Math 112 Problem Set 5

Due Feb. 28, 2018

February 22, 2018

Core problems:

1. Give examples of:

   (a) An infinite set in $\mathbb{R}$ with no accumulation points.

   (b) A nonempty subset of $\mathbb{R}$ contained in its set of accumulation points.

   (c) A subset of $\mathbb{R}$ that has infinitely many accumulation points, but contains none of them.

   (d) $A \subset \mathbb{R}$ such that $\partial A = cl(A)$

   You don't need to give formal proofs that your example works, but justify your selection. (2, #28)

2. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally, but not absolutely.

3. Prove the Comparison Test for series. Yes, this appears in the book. But I'd like you to either try it without looking at the book, or write the proof in your own words and supply any details the book omits.

4. Recall the following definition from the midterm:

   Definition. *Given a sequence $\{x_n\}_n \subset \mathbb{R}$, we say that x_n converges to infinity, written $\lim_{n\to\infty} x_n = \infty$, iff for all M, there is a natural number N so $n > N$ implies that $x_n > M$*

   Let $\{a_n\}_n$ be a sequence of nonzero numbers such that $\lim_{n\to\infty} a_n = \infty$. Show that $\lim_{n\to\infty} \frac{1}{a_n} = 0$ (note that Theorem 1.2.7.(iv) only applies to sequences that converge to a number, not infinity).

   Does the converse hold? Prove or find a counterexample.

5. Test the following series for convergence:

   (a) $\sum_{k=0}^{\infty} \frac{e^{-k}}{\sqrt{k}}$ (you may use basic facts about e^x)

   (b) $\sum_{k=0}^{\infty} \frac{k}{k^2+1}$

   (2, #52.(a), (b))

6. Test the following series for convergence:

   (a) $\sum_{k=0}^{\infty} \frac{\sqrt{k+1}}{k^2-3k+1}$

   (b) $\sum_{k=1}^{\infty} \frac{k^3}{3^k}$

   (2, #52.(c), (f))

7. Show directly that, for any $A \subset \mathbb{R}$, A is totally bounded iff A is bounded.

Niche problems:

1. Suppose that $s_n = \sum_{k=1}^{n} a_k$ is a conditionally convergent, but not absolutely convergent series. Show that for every $a \in \mathbb{R}$, there is a rearrangement of the terms whose sum is a. A formal way of writing this is: for all $a \in \mathbb{R}$, there is a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$a = \lim_{n\to\infty} \sum_{k=1}^{n} a_{\pi(k)}$$

2. Recall we defined $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Show that, for each $x \in \mathbb{R}$,

$$e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

   You can use the binomial theorem for this problem (and probably should)

3. To really finish the sequence of problems on Dedekind cuts, we need to do the following: We have defined the collection of Dedekind cuts $\mathcal{D}$ and operations $\oplus$, $\otimes$, and $\lhd$ on them solely in terms of $\mathbb{Q}$.
   Without making any reference to $\mathbb{R}$ *or the function* $A \mapsto \sup A$, show that these operations turn $\mathcal{D}$ into a complete ordered field.
   (This is rather involved, so is worth 20 pts and replaces 2 Core problems)

4. Lest you think we were done, here is an alternate way of constructing $\mathbb{R}$ from $\mathbb{Q}$ using Cauchy sequences. This method has the advantage that it's actually a general method of taking a metric space (X, d) and enlarging it to a complete metric space $(\bar{X}, \bar{d})$.

   Definition. *Let A be a set and $R \subset A \times A$. We say that R is* an equivalence relation *on A iff the following three conditions hold:*

   (a) for all $x \in A$, $(x, x) \in R$ (reflexivity)

   (b) for all $x, y \in A$, if $(x, y) \in R$, then $(y, x) \in R$ (symmetry)

   (c) for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$ (transitivity)

   Do the following:

(a) If A is any set and we use $=$ to denote the set $\{(a,a) \mid a \in A\}$, show that $=$ is an equivalence relation on A.

This is the motivation for the name. Generally, we write equivalence relation using infix notation (xRy as opposed to $(x,y) \in R$) and they are a sort of 'generalized equality.'

(b) Let A be the collection of Cauchy sequences in $\mathbb{Q}$ and define $\equiv \subset A \times A$ by

$$\{x_n\}_n \equiv \{y_n\}_n \text{ iff } \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$$

Show that $\equiv$ is an equivalence relation on A.

Doc Brown problems:

1. Prove that $\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x < 1, 0 \leq y \leq 1\}$ is not compact. (3.1, #2)
2. Let $A \subset \mathbb{R}^n$ be bounded. Show that $cl(A)$ is compact. (3.2, #4)
3. Let $A \subset \mathbb{R}$ be infinite with a single accumulation point in A. Must A be compact?(3.2, #5)

For problems from the book, something like 1, #8 refers to #8 from the exercises at the end of Chapter 1, while something like 1.3, #1 refers to #1 from the exercises at the end of section 1.3.