Math 21a

Gradients & Directional Derivatives

1. Warm up: A direction vector is another name for a unit vector.

(a) Find the direction vector in the same direction as $\langle 3, 4\rangle$.

(b) Given a vector $\mathbf{v}$, what direction vector(s) $\mathbf{u}$ maximizes $\mathbf{v}\cdot\mathbf{u}$? Minimizes it? Makes it 0?
Hint: this problem can be done with minimal computation.

2. If $f(x,y) = 3x + 7y^2$ and $\mathbf{u} = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$, find the directional derivative $D_{\mathbf{u}}f$ at $(1,1)$.

3. If $f(x,y) = \sin(xy)$, find the directional derivative $D_{\mathbf{u}}f$ where $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\rangle$.

4. Since $\mathbf{u}$ is a unit vector, we can write the directional derivative $D_{\mathbf{u}}f$ as

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}|\cos\theta = |\nabla f|\cos\theta.$$

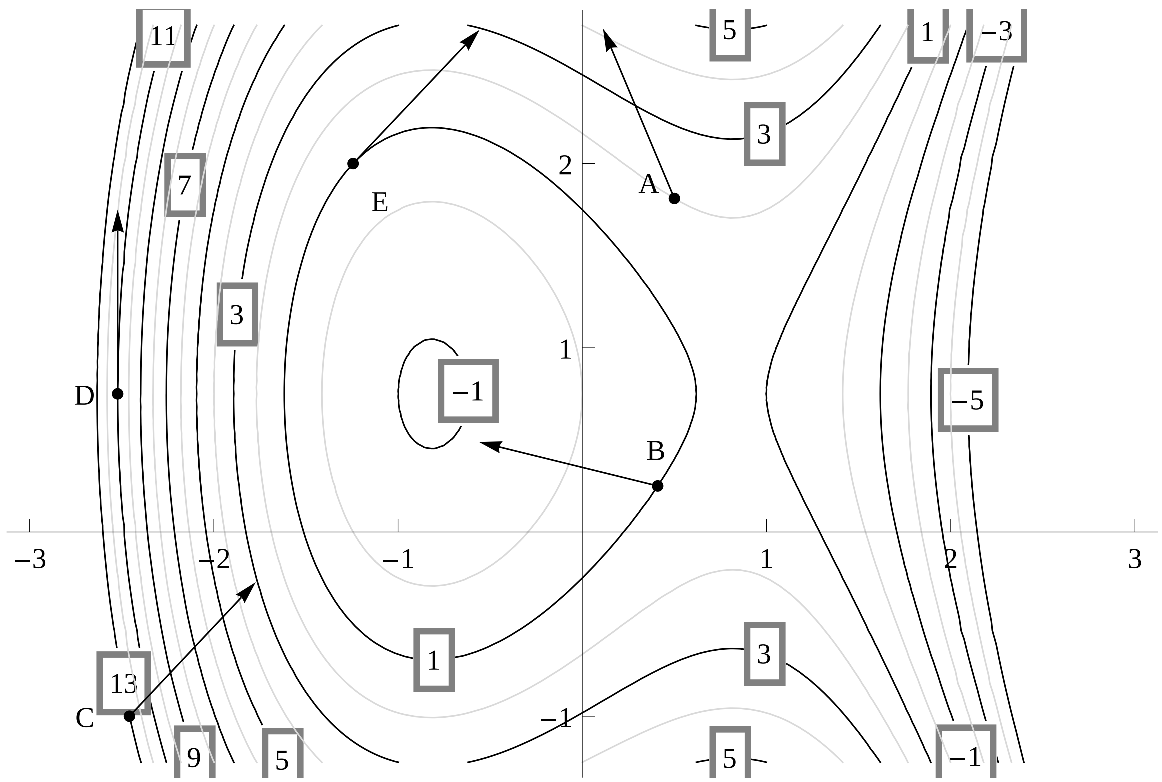
(a) What is the largest possible value of $D_{\mathbf{u}}f$? In what direction does it occur?

