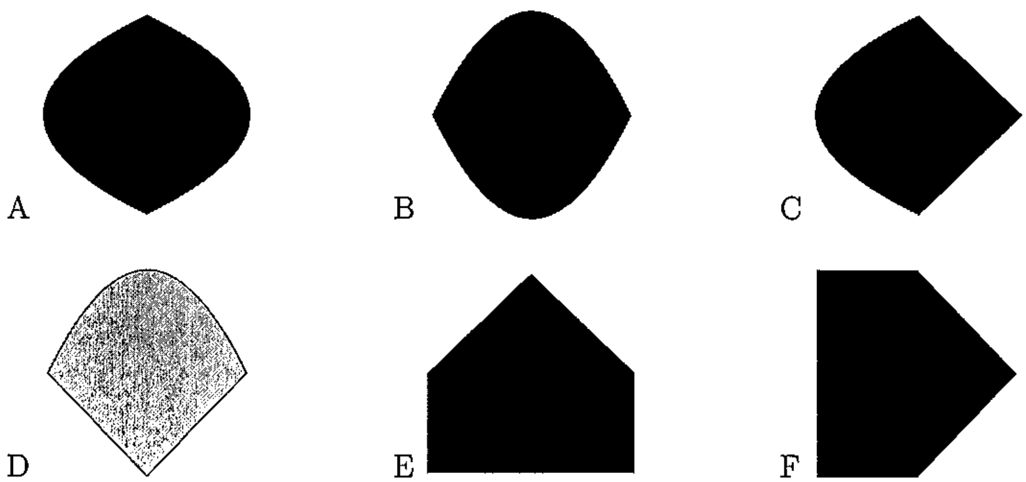
| | | | |
|---|---|---|---|
| 1) | T | F | For any continuous function $f(x,y)$, we have $\int_0^1 \int_1^2 f(x,y)\,dx\,dy = \int_1^2 \int_0^1 f(x,y)\,dx\,dy$. |
| 2) | T | F | If $\vec{u}$ is a unit vector tangent to $f(x,y)=1$ at $(0,0)$ and $f(0,0)=1$, then $D_{\vec{u}} f(0,0)$ is zero. |
| 3) | T | F | Assume $f$ is zero on $x=y$ and $x=-y$, then $(0,0)$ is a critical point of $f$. |
| 4) | T | F | If $(0,0)$ is the only local minimum of a function $f$ and $f$ has no local maxima, then $(0,0)$ is a global minimum. |
| 5) | T | F | If $(0,0)$ is a critical point for $f$, and $f_{yy}(0,0) < 0$ then $(0,0)$ is not a local minimum. |
| 6) | T | F | If $f(x,y)$ and $g(x,y)$ have the same non-constant linearization $L(x,y)$ at $(0,0)$ and $f(0,0)=g(0,0)=0$, then the level sets $f=0$ and $g=0$ have the same tangent line at $(0,0)$. |
| 7) | T | F | There are saddle points with positive discriminant $D > 0$. |
| 8) | T | F | If $R$ is the unit disc, then $\int\int_R x^2 - y^2\,dxdy$ is zero. |
| 9) | T | F | There is a nonzero function $f(x,y)$ for which the linearization $L(x,y)$ is equal to $2f(x,y)$. |
| 10) | T | F | The directional derivative at a local minimum $(0,0)$ is positive in every direction. |
| 11) | T | F | If $\vec{r}(t)$ is a curve on the surface $g(x,y,z)=1$, then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$. |
| 12) | T | F | If $\lvert\nabla f(0,0)\rvert = 2$, there is a direction in which the directional derivative at $(0,0)$ is 2. |
| 13) | T | F | If $D > 0$ at $(0,0)$ and $\nabla f(0,0) = 0$ and $f_{xx}(0,0) < 0$ then $f_{yy}(0,0) < 0$. |
| 14) | T | F | $\int_0^1 \int_0^x f(x,y)\,dydx = \int_0^1 \int_y^1 f(x,y)\,dxdy$. |
| 15) | T | F | The surface area of the sphere of radius $L$ is $\int_0^\pi L^2 \sin(\phi)\,d\phi$. |
| 16) | T | F | If $f(x,y) = g(x)$ is a function of $x$ only, then $D=0$ at every critical point. |
| 17) | T | F | The gradient vector $\nabla f(x_0, y_0)$ is a vector which is perpendicular to the surface $z = f(x,y)$. |
| 18) | T | F | If $\lvert\nabla f(0,0)\rvert = 2$, then there is a unit vector $\vec{v}$ such that $D_{\vec{v}} f(0,0) = 1$. |
| 19) | T | F | The gradient of the function $f(x,y) = \int_x^y \sin(t)\,dt$ is $\langle -\sin(x), \sin(y) \rangle$. |
| 20) | T | F | Assume $f(x,y) = x^2 + y^4$ and a curve $\vec{r}(t)$ satisfies $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$. |

