

Math 112 Problem Set 3

Due Feb. 14, 2018

February 4, 2018

Core problems:

1. Prove that the cluster points of the sequence defined in 1.5.4.(e) are -1, 0, and 1.
2. Define a sequence with an infinite number of cluster points.
3. Fix a sequence $\{x_n\}_n$ with a bounded and nonempty set of cluster points. Show that the infimum of the set of cluster points is a cluster point.
4. Show that $x_n \rightarrow x$ iff every subsequence of $\{x_n\}_n$ converges to x .
5. For a bounded sequence $\{x_n\}$, show that $\liminf_n x_n \leq \limsup_n x_n$.
6. If $\{x_n\}_n$ and $\{y_n\}_n$ are bounded sequences in \mathbb{R} , show that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

Find an example that shows equality does not always hold.

7. If $\{\vec{x}_k\}_n$ is a sequence in \mathbb{R}^n so $\vec{x}_k \rightarrow \vec{x}$, show that $\|\vec{x}_k\| \rightarrow \|\vec{x}\|$ in \mathbb{R} .
8. Show that any open set in \mathbb{R}^n can be written as a union of balls. (This actually holds in any metric space)
9. Find an infinite collection of open sets with a non-open intersection.

Niche problems:

1. Let $A, B \subset \mathbb{R}$ be open and set $AB = \{xy \in \mathbb{R} \mid x \in A \text{ and } y \in B\}$. Show that AB is open. (2.1, #5)
2. For $\vec{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, show that

$$|a_i| \leq \|\vec{a}\| \leq \sum_{j=1}^n |a_j|$$

3. This is the second in a sequence of Niche problems that will explore the formal construction of the real numbers. As it progresses, feel free to use previous parts of it if you want to jump in at a later stage.

Last time, we showed that $f(A) : \mathcal{D} \rightarrow \mathbb{R}$ given by $f(A) = \sup A$ is a bijection. Define the following operations on Dedekind cuts:

$$\begin{aligned} A \oplus B &= \{x + y \mid x \in A, y \in B\} \\ A \triangleleft B &\text{ iff } A \subsetneq B \end{aligned}$$

Show that $A \oplus B$ is a Dedekind cut and that these operations “commute” with f . This means that

$$\begin{aligned} f(A \oplus B) &= f(A) + f(B) \\ A \triangleleft B &\text{ iff } A \subsetneq B \end{aligned}$$

4. Define a multiplication \otimes on Dedekind cuts and show $f(A \otimes B) = f(A) \cdot f(B)$. Note that you should define the multiplication in terms of Dedekind cuts and **not** as $A \otimes B = f^{-1}(f(A) \cdot f(B))$.

Doc Brown problems:

1. For $\vec{x}_1, \dots, \vec{x}_k \in \mathbb{R}^n$, show that

$$\mathbb{R}^n - \{\vec{x}_1, \dots, \vec{x}_k\}$$

is open. (2.3, #3)

2. Let $A \subset \mathbb{R}$ be nonempty and bounded above. Must $\sup A$ be an accumulation point of A ? (2.4, #4)
3. Find the closure of $\{(x, y) \in \mathbb{R}^2 \mid x > y^2\}$. (2.5, #1)

For problems from the book, something like 1, #8 refers to #8 from the exercises at the end of Chapter 1, while something like 1.3, #1 refers to #1 from the exercises at the end of section 1.3.