Geometry^3

0. **Warm-up:** Find the vector of length 3 in the same direction as $\vec{v} = \langle 1, 2, 3 \rangle$.

1. Find the angle between $\langle 1, 2, 1 \rangle$ and $\langle 1, -1, 1 \rangle$. Draw a rough sketch of the two vectors (no need for axes, just indicate what the angle means graphically).

2. Find the angle between $\langle 2, -3, 0 \rangle$ and $\langle 1, 0, 5 \rangle$.

3. (a) Find the vector projection of $\langle 0, 0, 1 \rangle$ onto $\langle 1, 2, 3 \rangle$.

   (b) Write $\langle 1, 2, 3 \rangle$ as a sum of two orthogonal vectors so that one of them is in the same direction as as $\langle 0, 0, 1 \rangle$ and the other is orthogonal to it.

4. When is $\mathrm{Pr}_{\vec{v}}(\vec{u}) = \vec{0}$?

Algebraic definition of the cross product. If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then we define $\vec{v} \times \vec{w}$ to be $\langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$.

There is a handy way of remembering this definition: the cross product $\vec{v} \times \vec{w}$ is equal to the determinant

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k}$$

where $\vec{i} = \langle 1, 0, 0 \rangle$ and $\vec{j} = \langle 0, 1, 0 \rangle$ and $\vec{k} = \langle 0, 0, 1 \rangle$.
Note: The cross product is only defined for three-dimensional vectors.

5. For this problem, let $\vec{v} = \langle 1, 2, 1 \rangle$ and $\vec{w} = \langle 0, -1, 3 \rangle$.

   (a) Compute $\vec{v} \times \vec{w}$.

   (b) Compute $\vec{w} \times \vec{v}$.

   (c) Let $\vec{u} = \vec{v} \times \vec{w}$, the vector you found in (a). What is the angle between $\vec{u}$ and $\vec{v}$? $\vec{u}$ and $\vec{w}$?

6. For this problem, let $\vec{a} = \langle 3, 5, -2 \rangle$ and $\vec{w} = \langle -2, 7, 1 \rangle$.

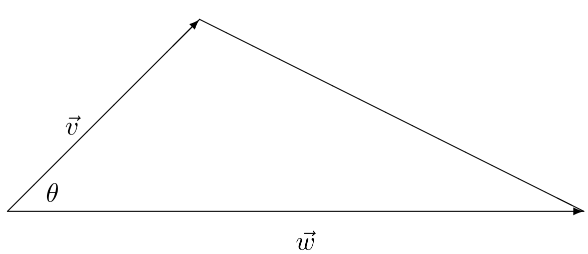
(a) Compute $\vec{a} \times \vec{b}$.

(b) What is the angle between $\vec{a} \times \vec{b}$ and $\vec{a}$? $\vec{a} \times \vec{b}$ and $\vec{b}$?

7. In general, what is the relationship between $\vec{v} \times \vec{w}$ and $\vec{w} \times \vec{v}$?

8. If $\vec{a}$ and $\vec{b}$ are vectors, what is the angle between $\vec{a}$ and $\vec{a} \times \vec{b}$?

9. Any two vectors $\vec{v}$ and $\vec{w}$ which are not parallel determine a triangle, as shown. What is the relationship between the area of the triangle and $\vec{v} \times \vec{w}$?



10. If $\vec{v}$ and $\vec{w}$ are parallel, what is $\vec{v} \times \vec{w}$?

11. What is the scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ of $\vec{u} = \langle 2, 3, 1 \rangle$, $\vec{v} = \langle -2, 0, 1 \rangle$, $\vec{w} = \langle 0, 3, -2 \rangle$?

12. If the scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ is equal to 0, what can you say about the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$?

13. True or false: If $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, then $\vec{v} = \vec{w}$.

14. True or false: If $\vec{v} \times \vec{w} = \vec{0}$ and $\vec{v} \cdot \vec{w} = 0$, then at least one of $\vec{v}$ and $\vec{w}$ must be $\vec{0}$.

15. Find an equation for the plane which passes through the points $(1, 0, 1)$, $(0, 2, 0)$, and $(2, 1, 0)$.

16. Find an equation describing the plane which passes through the points $(2, 2, 1)$, $(3, 1, 0)$, and $(0, -2, 1)$.

17. Find an equation describing the plane which goes through the point $(1, 3, 5)$ and is perpendicular to the vector $\langle 2, 1, -3\rangle$.

18. Let L be the line which passes through the points $(1, -2, 3)$ and $(4, -5, 6)$. Find a parametric vector equation for L.

19. Let L_1 be the line with parametric vector equation $\vec{r}_1(t) = \langle 7, 1, 3\rangle + t\langle 1, 0, -1\rangle$ and L_2 be the line described parametrically by $x = 5, y = 1 + 3t$, $z = t$. How many planes are there which contain L_2 and are parallel to L_1? Find an equation describing one such plane.

20. Find the distance from the point $(0, 1, 1)$ to the plane $2x + 3y + 4z = 15$.

21. Find the distance from the point $(1, 3, -2)$ to the line $\{\langle x, y, z\rangle = t\langle 3, 1, 1\rangle + \langle 0, 1, -2\rangle, t \in \mathbb{R}\}$.

22. True or false: The line $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$ is parallel to the plane $x - 2y + z = 7$.

23. True or false: Let S be a plane normal to the vector $\vec{n}$, and let P and Q be points not on the plane S. If $\vec{n} \cdot \overrightarrow{PQ} = 0$, then P and Q lie on the same side of S.