Problem 1) True/False questions (20 points), no justifications needed

| | | |
|---|---|---|
| 1) | T F | The length of the gradient $\nabla f(0,0)$ is the maximal directional derivative $\lvert D_{\vec{v}} f(0,0)\rvert$ among all unit vectors $\vec{v}$. |
| 2) | T F | The relation $f_{xxyyxx} = f_{xxyxyx}$ holds everywhere for $f(x,y) = \cos(\exp(x^{10}) + \sin(x-y))$. |
| 3) | T F | $\int_0^4 \int_0^{4x} f(x,y)\, dydx = \int_0^{16} \int_{y/4}^{16} f(x,y)\, dxdy$. |
| 4) | T F | $g(x,y) = \int_y^0 \int_0^x f(s,t)\, dsdt$ satisfies $g_{xy} = -f(x,y)$. |
| 5) | T F | If $\vec{r}(u,v)$ is a parametrization of the level surface $f(x,y,z) = c$, then $\nabla f(\vec{r}(u,v)) \cdot \vec{r}_v(u,v) = 0$. |
| 6) | T F | If $D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle} f(a,b) = 3$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle} f(a,b) = 5$, then $D_{\vec{v}} f(a,b) \geq 0$ for all unit directions $\vec{v}$. |
| 7) | T F | Given a parametrization $\vec{r}(t)$ of a curve and a function $f(x,y)$ we have $\frac{d}{dt} f(\vec{r}(2t)) = 2\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ at $t=0$. |
| 8) | T F | If $u(t,x)$ solves both the heat and wave equation, then $u_t = c\, u_{tt}$ for some constant $c$. |
| 9) | T F | If the Lagrange multiplier $\lambda$ at a solution to a Lagrange problem is negative then this point is neither a maximum nor a minimum. |
| 10) | T F | The equation $f_x^2 + f_y^2 + f_z^2 = 1$ is an example of a partial differential equation. |
| 11) | T F | If the discriminant $D$ of $f(x,y)$ is zero at $(0,0)$ then $\nabla f(0,0) = \langle 0,0\rangle$. |
| 12) | T F | If $f(x,y,z) = 0$ describes the unit sphere, then the gradient $\nabla f$ points outwards. |
| 13) | T F | If $f(x,y)$ is a continuous function then $\int_0^2 \int_0^1 f(x,y)\, dxdy = \int_0^2 \int_0^1 f(y,x)\, dxdy$. |
| 14) | T F | The point $(5,5,5)$ is a critical point of $f(x,y,z) = x+y+z$. |
| 15) | T F | Assume $\nabla f(0,0) = \langle 0,0\rangle$ with discriminant $D>0$, then $-f(x,y)$ has the same critical point $(0,0)$ with discriminant $D<0$. |
| 16) | T F | $\int\int_R \lvert\nabla f\rvert^2\, dxdy$ is the surface area of the cubic paraboloid $z = f(x,y) = x^3 + y^3$ defined over the region $R$. |
| 17) | T F | If $D(x,y)$ is the discriminant of $f$ at $(x,y)$ then the following poetic formula of the directional derivative of the discriminant holds: $D_{\langle 1,0\rangle} D = \partial_x D$. |
| 18) | T F | Assume $f(x,y) = -x^2 + y^4$ and a curve $\vec{r}(t)$ satisfying $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$ for all $t$. |
| 19) | T F | The Lagrange equations for extremizing $f(x,y)$ under the constraint $g(x,y) = c$ have the same solutions as the Lagrange equations for extremizing $F = f + g$ under the constraint $g = c$. |
| 20) | T F | If $f$ is a maximum under the constraint $g = 1$ at $(0,0)$, and $(0,0)$ is not a critical point for both $f$ and $g$, then the level curves of $f$ and $g$ have the same tangent line at $(0,0)$. |

| Problem 2) (10 points) No justifications needed |
|---|

a) (6 points) Match the regions with the integrals. Each integral matches one region $A-F$.




