Points, Vectors, and Geometry (Oh my!)
September 10, 2015

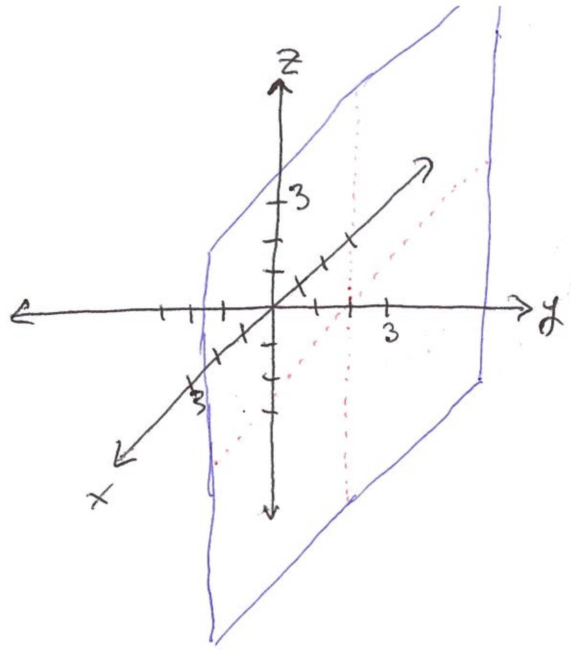
1. (a) Graph the points $P = (1, 2, 3)$ and $Q = (3, -1, -3)$ in $\mathbb{R}^3$ on the same axis. What is the distance between them?

   (b) Find the equation of a sphere centered at the first point that contains the second. What is another point on that plane?

   (c) Find the equation of a sphere centered at $(0, 0, 0)$ that contains $(3, 4, 0)$. What is a difference between the equation you found and $x^2 + y^2 = 5^2$? (There are many ways to answer this, but, as a start, you could try graphing the two equations.)

2. (a) Draw the set in $\mathbb{R}^3$ represented by $\{z = -1\}$. How can you write this in the other set notation?

   (b) What set is depicted below?



(c) Draw the sets $\{x = 2\}$ and $\{z = 1\}$ on the same axes. How would you describe their intersection (the points lying in both) both graphically and geometrically?

3. Are the following better described by vectors or scalars?

   (a) The cost of a Super Bowl ticket.

   (b) The wind at a particular point outside.

   (c) The number of students at Harvard.

   (d) The velocity of a car.

   (e) The speed of a car.

4. Bert and Ernie are trying to drag a large box on the ground. Bert pulls the box toward the north with a force of 30 N, while Ernie pulls the box toward the east with a force of 40 N. What is the resultant force on the box?

5. Oscar is traveling around to pick up some trash. His route instructions[1] are to start at the Trash Collecting Center and

   (a) go $\langle 1, -2, 3 \rangle$,

   (b) go $\langle 0, 1, -2 \rangle$,

   (c) go half of $\langle 2, -4, 10 \rangle$,

   (d) go the opposite of $\langle 0, -5, 4 \rangle$, and

   (e) go twice the opposite of $\langle 1, 1, 1 \rangle$.

   How far away from his starting point is Oscar after the 2nd step? Show that Oscar ends his route at the same place as he begins using vector addition.

Definition. The dot product $\vec{v} \cdot \vec{w}$ of two vectors $\vec{v}$ and $\vec{w}$ is defined as follows.

- If $\vec{v}$ and $\vec{w}$ are two-dimensional vectors, say $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2$.
- If $\vec{v}$ and $\vec{w}$ are three-dimensional vectors, say $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$, then their dot product is $v_1 w_1 + v_2 w_2 + v_3 w_3$.

It is not possible to dot a two-dimensional vector with a three-dimensional vector!

6. (a) What is $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$?

   (b) What is $\langle 1, 2, 3 \rangle \cdot \langle 4, -5, 6 \rangle$?

[1] It's on a hill, so three-dimensional vectors make sense! Or imagine that Oscar is a space garbage man!

Here are some basic algebraic properties of the dot product. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors of the same dimension and c is a scalar, then

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
3. $(c\vec{v}) \cdot \vec{w} = c(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (c\vec{w})$.

7. True or false: if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors of the same dimension, then $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$.

8. What is the relationship between $\vec{v} \cdot \vec{v}$ and $|\vec{v}|$? (Hint: Try it!)