(b) What is the smallest possible value of $D_{\mathbf{u}}f$? In what direction does it occur?

(c) When is $D_{\mathbf{u}}f = 0$? What direction (or directions) does this occur?

5. A fly is flying around a room in which the temperature is given by $T(x, y, z) = x^2 + y^4 + 2z^2$. The fly is at the point $(1, 1, 1)$ and realizes that he's cold. In what direction should he fly to warm up most quickly?

6. For each of the points A through E, determine if the directional derivative in the indicated direction is positive, negative, or zero.



7. At points A, B and E in Problem 6, sketch the gradient ∇f on the graph.

8. In this problem, we explore the directional derivative of a non-differentiable function $f(x,y) = \frac{y^3}{x^2+y^2}$ at $(0,0)$. Recall, we have seen that $f_x(0,0) = 0$ and $f_y(0,0) = 1$.

(a) Parameterize the line (in $\mathbb{R}^2$) through $(0,0)$ in the direction $\mathbf{u} = \langle a, b \rangle$ so $\mathbf{r}(0) = (0,0)$.

(b) Write out $f(\mathbf{r}(t))$ and compute $\frac{d}{dt} f(\mathbf{r}(t))$ directly.

(c) Compute $\nabla f(0,0) \cdot \mathbf{u}$.

(d) Notice your last two answers are different. What does this mean?

Gradients & Directional Derivatives – Answers / Solutions

1. (a) The length of this vector is 5, so to get a vector of length 1

(b)

2. Here $\nabla f = \langle 3, 14y \rangle$, so the directional derivative is $D_{\mathbf{u}}f = \langle 3, 14y \rangle \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{1}{5}(9 + 56y)$. At the point $(1, 1)$, this is $D_{\mathbf{u}}f(1, 1) = \frac{1}{5}(9 + 56) = 13$.

3. Here $\nabla f = \langle y\cos(xy), x\cos(xy) \rangle$, so $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1}{\sqrt{2}}\cos(xy)\,(y - x)$. Note that we aren't given a point, so the directional derivative still depends on x and y. (This makes sense: how fast f is changing depends on which point (x, y) we are talking about.)

4. (a) Since $D_{\mathbf{u}}f = |\nabla f|\cos\theta$, the largest value of $D_{\mathbf{u}}f$ is $|\nabla f|$ which occurs when $\cos\theta = 1$. This happens when ∇f and $\mathbf{u}$ point in the same direction.

(b) Since $D_{\mathbf{u}}f = |\nabla f|\cos\theta$, the smallest value of $D_{\mathbf{u}}f$ is $-|\nabla f|$ which occurs when $\cos\theta = -1$. This happens when ∇f and $\mathbf{u}$ point in opposite directions.

(c) Since $D_{\mathbf{u}}f = |\nabla f|\cos\theta$, we find $D_{\mathbf{u}}f = 0$ when $\cos\theta = 0$, or when $\mathbf{u}$ and ∇f are perpendicular. That is, the directional derivative is zero when $\mathbf{u}$ is perpendicular to ∇f.

The key fact(s) that come out of the previous problems are:

- The gradient is perpendicular to the level curves (contours);
- The gradient points toward the direction of greatest increase of the function.