| Enter A-F | Integral |
|---|---|
| | $\int_{-1}^{1} \int_{-1}^{2-\lvert y\rvert} f(x,y)\; dxdy$ |
| | $\int_{-1}^{1} \int_{y^2}^{2-\lvert y\rvert} f(x,y)\; dxdy$ |
| | $\int_{-1}^{1} \int_{x^2}^{2-x^2} f(x,y)\; dydx$ |
| | $\int_{-1}^{1} \int_{\lvert x\rvert}^{2-x^2} f(x,y)\; dydx$ |
| | $\int_{-1}^{1} \int_{y^2}^{2-y^2} f(x,y)\; dxdy$ |
| | $\int_{-1}^{1} \int_{-1}^{2-\lvert x\rvert} f(x,y)\; dydx$ |

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

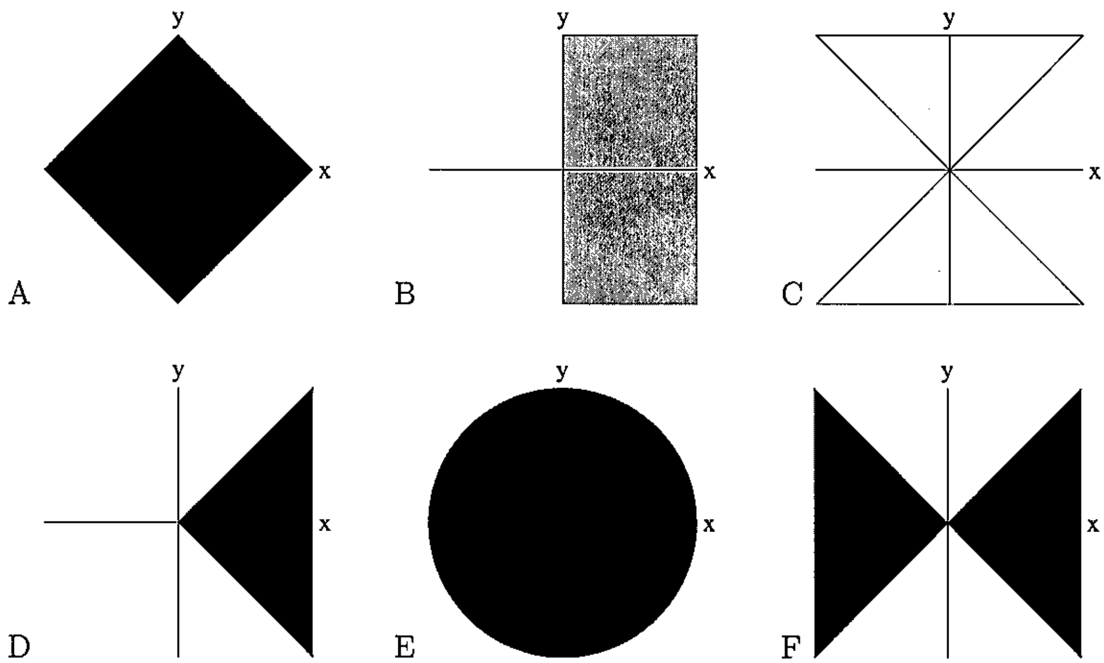
| Fill in 1-4 | Name |
|---|---|
| | Laplace |
| | Wave |
| | Transport |
| | Heat |

| Equation Number | PDE |
|---|---|
| 1 | $g_x - g_y = 0$ |
| 2 | $g_{xx} - g_{yy} = 0$ |
| 3 | $g_x - g_{yy} = 0$ |
| 4 | $g_{xx} + g_{yy} = 0$ |

| Problem 3) (10 points) |
|---|

Problem 2) (10 points)

a) (6 points) Match the regions with the integrals. Each integral matches exactly one region $A - F$.




| Enter A-F | Integral |
|---|---|
| | $\int_{-\pi}^{\pi} \int_{-\lvert y\rvert}^{\lvert y\rvert} f(x,y)\, dxdy$ |
| | $\int_{-\pi}^{\pi} \int_{0}^{\pi} f(r,\theta) r\, dr d\theta$ |
| | $\int_{-\pi}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\, dydx$ |
| | $\int_{0}^{\pi} \int_{-\lvert x\rvert}^{\lvert x\rvert} f(x,y)\, dydx$ |
| | $\int_{-\pi}^{\pi} \int_{0}^{\pi} f(x,y)\, dxdy$ |
| | $\int_{-\pi}^{\pi} \int_{-\pi+\lvert x\rvert}^{\pi-\lvert x\rvert} f(x,y)\, dydx$ |

