Math 112 Problem Set 8

Due Wednesday, April 4, 2018

March 23, 2018

Core problems:

1. The following is a further exploration of the properties of images and inverse images. Let $f : X \to Y$ and show the following: (this is worth 2 problems)

   (a) Given $A, A' \subset X$, if $A \subset A'$, then $f(A) \subset f(A')$. Find a counter-example showing the converse doesn't hold.

   (b) Given $A \subset X$, $A \subset f^{-1}(f(A))$. Find an example showing equality doesn't hold.

   (c) Given $B \subset Y$, $f(f^{-1}(B)) = B \cap f(X)$.

2. Find a continuous function f and an open subset A of its domain such that $f(A)$ is not open. (You can use the functions from calculus if you want, although this is not necessary.)

3. Let $f : [0,1] \to [0,1]$ be continuous. Show that f has a fixed point; that is, there is $x \in [0,1]$ such that $f(x) = x$. (4.5, #3)

4. (a) Find $f : \mathbb{R}^2 \to \mathbb{R}$ such that

   $$\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

   although both exist.

   (b) Find $f : \mathbb{R}^2 \to \mathbb{R}$ such that

   $$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

   but f is not continuous at 0.

   (c) Find $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous on every line through $(0,0)$ but not continuous at $(0,0)$. Being continuous on every line means that, for every $m \in \mathbb{R}$, $\lim_{t\to 0} f(t, mt) = f(0,0)$ and $\lim_{t\to 0} f(0,t) = f(0,0)$.

   (You may use the various functions from calculus for this exercise; 4, #14)

5. Let $K \subset \mathbb{R}^n$ be compact and let $f : K \to \mathbb{R}^m$ be continuous and one-to-one. Show that $f^{-1} : f(K) \to K$ is continuous. Find an example that this can fail if K is connected but not compact. (4, #7)

6. (a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is connected, is $f^{-1}(K)$ connected?
   (b) Show that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$ and $B \subset \mathbb{R}^n$ is bounded, then $f(B)$ is bounded.

   (4, #3)

7. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $K \subset \mathbb{R}$ is compact, is $f^{-1}(K)$ compact? Prove or find a counter example.

Niche problems:

1. Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Show that f is continuous iff for every $B \subset A$,

$$f\left(cl(B) \cap A\right) \subset cl\left(f(B)\right)$$

   (4, #10)

2. Show that closed intervals $[a, b]$ are the only subsets of $\mathbb{R}$ that are both compact and connected.

3. We proved in class that continuous functions satisfy the intermediate value theorem. This exercise shows the converse fails.

   Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & x = 0 \\ \sin \frac{1}{x} & x \neq 0 \end{cases}$$

   Show that f satisfies the Intermediate Value Property: for any $a < c \in \mathbb{R}$ and $z \in \mathbb{R}$ such that z is between $f(a)$ and $f(c)$, there is a $b \in [a, c]$ such that $f(b) = z$.

   Find a value $x_0 \in \mathbb{R}$ such that f is not continuous at x_0 (and show this).

   You can use facts about sin for this exercise.

4. We've built complete, ordered fields from Dedekind cuts and from equivalence classes of Cauchy sequences of rationals. But maybe this seems strange since we like to think of real numbers as, well, numbers. Specifically, we like to write down numbers as decimals in base 10. This exercise will attempt to make the point that this is a hard thing to make precise.

   A base 10 representation should consist of $(k; a_k, \ldots, a_1; \{b_n\})$ where

   - k is a nonnegative integer;
   - each a_i is an integer between 0 and 9; and
   - $\{b_n\}_{n=1}$ is a (infinite) sequence of numbers between 0 and 9.

   This is a weird way of writing it, but we imagine that $(k; a_k, \ldots, a_0; \{b_n\})$ represents the real number

$$\sum_{i=0}^{k} a_i 10^i + \sum_{j=0}^{\infty} b_i 10^{-i}$$

(a) Show that this sequence converges for any choice of a base 10 representation.

(b) Define an ordering $(k; a_k, \dots, a_1; \{b_n\}) \prec (k'; a'_{k'}, \dots, a'_1; \{b'_n\})$ that agrees with this ordering of real numbers.

(c) Can you define addition for these representations? What about multiplication? You don't have to succeed here, but at least talk about why this is a very hard task.

Doc Brown problems:

1. Give an example of a function defined on $\mathbb{R}$ which is continuous everywhere but fails to be differentiable exactly at $x_1, \dots, x_n$. (4.7, #1)

2. Does the mean value theorem apply to $f(x) = \sqrt{x}$ on $[0, 1]$? Does it apply to $g(x) = \sqrt{|x|}$ on $[-1, 1]$? (4.7, #2)

For problems from the book, something like 1, #8 refers to #8 from the exercises at the end of Chapter 1, while something like 1.3, #1 refers to #1 from the exercises at the end of section 1.3.