Math 112 Problem Set 6

Due **Friday** March 9, 2018

March 1, 2018

Core problems:

1. Find an open cover of the following with no finite subcover

   (a) $\{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| < 1\}$

   (b) $\mathbb{Z}$ in $\mathbb{R}$

   (3, #5)

2. We generalize a previous problem to $\mathbb{R}^n$: Let $\{F_k\}_k$ be a sequence of compact sets in $\mathbb{R}^n$ such that

   (a) $diam(F_k) \to 0$

   (b) $F_{k+1} \subset F_k$ for $k \in \mathbb{N}$

   Show that there is exactly one point in $\cap_{k=1}^{\infty} F_k$. (3, #6; $diam(A) = \sup\{\|\vec{x} - \vec{y}\| \mid x, y \in A\}$)

3. Prove or find a counter-example to each of the following for $A \subset \mathbb{R}^n$:

   (a) A is compact implies $\mathbb{R}^n - A$ is connected

   (b) A is connected implies $\mathbb{R}^n - A$ is connected

   (c) A is connected implies $\mathbb{R}^n - A$ is open or closed

   (d) $\mathbb{R}^n - \{\vec{x} \mid \|x\| \leq 1\}$ is connected

   (3, # 9)

4. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be compact. Show that $A \times B \subset \mathbb{R}^{n+m}$ is compact. (3, # 15)

5. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be path-connected. Show that $A \times B \subset \mathbb{R}^{n+m}$ is path-connected. (3, # 15)

6. $A \subset \mathbb{R}^n$ is called *nowhere dense* iff $int(\overline{cl(A)}) = \emptyset$. Show that there does **not** exist a sequence $\{A_n\}_n$ of nowhere dense subsets of $\mathbb{R}^n$ such that $\mathbb{R}^n = \cup_{n=1}^{\infty} A_n$.(3, #33)

7. Let $A, B \subset \mathbb{R}^n$ with A compact and B closed such that $A \cap B = \emptyset$. We previously defined the distance between a point and a set and now define the distance between these sets as

$$dist(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}$$

   (a) Show that $dist(A, B) \neq 0$.

   (b) Can you find closed A, B that are disjoint but the distance between them is zero?

   (3, #37)

8. Recall the definition of the Cantor set C from class (or see the book). Prove the following things about it:

   (a) C is compact

   (b) C has infinitely many points

   (c) $int(C) = \emptyset$

   (d) C is closed and has no isolated points (this is called being *perfect*)

   (e) for any $x \neq y \in C$, there are $U, V \subset C$ that are disjoint and open such that $x \in U$ and $y \in V$ and $C \subset U \cup V$ (this is called being *totally disconnected*)

   (3, #39)

Niche problems:

1. Define the 'Topologist's sine curve' to be

$$\mathbb{T} = \{(0, y) \mid y \in [-1, 1]\} \cup \left\{ \left(x, \sin \frac{1}{x}\right) \mid x > 0 \right\} \subset \mathbb{R}^2$$

   In class, we claimed that this is connected but not path-connected. Prove this.

   (You can use your knowledge about sin for this problem.)

2. We expand the above discussion of the Cantor set C. We take for granted that every number in $[0, 1]$ has a representation in base 3 (and you can use facts about bases in this problem). That is, given $x \in [0, 1]$, there is some sequence $\{b_n\}_n$ whose terms consist of $0, 1$, or 2 such that $x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$ (although this sequence is not unique). Show that $x \in C$ iff it has a base 3 representation consisting only of 0's and 2's.

3. Lest you think we were done, here is an alternate way of constructing $\mathbb{R}$ from $\mathbb{Q}$ using Cauchy sequences. This method has the advantage that it's actually a general method of taking a metric space (X, d) and enlarging it to a complete metric space $(\bar{X}, \bar{d})$. Recall the definition of an equivalence relation from last time.

   Let R be an equivalence relation on a set A. Given $a \in A$, the equivalence class of a is the subset of A consisting of everything that is R-equivalent to a:

$$[a]_R = \{b \in A \mid (a, b) \in R\}$$

Then we can form the set of *equivalence classes*

$$A/R = \{[a]_R \mid a \in A\}$$

which has equivalence classes as its elements.

Show that A/R is a partition of A. That is, every point in a is in one of the sets and that different classes are disjoint.

4. Let $\mathcal{C}$ be the set of Cauchy sequence with elements in $\mathbb{Q}$. Define an equivalence relation $\sim$ on $\mathcal{C}$ as follows: given Cauchy sequences $\{a_n\}_n, \{b_n\}_n \subset \mathbb{Q}$, we set

   $$\{a_n\}_n \sim \{b_n\}_n \text{ iff for every rational } \epsilon > 0, \text{ there is } N \text{ such that if } n > N, \text{ we have } |a_n - b_n| < \epsilon$$

   This is an equivalence relation (you don't have to show this, it's the equivalence relation from the previous assignment written to avoid $\mathbb{R}$).

   Similar to Dedekind cuts, we wish to turn $\mathcal{C}/\sim$ into a complete ordered field. Define $\oplus, \otimes, \lhd$ on equivalence classes of Cauchy sequences as follows:

   $$\begin{aligned} [\{a_n\}]_\sim \oplus [\{b_n\}]_\sim &= [\{a_n + b_n\}]_\sim \\ [\{a_n\}]_\sim \otimes [\{b_n\}]_\sim &= [\{a_n b_n\}]_\sim \\ [\{a_n\}]_\sim \lhd [\{b_n\}]_\sim &\text{ iff } \text{there is } \epsilon > 0 \text{ and } N \text{ such that } n > N \text{ implies } a_n + \epsilon < b_n \end{aligned}$$

   But there's a (potential) problem! We're defining functions on equivalence classes of Cauchy sequences, but using the particular sequence that we've picked as a 'representative' to define the function. This is a problem if equivalent Cauchy sequences sum to inequivalent sequences. Show that this can't happen and that the functions are *well-defined*.

   In particular, let $\{a_n\} \sim \{a'_n\}$ and $\{b_n\} \sim \{b'_n\}$ be Cauchy sequences. Show that

   (a) $\{a_n + b_n\} \sim \{a'_n + b'_n\}$
   (b) $\{a_n b_n\} \sim \{a'_n b'_n\}$
   (c) $[\{a_n\}]_\sim \lhd [\{b_n\}] \iff [\{a'_n\}]_\sim \lhd [\{b'_n\}]_\sim$

Doc Brown problems:

1. Have a good Spring Break! (this is worth no class points, but lots of life points)

For problems from the book, something like 1, #8 refers to #8 from the exercises at the end of Chapter 1, while something like 1.3, #1 refers to #1 from the exercises at the end of section 1.3.