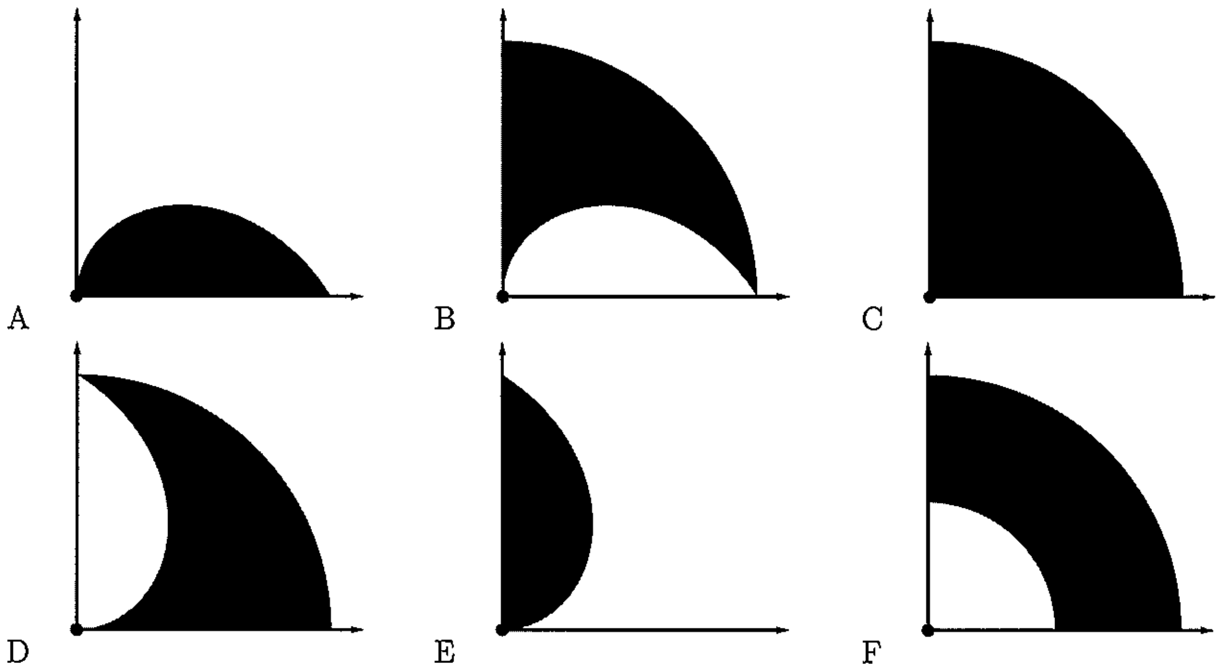
b) (4 points) Name the partial differential equations correctly. Each equation appears once to the left.

| Fill in 1-4 | Order |
|---|---|
| | Burgers |
| | Transport |
| | Heat |
| | Wave |

| Equation Number | PDE |
|---|---|
| 1 | $u_x - u_y = 0$ |
| 2 | $u_{xx} - u_{yy} = 0$ |
| 3 | $u_x - u_{yy} = 0$ |
| 4 | $u_x + uu_x - u_{xx} = 0$ |

| Problem 2) (10 points) |
|---|

a) (6 points) Match the regions with the corresponding polar double integrals




| Enter A-F | Integral of $f(r,\theta)$ | Enter A-F | Integral of $f(r,\theta)$ |
|---|---|---|---|
| | $\int_0^{\pi/2}\int_0^{\pi/2} f(r,\theta) r\, dr d\theta$ | | $\int_0^{\pi/2}\int_\theta^{\pi/2} f(r,\theta) r\, dr d\theta$ |
| | $\int_0^{\pi/2}\int_0^{\theta} f(r,\theta) r\, dr d\theta$ | | $\int_0^{\pi/2}\int_{\pi/2-\theta}^{\pi/2} f(r,\theta) r\, dr d\theta$ |
| | $\int_0^{\pi/2}\int_0^{\pi/2-\theta} f(r,\theta) r\, dr d\theta$ | | $\int_0^{\pi/2}\int_{\pi/4}^{\pi/2} f(r,\theta) r\, dr d\theta$ |