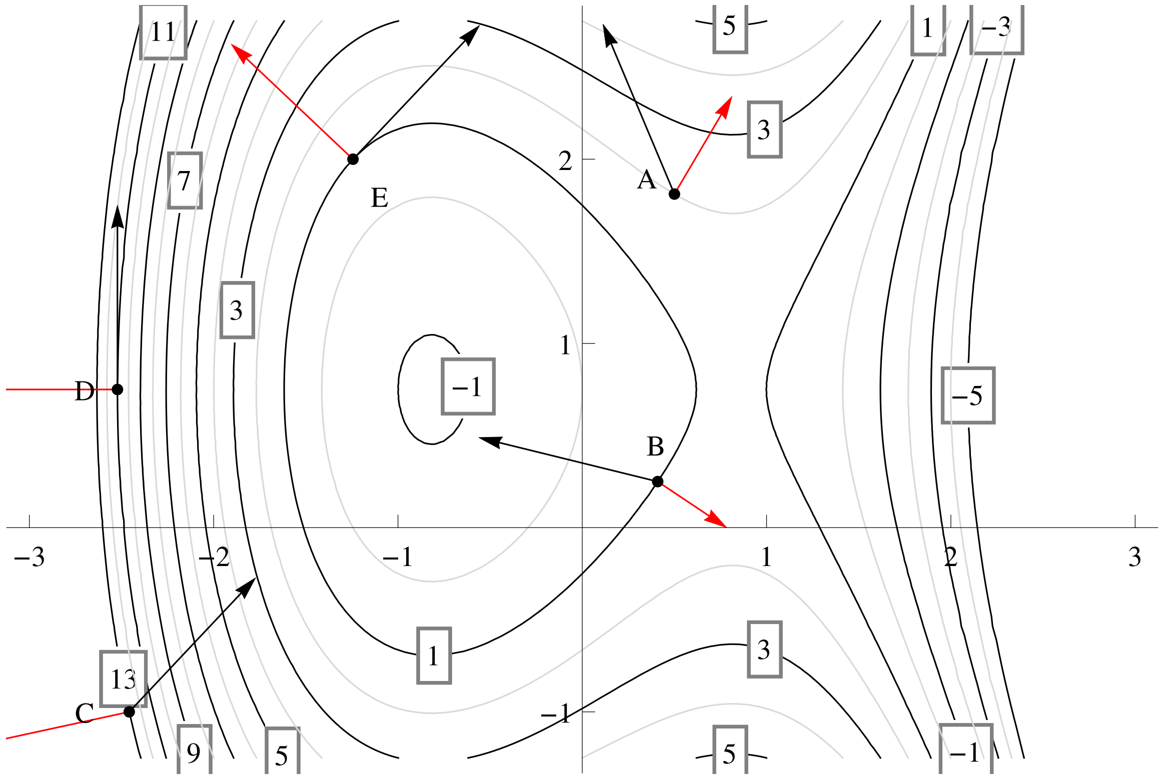
5. The idea here is that T increases fastest in the direction of ∇T. Since $\nabla T = \langle 2x, 4y^3, 4z \rangle$, at the point $(1, 1, 1)$ the fly should go in the direction $\langle 2, 4, 4 \rangle$. Since we want our direction vectors to be unit length, let's divide this by $|\langle 2, 4, 4 \rangle| = \sqrt{36} = 6$ to get $\mathbf{u} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$.

6. Think about the question this way: How is f changing if we are at the given point and headed in the given direction along a line through that point. Some of these are tricky. For example, consider D. The vertical line through D is tangent to the level curve. This means that, traveling along this line, f reaches a local minimum at D. That is, the derivative $\frac{d}{dt}f(\mathbf{r}(t))$ should be zero at D. This is simply $\nabla f \cdot \mathbf{r}'$ at D, which is $\nabla f \cdot \mathbf{u}$.

We get the following signs:

Point	Sign of $D_{\mathbf{u}}f$
A	$+$
B	$-$
C	$-$
D	0
E	0

7. Here we added the gradient vector (in red) at each point. The gradients are actually shown at one-quarter their true length (and the gradients for C and D move off the picture fairly quickly).



8. (a) $\mathbf{r}(t) = \langle at, bt \rangle$

(b) We have

$$f(\mathbf{r}(t)) = \frac{(bt)^3}{(at)^2 + (bt)^2} = t\frac{b^3}{a^2 + b^2} = tb^3$$

Then

$$\frac{d}{dt} tb^3 = b^3$$

(c) We already know that $\nabla f(0,0) = \langle f_x(0,0), f_y(0,0) \rangle = \langle 0, 1 \rangle$. Then

$$\nabla f \cdot \mathbf{u} = b$$

(d) This is different from the directional derivative if $b \neq 0, \pm 1$. This means that our formula for computing the directional derivative *does not work* for this function or nondifferentiable functions in general.