Math 113 Problem Set 6

Due March 8, 2018

February 27, 2018

1. Compute the Taylor expansion of $\zeta(z) = \sum_{n=1}^{\infty}$ around $z = 2$. (3.2, #24)
2. Find the Laurent series of the following around 0 in the indicated regions:
   (a) $\sin \frac{1}{z}, \mathbb{C} - \{0\}$
   (b) $\frac{1}{z(z+1)}, 0 < |z| < 1$
   (c) $\frac{z}{z+1}, 0 < |z| < 1$
   (d) $\frac{e^z}{z^2}, 0 < |z| < \infty$

   (3.3, #1)
3. Let f have a pole at z_0 of order $k \geq 1$. Show that $f(z) \to \infty$ as $z \to z_0$. (3.3, #7)
4. Let $f(z) = \frac{z^2-1}{\cos(\pi z)+1}$ have Taylor series expansion $\sum_{n=0}^{\infty} a_n z^n$ around 0.
   (a) Find a_0, a_1, a_2.
   (b) Find the singularities of f and classify them as essential or a pole of a specified order.
   (c) What is the radius of convergence of this series?

   (3, #29)
5. Define the Bernoulli numbers B_n to be the coefficients of the Taylor series of $\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$ at 0.
   (a) What is the radius of convergence of this series?
   (b) Using the Cauchy Integral Formula and γ as the unit circle to find an expression for B_n of the form
   $$B_n = \int_0^{2\pi} g_n(\theta) d\theta$$
   for an appropriately chosen g_n.

   (3, #33)

6. Find the fault in the following argument:

   Set $f(z) = \sum_{n=0}^{\infty} z^n + \sum_{n=1}^{\infty} \frac{1}{z^n}$. Then from other power series, we know that

$$\begin{aligned} \sum_{n=1}^{\infty} z^n &= \frac{z}{1-z} \\ \sum_{n=0}^{\infty} \frac{1}{z^n} &= \frac{1}{1-\frac{1}{z}} = \frac{-z}{1-z} \end{aligned}$$

   (3, #38) The above left-hand sides cover all the terms of f, so

$$f(z) = \frac{z}{1-z} + \frac{-z}{1-z} = 0$$

   for all z.

For problems from the book, something like 1, #8 refers to #8 from the exercises at the end of Chapter 1, while something like 1.3, #1 refers to #1 from the exercises at the end of section 1.3.