Math 112 Problem Set 7

Due Wednesday, March 28, 2018

March 18, 2018

Core problems:

1. It will be useful to have some rules of inverse images of functions. To set notation, if $f : X \to Y$ is a function and $A \subset X$ and $B \subset Y$, then set

$$\begin{aligned} f^{-1}(B) &= \{x \in X \mid f(x) \in B\} \\ f(A) &= \{f(x) \in Y \mid x \in A\} \end{aligned}$$

Show that, for any $B, B' \subset Y$,

$$f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$$

Does this hold for images instead of inverse images? I. e., given $A, A' \subset X$, do we have

$$f(A \cap A') = f(A) \cap f(A)$$

Prove or find a counterexample.

2. Show that, for any $B, B' \subset Y$,

$$f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$$

Does this hold for images instead of inverse images? I. e., given $A, A' \subset X$, do we have

$$f(A \cup A') = f(A) \cup f(A)$$

Prove or find a counterexample.

3. Given $f : \mathbb{R}^n \to \mathbb{R}^m$, we can think of f as a vector of m functions from $\mathbb{R}^n$ to $\mathbb{R}$, that is, write $f = (f_1, \ldots, f_m)$ where each $f_k : \mathbb{R}^n \to \mathbb{R}$ and, for every $\vec{x} \in \mathbb{R}^n$,

$$f(\vec{x}) = (f_1(\vec{x}), \ldots, f_m(\vec{x}))$$

Given $\vec{a} \in \mathbb{R}^n$, show that f is continuous at $\vec{a}$ iff each f_k is continuous at $\vec{a}$.

4. Show that $f(x) = \sqrt{x}$ is continuous at any $a > 0$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Show that

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq f(x, y) \leq 1\}$$

is closed. (4.1, #3)

6. The following function is sometimes called Dirichlet's function:

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

Show that f is discontinuous everywhere.

7. Define maps $s : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as $s(\vec{x}, \vec{y}) = \vec{x} + \vec{y}$ and $m(r, \vec{x}) = r\vec{x}$. Show that these mappings are continuous.(4, #8)

8. Let $f : \mathbb{R} \to \mathbb{R}$ be increasing and bounded above. Show that

$$\lim_{x \to 0^+} f(x)$$

exists. (4, #35)

Niche problems:

1. In proving sequences converge, we often let an $\epsilon > 0$, do some stuff, and end up showing the desired absolute value is less than some nice function of ϵ, like $\frac{2\epsilon}{3}$. We're basically done, but formally we have to go back and pick another epsilon. The following problem tells us we don't need to worry about this any more!

   Let $\{x_n\} \subset \mathbb{R}$ be a sequence and $x \in \mathbb{R}$. Then $x_n \to x$ iff there is a function $f : (0, \infty) \to (0, \infty)$ such that

   - $\lim_{x \to 0^+} f(x) = 0$; and
   - for every $\epsilon > 0$, there is N such that $n > N$ implies $|x - x_n| < f(\epsilon)$.

2. The following function is sometimes called the raindrop function:

$$f(x) = \begin{cases} \frac{1}{q} & x \neq 0 \text{ is rational and written in lowest terms as } \frac{p}{q} \\ 0 & \text{otherwise} \end{cases}$$

Show that f is discontinuous at each nonzero rational and continuous at each irrational and 0.

3. Given $f : \mathbb{R} \to \mathbb{R}$, set $\Gamma \subset \mathbb{R}^2$ to be the graph of the function, that is,

$$\Gamma = \{(x, f(x)) \mid x \in \mathbb{R}\}$$

Show that f is continuous iff Γ is connected.

4. Recall that last problem set, we defined an equivalence relation $\sim$ on the set $\mathcal{C}$ of rational Cauchy sequences, and then defined addition, multiplication, and order on $\mathcal{C}/\sim$, the set of equivalence classes. Show that the turns $\mathcal{C}/\sim$ into a complete ordered field. (As before, this is worth 20 points, replaces 2 Core problems, and counts as 2 Niche problems)

Doc Brown problems:

1. Set $A = \{x \in \mathbb{R} \mid \sin x = 0.56\}$. Show that A is a closed set. Is it compact? (You can use facts about sin like that it is continuous., 4.3, #3)

2. Give an example of a continuous and bounded function on all of $\mathbb{R}$ that does not attain its max or min. (4.4, #1)

3. Prove that there is no continuous map from $[0, 1]$ onto $(0, 1)$. (4.5, #5)

For problems from the book, something like 1, #8 refers to #8 from the exercises at the end of Chapter 1, while something like 1.3, #1 refers to #1 from the exercises at the end of section 1.3.