

**Definition.** The *limit* of  $f(x, y)$  as  $(x, y)$  approaches  $(a, b)$  is  $L$  if we can make the values of  $f(x, y)$  as close to  $L$  as we like by taking the point  $(x, y)$  sufficiently close to the point  $(a, b)$ , but not equal to  $(a, b)$ . We write this as  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$ .

**Strategy.**

- To show that a limit  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$  does *not* exist, we usually try to find two different paths approaching  $(a, b)$  on which  $f(x, y)$  has different limits.
  - Showing that a limit  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$  *does* exist is generally harder. If the point  $(a, b)$  is  $(0, 0)$ , one strategy is to rewrite the limit in polar coordinates. Then, no matter how  $(x, y)$  approaches  $(0, 0)$ ,  $r$  tends to 0, so if the limit  $\lim_{r \rightarrow 0^+} f(r \cos \theta, r \sin \theta)$  exists, then the original limit  $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$  also exists.
1. Determine if the following functions have limits at  $(0, 0)$ . Show that you are correct using a strategy described above or discussed in class.

(a)  $f(x, y) = 5x^3 - x^2y^2$

(b)  $f(x, y) = \frac{y^2}{x^2 + y^2}$

(c)  $f(x, y) = -\frac{xy^2}{x^2 + y^2}$

(d)  $f(x, y) = \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

Define the partial derivative of  $f(x, y)$  with respect to  $x$  by

$$f_x = \frac{\partial}{\partial x} (f) = \frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and similarly the partial derivative of  $f(x, y)$  with respect to  $y$  by

$$f_y = \frac{\partial}{\partial y} (f) = \frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

For each of the following functions, compute both first partial derivatives  $f_x$  and  $f_y$  (or  $f_t$ ):

2.  $f(x, y) = e^x \cos(y)$

3.  $f(x, y) = x^3 - 3xy^2$

4.  $f(x, t) = e^{-(x+t)^2}$

5.  $f(x, t) = \sin(x-t) + \sin(x+t)$

We can compute higher order derivatives by simply repeating the process. For example,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}.$$

Compute the four second partial derivatives  $f_{xx}$ ,  $f_{xy}$ ,  $f_{yx}$ , and  $f_{yy}$  for the four functions above.

6. The four functions in (2)-(4) were selected because they solve some *partial differential equations* or PDEs. Listed below are four common PDEs, some of which you will see in the homework. Determine which function is a solution of which PDE by substituting in derivatives.

Laplace Equation:  $f_{xx} + f_{yy} = 0$

Advection (Transport) Equation:  $f_t = f_x$

Wave Equation:  $f_{tt} = f_{xx}$

Heat Equation:  $f_t = f_{xx}$

7. You may notice that  $f_{xy} = f_{yx}$  in all of the above cases. This is a consequence of *Clairaut's Theorem*. Use Clairaut's Theorem to compute the requested derivatives of the following functions:

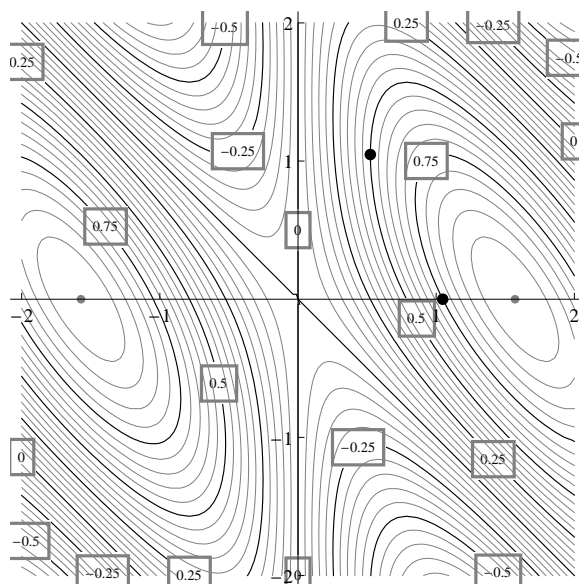
(a)  $f_{xyxyxy}$  if  $f(x, y) = x^2 \cos(e^y + y^2)$       (b)  $f_{xxxxyy}$  if  $f(x, y) = x^3 y^2 - \frac{y}{x + \ln(x)}$

8. Show that the Cobb-Douglas production function  $Y = AK^\alpha L^\beta$  satisfies the equation

$$L \frac{\partial Y}{\partial L} + K \frac{\partial Y}{\partial K} = (\alpha + \beta) Y.$$

The constants  $\alpha$  and  $\beta$  are fixed,  $L$  is labor and  $K$  is capital.

9. Here is a contour plot for the function  $f(x, y) = \sin(x) \sin(x + y)$ .



Without actually computing the derivatives, answer the following questions:

(a) What is the sign of  $f_x$  at  $(x, y) = (\frac{\pi}{3}, 0)$ ?

(b) What is the sign of  $f_y$  at  $(x, y) = (\frac{\pi}{3}, 0)$ ?

(c) What is the sign of  $f_{xx}$  at  $(x, y) = (\frac{\pi}{6}, \frac{\pi}{3})$ ?

(d) What is the sign of  $f_{xy}$  at  $(x, y) = (\frac{\pi}{6}, \frac{\pi}{3})$ ?

**10. Lines are not enough!** Consider  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^4}$ .

(a) Show that the limit along all straight line paths through  $(0, 0)$  are the same.

(b) What other curves go through the origin? Try these and see if you can get a different limit. (Hint: you can!)

(c) Maybe now you conjecture that straight lines and the curves you found above are enough. Does that conjecture hold for  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^3}{x^2+y^6}$ ?

(d) Fix a natural number  $k$ . Does  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^k}{x^2+y^{2k}}$  exist?