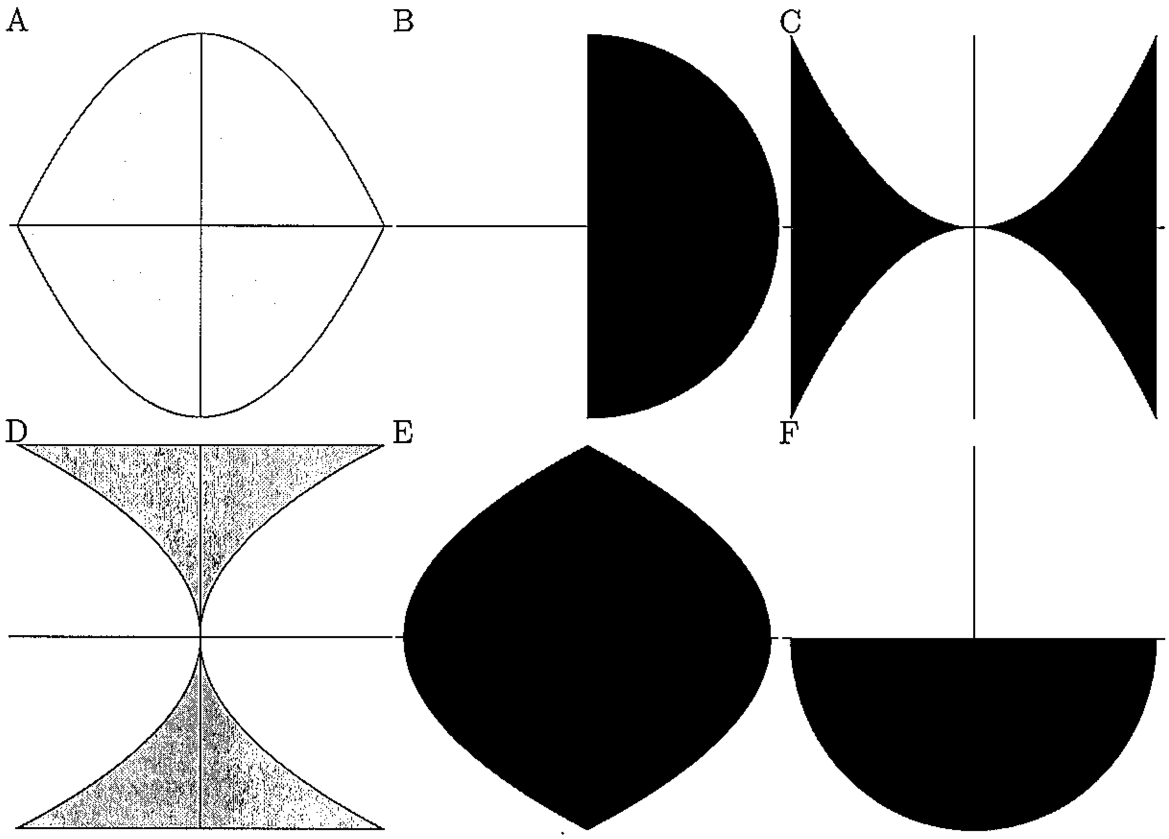
b) (4 points) Match the partial differential equations (PDE's) for the functions $u(t,s)$ with their names. No justifications are needed.

| Enter A,B,C,D here | PDE | Enter A,B,C,D here | PDE |
|---|---|---|---|
| | $u_t + uu_s - u_{ss} = 0$ | | $u_{tt} - u_{ss} = 0$ |
| | $u_{tt} + u_{ss} = 0$ | | $u_t - u_{ss} = 0$ |

| A) Wave equation | B) Heat equation | C) Burgers equation | D) Laplace equation |
|---|---|---|---|

| Problem 2) (10 points) |
|---|

a) (6 points) Match the integration regions with the integrals. Each integral matches exactly one region $A - F$.




| Enter A-F | Integral |
|---|---|
| | $\int_{-1}^{1} \int_{-x^2}^{x^2} f(x,y)\ dydx.$ |
| | $\int_{-1}^{1} \int_{-y^2}^{y^2} f(x,y)\ dxdy.$ |
| | $\int_{-1}^{1} \int_{y^2-1}^{1-y^2} f(x,y)\ dxdy.$ |
| | $\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} f(x,y)\ dxdy.$ |
| | $\int_{-1}^{1} \int_{x^2-1}^{1-x^2} f(x,y)\ dydx.$ |
| | $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} f(x,y)\ dydx.$ |

b) (4 points) Fill in one word names (like "Heat","Wave" etc) for the partial differential equations:

| Enter one word | PDE |
|---|---|
| | $g_x = g_y$ |
| | $g_{xx} = g_{yy}$ |
| | $g_{xx} = -g_{yy}$ |
| | $g_x = g_{yy}$ |

| Problem 3) (10 points) |
|---|